SOME ASYMPTOTIC RESULTS ON
RANDOM RANK STATISTICS

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This paper deals with two different problems. The first one deals with asymptotic normality of simple linear rank statistics based on random number of observations $X_i$, henceforth called random rank statistics, under the alternative where each $X_i$ has a different distribution $F_i$. The second problem deals with showing that the random rank statistics as a function of the regression parameter in the simple linear regression model is asymptotically uniformly linear (hence continuous) in that parameter. Obviously the two problems are different and could be solved in separate papers but for certain lemmas which are common to the solution of both of these problems. It is suggested not to try to apply the result of Section 3 to Section 4, unless mentioned explicitly. The results of Section 2 are the results which are common, to some extent, to the solution of these two problems.

Pyke and Schorack [11] proved asymptotic normality of a class of two sample random rank statistics under two sample alternatives. Our theorem 3.1 could be thought of as a generalization of the result of [11] to more than two samples situation. Our score function $\phi$ is in smaller class than that of [11]. On the other hand our methods yield the asymptotic normality for random rank-sign statistics. This is also contained in Section 3. Section 2 proves a basic lemma about weak convergence of random weighted empirical cumulatives to a tied down continuous Gaussian process.

In [5] and [7] asymptotic uniform linearity of rank statistics based on nonrandom number of observations was proved. In [5] conditions are very general on $\phi$ and underlying distribution $F$ whereas conditions in [7] are quite stringent. But in [7] we do not need any artificial condition like (2.1) of [5] on underlying regression constants. However in both of these references, regression scores were assumed to be bounded. In Section 4 here we extend the results of [7] to random rank and random rank-sign statistics and to the case where regression scores need not be bounded. In Section 5 we show how the results of Section 4 can be used to construct a bounded length confidence interval for a regression parameter using rank sign statistics with asymptotic (as length $\to 0$) coverage probability achieved.

Apart from applying Theorem 3.1 to the i.i.d. case, as is mentioned in the remark at the end of Section 5, it is hoped that Theorem 3.1 can be found applicable in some other interesting situations.

1. Assumptions and notation. Suppose $X_i$, $i \geq 1$ are independent random variables with cdf's $F_i$, $i \geq 1$, $c_i$, $i \geq 1$ are some real numbers and $\{N_i\}$ a sequence of positive integer valued random variables. All the random variables are assumed to be defined on the same probability space $(\Omega, \mathcal{A}, P)$ and all probability in this paper will be computed under the probability measure $P$, unless otherwise specified. Let

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(1.1) \[ u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}. \]

Define for an integer \( n \)
\[
(1.2) \quad R_n = \sum_{j=1}^{n} u(X_i - X_j). 
\]

Let, for a real number \( b > 0 \),
\[
(1.3) \quad \mathcal{C} = \{ \varphi : \mathbb{R} \to \mathbb{R} \mid \text{ absolutely continuous,} \}
\]
\[
\varphi' \text{ exists a.e. such that } \| \varphi' \|_1 = \int \varphi'(u)\, du \leq b < \infty, \\
\varphi(u) = \int_{0}^{u} \varphi'(v)\, dv \quad \forall 0 \leq u \leq 1, \text{ and} \\
\varphi' \text{ continuous on } [0, 1].
\]

Clearly \( \varphi \in \mathcal{C} \Rightarrow \varphi \) is bounded, uniformly continuous, square integrable and \( \varphi' \) is uniformly continuous.

Define for a \( \varphi \in \mathcal{C} \)
\[
(1.4) \quad S_n = n^{-1} \sum_{i=1}^{n} c_i \varphi(R_n/(n + 1)) 
\]
\[
(1.5) \quad \mu_n = n^{-1} \sum_{i=1}^{n} c_i \varphi(n^{-1} \sum_{j=1}^{n} F_j(x)) \, dF_i(x). 
\]

Our problem is to show that under suitable conditions on the underlying objects \( \mathcal{L}(N, r(s_n - \mu_n) / s_n^{-1}) \to N(0, 1) \) for some sequence \( \{s_n\} \) of positive real numbers.

We begin by stating certain assumptions. To begin with all the limits in the sequel will be taken as \( r \to \infty \), unless otherwise specified. By \( \mathcal{L}(X) \) we will mean the law of random variable \( X \).

About \( \{N_r\} \) we assume that there exist positive integers \( \{a_r\} \) and \( \{b_r\} \) such that \( a_r < b_r \),
\[
(1.6) \quad a_r \to \infty, \quad b_r \to \infty, \quad b_r/a_r \to 1
\]
and if \( A_r = [a_r \leq N_r \leq b_r] \), then
\[
(1.7) \quad P_r[A_r] \to 1.
\]

\( A_r^c \) will denote complement of \( A_r \).

About \( \{c_i\} \) and \( \{F_i\} \) we assume that all \( \{F_i\} \) are continuous and that
\[
(1.8) \quad \lim_{n \to \infty} \max_{1 \leq i \leq n} c_i^2 / \sum_{i=1}^{n} c_i^2 \to 0.
\]

Let \( \sigma_{\alpha}^2 = n^{-1} \sum_{i=1}^{n} b_i^2. \)

We further assume that \( \forall x, y \)
\[
(1.9) \quad \lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} c_i^2 [F_i(x) \wedge F_i(y)] [1 - F_i(x) \vee F_i(y)] = K_s(x, y)
\]
eexists. Let \( G_i(u) = F_i(\tan (u - \frac{1}{2})) \) for \( 0 \leq u \leq 1 \). Assume, for \( 0 \leq u \leq 1, \)
\( 0 \leq v \leq 1 \), that
\[
(1.10) \quad \lim_{\varepsilon \to 0} \max_{1 \leq i \leq \infty} \sup_{|u| < \varepsilon, |v| < \varepsilon} |G_i(u) - G_i(v)| = 0.
\]

Furthermore assume that
\[
(1.11) \quad \sigma_{\beta}^2 / \sigma_{\alpha}^2 \to 1
\]
or equivalently
\[(1.12) \quad \sum_{i=1}^t c_i^2 / \sum_{i=1}^t c_i^2 \to 1 .\]
This implies in view of (1.6) and (1.7) that
\[(1.13) \quad \sigma^2_{v_i, e} / \sigma^2_{v_{v_i, e}} \quad \text{and} \quad \sigma^2_{v_i, e} / \sigma^2_{v_{v_i, e}} \to 1 \quad \text{in probability.}\]
We also remark here that
\[(1.14) \quad P[ \max_{1 \leq i \leq N_r} c_i^2 / \max_{1 \leq i \leq t_r} c_i^2 \leq 1 ] \to 1 .\]
A condition like (1.11) will be used in Section 2 whereas conclusions like (1.14) and (1.13) will be used in the proof of Theorem 3.1.
Next we introduce some notation so that we can represent $S_{N_r}$ as a Chernoff-Savage type statistic.
For a real number $x$ and an integer $n$ define
\[(1.15) \quad m_n(x) = n^{-1} \sum_{i=1}^n c_i u(x - X_i) , \quad \bar{m}_n(x) = n^{-1} \sum_{i=1}^n c_i F_i(x) \]
and
\[(1.16) \quad H_n(x) = n^{-1} \sum_{i=1}^n u(x - X_i) , \quad \bar{H}_n(x) = n^{-1} \sum_{i=1}^n F_i(x) .\]
Under the above notation we have
\[S_{N_r} = \int \varphi \left( \frac{N_r}{N_r + 1} H_n(x) \right) \, dm_{N_r}(x) ,\]
and
\[\mu_{N_r} = \int \varphi(\bar{H}_n(x)) \, d\bar{m}_{N_r}(x) .\]
Also define
\[(1.17) \quad L_n = n^4 (m_n - \bar{m}_n) , \quad Z_n = n^4 (H_n - \bar{H}_n) .\]
Right away, since $N_r / (N_r + 1) \to 1$ in probability, we may without loss of generality write
\[(1.18) \quad S_{N_r} = \int \varphi(H_n(x)) \, dm_{N_r}(x) .\]
Also define
\[(1.19) \quad l_r(x) = a_r^{-1} \sum_{i=1}^r (c_i - c_i) \left\{ [u(y - x) - F_i(y)] \varphi'(H_r(y)) \right\} dF_i(y)\]
where $H_r = \bar{H}_{a_r}$.
Furthermore let
\[(1.20) \quad \hat{S}_n = n^{-1} \sum_{i=1}^n l_r(X_i) , \quad s_r^2 = \text{Var} (a_r \hat{S}_n) = a_r^{-1} \sum_{i=1}^r \text{Var} (l_r(X_i)) .\]
We will need these definitions in proving our main theorem. Before closing this section we state one of the important conditions that is needed.
We need an existence of a constant $0 < k < \infty$ such that
\[(1.22) \quad \lim \sup \max_{1 \leq i \leq t_r} c_i^2 / s_r^2 \leq k < \infty .\]
It may be noted that (1.22) implies that
\begin{equation}
\lim \sup_{r \to \infty} \max_{1 \leq i \leq r} c_i^2 \sigma_{i}^2 < \infty
\end{equation}
which in turn implies (1.8).

For any cdf \( G \) we define
\[ G^{-1}(u) = \inf \{ x; G(x) \geq u \} \quad 0 \leq u \leq 1 \]

Finally for any function \( f \) on the real line, \( ||f|| \) will denote the usual "sup" norm and \( ||fG^{-1}|| \) will stand for \( \sup_{\| x \| \leq 1} |f(G^{-1}(u))| \).

2. Some weak convergence results. Let
\begin{equation}
(2.1) \quad W_n(x) = n^{-1} \sum_{i=1}^{n} d_i [u(x - X_i) - F_i(x)] = n^{-1} V_n(x) \quad \text{(say)}
\end{equation}
where \( n \) is an integer, \( \{d_i\} \) are some constants.

From [8] we recall the following

**Lemma 2.1.** If \( \{d_i\} \) and \( \{F_i\} \) satisfy conditions like (1.8), (1.9) and (1.10) then \( \forall \varepsilon > 0 \)
\begin{equation}
(2.2) \quad \lim_{\varepsilon \to 0} \lim_{n \to \infty} P[\sup_{|x - y| \leq \varepsilon} |W_n(x) - W_n(y)| > \varepsilon \sigma_n] = 0
\end{equation}
and
\begin{equation}
(2.3) \quad \lim_n \mathcal{L}(\sigma_n^{-1} W_n(x), -\infty \leq x \leq +\infty) = \mathcal{L}(W_d^0(x), -\infty \leq x \leq +\infty)
\end{equation}
where \( W_d^0 \) is a continuous Gaussian process with \( W_d^0(\pm \infty) = 0 \) wp 1, \( E W_d^0 = 0 \) and \( \text{Cov}(W_d^0(x), W_d^0(y)) = K_d(x, y) \).

**Remark.** In [8], the above lemma was proved for the processes defined on \([0, 1] \). The above \( W_n \) processes may be transformed to the ones on \([0, 1] \) by taking \( X_i \) to \( U_i = (1/\pi) \tan^{-1} X_i + \frac{1}{2} \). With this transformation the weak convergence of the processes on \([0, 1] \) is equivalent to that of the processes defined on the extended real line, above. Also the condition (2.9) of [8] is implied by the condition (1.10) above because \( F_i \) get transformed to \( G_i \) defined above.

Our objective is to show that the process \( W_{\lambda_n} \) are relatively compact as \( r \to \infty \). Before we proceed further we state an inequality here, the proof of which is straightforward generalization of one given in real random variable case on page 45 in [1]. Also see [2] for measurability considerations.

**Inequality.** Let \( D[-\infty, +\infty] \) be the space of functions on \([-\infty, +\infty] \), the elements of which are right continuous and have left limit. Let \( \{Y_i\} \) be sequence of independent rv's in \( D[-\infty, +\infty] \) and \( T_j = \sum_{i=1}^{j} Y_i \). Then \( \forall \varepsilon > 0 \)
\begin{equation}
(2.4) \quad \Prob[\max_{1 \leq j \leq n} ||T_j|| > 2\varepsilon] \leq \Prob[||T_n|| > \varepsilon](1 - \gamma_n)^{-1}
\end{equation}
where
\begin{equation}
(2.5) \quad \gamma_n = \max_{1 \leq j \leq n} \Prob[||T_n - T_j|| > \varepsilon].
\end{equation}
Lemma 2.2. In addition to the condition of Lemma 2.1 assume that \( \{a_r\}, \{b_r\} \) and \( \{N_r\} \) satisfy (1.6) and (1.7) and that \( \{d_i\} \) satisfies a condition like (1.11). Then for \( \forall \varepsilon > 0 \)
\[
\lim_{\delta \to 0} \lim_{n \to \infty} P[\sup_{|x-y| \leq \delta} |W_{N_r}(x) - W_{N_r}(y)| > \varepsilon \sigma_{N_r}] = 0
\]
and
\[
\mathcal{L}(\sigma_{N_r}^{-1} W_{N_r}(x), -\infty \leq x \leq +\infty) \to \mathcal{L}(W_d^0(x), -\infty \leq x \leq +\infty).
\]
Consequently
\[
\mathcal{L}(\sigma_{N_r}^{-1} \|W_{N_r}\|) \to \mathcal{L}(\|W_d^0\|)
\]
where \( W_d^0 \) is the same as in Lemma 2.1 and \( \|W_d^0\| \) is bounded rv’s.

Proof. The proof consists of comparing random processes with nonrandom processes.

In what follows \( \sigma_r \equiv \sigma_{a_r} \).

First we show that
\[
a_r^{-1} \sigma_r^{-1} \|V_{N_r} - V_{a_r}\| \to 0 \quad \text{in probability.}
\]
Let
\[
Y_i(x) = d_i[u(x - X_i) - F_i(x)] \quad a_r < i \leq b_r,
\]
\[
T_j(x) = \sum_{i=a_r + 1}^{b_r} d_i[u(x - X_i) - F_i(x)] \quad a_r < j \leq b_r.
\]
Then \( \{Y_i\} \) are independent rv’s and \( Y_i \in D[-\infty, +\infty] \) and (2.4) is applicable to
\[
T_j(x) = V_j(x) - V_{a_r}(x).
\]
Therefore
\[
P(\|V_{N_r} - V_{a_r}\| > 2\varepsilon a_r^{-1} \sigma_r) \leq P[\max \|V_j - V_{a_r}\| > 2\varepsilon a_r^{-1} \sigma_r] + P(A_r^c)
\]
\[
\leq P(\|T_{b_r}\| > \varepsilon a_r^{-1} \sigma_r)(1 - \eta_r)^{-1} + P(A_r^c)
\]
where “max” in the first inequality is taken over \( a_r \leq j \leq b_r \) and
\[
\eta_r = \max_{a_r \leq j \leq b_r} P[\|V_{a_r} - V_j\| > \varepsilon a_r^{-1} \sigma_r].
\]
Using (1.11) it is easy to show that \( \text{Var}(a_r^{-1} \sigma_r^{-1} T_{b_r}(x)) \to 0 \) for each fixed \( x \). Using (2.2) and (1.11) one shows that
\[
\sup_{|x-y| \leq \delta} |T_{b_r}(x) - T_{b_r}(y)| a_r^{-1} \sigma_r^{-1} \to 0 \quad \text{in probability}
\]
as \( r \to \infty \) and then \( \delta \to 0 \).

Consequently \( a_r^{-1} \sigma_r^{-1} \|T_{b_r}\| \to 0 \) in probability.

Similarly one can show that \( \eta_r \to 0 \). Combining these conclusions with (1.7) and (2.10) we have (2.9).

In view of the fact that \( a_r^{-1} \sigma_r^{-1} \|V_{a_r}\| \) is bounded in probability in the limit, which follows in view of Lemma 2.1, (2.9) implies that the rv’s \( a_r^{-1} \sigma_r^{-1} \|V_{N_r}\| \) are bounded in probability in the limit. Using this conclusion, (1.13) and (1.7) it is easy to conclude that
\[
\|\sigma_{N_r}^{-1} W_{N_r} - a_r^{-1} \sigma_r^{-1} V_{N_r}\| \to 0 \quad \text{in probability.}
\]
Combining all the above arguments and using (1.11) we have essentially proved
\[ \| \sigma_{n,r}^{-1} W_{N_r} - \sigma_r^{-1} W_{\sigma_r} \| \to 0 \quad \text{in probability.} \]

Therefore (2.7) and (2.6) are proved in view of Lemma 2.1. □

**Corollary 2.1.** Under the conditions of Lemma 2.2 we have \( \forall \varepsilon > 0 \)

\[
\begin{align*}
(2.11) \quad & \lim_{\delta \to 0} \lim_{r \to \infty} P[\sup_{|x-y| \leq \delta} |L_{N_r}(x) - L_{N_r}(y)| > \varepsilon \sigma_{N_r}] = 0, \\
(2.12) \quad & \lim_{\delta \to 0} \lim_{r \to \infty} P[\sup_{|x-y| \leq \delta} |Z_{N_r}(x) - Z_{N_r}(y)| > \varepsilon] = 0, \\
(2.13) \quad & \lim P[|L_{N_r} H_{N_r}^{-1} - L_{N_r} H_{\sigma_r}^{-1}| > \varepsilon \sigma_{N_r}] = 0, \\
(2.14) \quad & \lim P[|L_{N_r} - L_{\sigma_r}| > \varepsilon \sigma_{N_r}] = 0, \\
(2.15) \quad & \lim P[|Z_{N_r} - Z_{\sigma_r}| > \varepsilon] = 0,
\end{align*}
\]

and
\[
(2.16) \quad \lim P[|L_{N_r} H_{N_r}^{-1} - L_{N_r} H_{\sigma_r}^{-1}| > \varepsilon \sigma_{N_r}] = 0.
\]

Furthermore \( \forall \varepsilon > 0 \exists b(\varepsilon) = b \) and \( r_0 \exists r \geq r_0 \Rightarrow \)

\[
(2.17) \quad P[|Z_{N_r}| > b] < \varepsilon, \\
(2.18) \quad P[|\sigma_{N_r}^{-1} L_{N_r}| > b] < \varepsilon.
\]

Also
\[
\mathcal{L}(\sigma_{N_r}^{-1} L_{N_r}) \to \mathcal{L}(||W_{\varepsilon}^0||) \quad \text{and} \quad \mathcal{L}(||Z_{N_r}||) \to \mathcal{L}(||W_{\varepsilon}^0||).
\]

**Proof.** Put \( d_i = c_i \) in \( W_n \), then \( W_n = L_n \) and (2.11), (2.14) and (2.18) are consequences of Lemma 2.2 and its proof.

Put \( d_i = 1 \) in \( W_n \) to get \( W_n = Z_n \) and then (2.12), (2.15) and (2.17) are consequences of Lemma 2.2 and its proof.

The proof of (2.13) uses (2.11) and (2.17) and is similar to that of Theorem A6 of [6].

The proof of (2.16) uses the fact
\[
||H_{N_r}^{-1} - H_{\sigma_r}^{-1}|| \to 0 \quad \text{in probability}
\]
(2.11) and is similar to that of (2.13). □

3. Asymptotic normality of random rank statistics. Here we will state and prove our main

**Theorem 3.1.** Under (1.3), (1.6), (1.7), (1.9), (1.10), (1.11) and (1.22) we have
\[
(3.1) \quad \mathcal{L}(s_r^{-1} N_r^4 (S_{N_r} - \mu_{N_r})) \to N(0, 1)
\]
uniformly in all \( \varphi \in M \) where \( M \) is a relatively compact subset of \( C \) with respect to the norm \( ||\varphi'||_1 \).

**Proof.** Our method of proof is decomposition method of \( S_{N_r} - \mu_{N_r} \) and then showing \( S_{N_r} - \mu_{N_r} \) behaves like \( \tilde{S}_{\sigma} \).
We have the following decomposition.
\[
N_r \{ (S_{N_r} - \mu_{N_r}) = N_r \frac{1}{2} \int \phi(H_{N_r}) d(m_{N_r} - \bar{m}_{N_r}) + N_r \frac{1}{2} \int \phi'(H_{N_r}) (H_{N_r} - \bar{H}_{N_r}) d\bar{m}_{N_r} \\
+ N_r \frac{1}{2} \int \phi(H_{N_r}) - \phi(\bar{H}_{N_r}) d(m_{N_r} - \bar{m}_{N_r}) \\
+ N_r \frac{1}{2} \int \phi(H_{N_r}) - \phi(\bar{H}_{N_r}) - (H_{N_r} - \bar{H}_{N_r}) \phi'(\bar{H}_{N_r}) d\bar{m}_{N_r} \}
= B_{1N_r} + B_{2N_r} + E_{1N_r} + E_{2N_r} \quad \text{(say)}.
\]

Note that conditions (1.6), (1.7), (1.8), (1.9) and (1.10) allow us to apply results of Corollary 2.1 and we will invoke these results without going through mention of assumptions, etc.

We first shown that
\[(3.2) \quad s_r^{-1} E_{i,N_r} \rightarrow 0 \quad \text{in probability} \quad i = 1, 2\]
and then show, with \(B_{a_r} = B_{1a_r} + B_{2a_r}\), that
\[s_r^{-1} (B_{N_r} - B_{a_r}) \rightarrow 0 \quad \text{in probability}\]
and then show that \(B_{a_r}\) is nothing but \(a_r \hat{\xi}_{a_r}\).

Before proceeding note that since
\[P[\sigma_{N_r}^2 = N_r^{-1} \sum_{i=1}^{N_r} c_i^2] \leq \max_{1 \leq i \leq r} |c_i|^2] \rightarrow 1 \]
we have in view of (1.22) that
\[(3.3) \quad P[\sigma_{N_r}/s_r \leq k^1] \rightarrow 1 \quad .\]

Using \(P[L_{N_r}(\pm \infty) = 0] = 1 \quad \forall r \geq 1\) and integrating \(R_{1N_r}\) term by parts one gets
\[s_r^{-1} |E_{iN_r}| \leq (s_r^{-1} \sigma_{N_r}^{-1}) |L_{N_r} H_{N_r}^{-1} \bar{H}_{N_r}^{-1}|||\phi'||_1 \rightarrow 0 \quad \text{in probability} \]
in view of (3.3) and (2.13).

To prove (3.2) for \(E_{2N_r}\) term one proceeds as follows. Note that \(P[\bar{m}_{N_r} \ll \bar{H}_{N_r}] = 1 \quad \forall r \geq 1\),
\[(3.4) \quad P[|d\bar{m}_{N_r}/d\bar{H}_{N_r}| \leq \max_{1 \leq i \leq r} |c_i|] \rightarrow 1 \quad ,\]
and that \(\forall \epsilon > 0 \exists b \equiv b(\epsilon) \) and \(r_0 \gg r \geq r_0 \Rightarrow \)
\[(3.5) \quad H_{N_r} = \bar{H}_{N_r} + za_r^{-1} \forall |z| \leq b \quad \]
with probability at least \(1 - \epsilon\). (3.5) follows from (2.17).

Using (1.22), (3.4) and (3.5) along with an argument used in ([3] page 629), we conclude (3.2) for \(E_{2N_r}\) term.

Now consider \(B_{1N_r}\). Denote \(L_{a_r}\) by \(L_r\) and \(\bar{H}_{a_r}\) by \(\bar{H}_r\). Integrating by parts we have
\[\lim P[\int \phi(\bar{H}_{N_r}) dL_{N_r} = -\int L_{N_r} d\phi(\bar{H}_{N_r})] = 1 \quad .\]
Consequently
\[s_r^{-1}|B_{1N_r} - B_{1a}r| \leq s_r^{-1} |L_{N_r} \bar{H}_{N_r}^{-1} - L_r \bar{H}_r^{-1}|||\phi'||_1 \leq s_r^{-1} \sigma_{N_r}^{-1} \bar{H}_{N_r}^{-1} |L_{N_r} \bar{H}_{N_r}^{-1} - L_{N_r} \bar{H}_r^{-1}|||\phi'||_1 \quad \]
\[+ |L_{N_r} \bar{H}_r^{-1} - L_r \bar{H}_r^{-1}|||\phi'||_1 \quad .\]
s_r^{-1} \sigma_{N,r}^{-1} \text{ is bounded in probability by (3.3). First term on the right-hand side of the above inequality tends to zero in probability by (2.15) while the second term tends to zero in probability by the proof of Lemma 2.1. Hence}

\begin{equation}
(3.6) \quad s_r^{-1} |B_{1N,r} - B_{1a,r}| \to 0 \quad \text{in probability.}
\end{equation}

Next consider \(B_{2N,r} - B_{2a,r}\):

\[
B_{2N,r} - B_{2a,r} = \int \varphi'(\tilde{H}_{N,r}) Z_{N,r} \, d\tilde{m}_{N,r} - \int \varphi'(\tilde{H}_r) Z_r \, d\tilde{m}_r \\
= \int \{ \varphi'(\tilde{H}_{N,r}) - \varphi'(\tilde{H}_r) \} Z_{N,r} \, d\tilde{m}_{N,r} + \int \varphi'(\tilde{H}_r) Z_{N,r} \, d(\tilde{m}_{N,r} - \tilde{m}_r) \\
+ \int \varphi'(\tilde{H}_r) (Z_{N,r} - Z_r) \, d\tilde{m}_r \\
= I_{1r} + I_{2r} + I_{3r}.
\]

Now observe that in view of (1.7)

\[ ||\tilde{H}_{N,r} - \tilde{H}_r|| \to 0 \quad \text{in probability.} \]

Using this fact along with uniform continuity of \(\varphi', (1.22)\) and (2.17) it is easy to see that

\begin{equation}
(3.7) \quad s_r^{-1} |I_{1r}| \to 0 \quad \text{in probability.}
\end{equation}

Again, using (1.22), (2.14), the fact that \(\tilde{m}_{a_r} \ll \tilde{H}_r\) and the assumption \(||\varphi'||_1 < \infty\), it is easy to see that

\begin{equation}
(3.8) \quad s_r^{-1} |I_{2r}| \to 0 \quad \text{in probability.}
\end{equation}

Finally write \(\tilde{m}_r = \tilde{m}_{a_r},\)

\[
I_{3r} = \int \varphi'(\tilde{H}_r) Z_{N,r} \, d(\tilde{m}_{N,r} - \tilde{m}_r) \\
= \int \varphi' Z_{N,r} \tilde{H}_r^{-1} \, d(m_{N,r} \tilde{H}_r^{-1} - \tilde{m}_r \tilde{H}_r^{-1})
\]

so that

\begin{equation}
(3.9) \quad |s_r^{-1} I_{3r}| \leq ||Z_{N,r} \tilde{H}_r^{-1}|| s_r^{-1} \int \varphi' \, d(\tilde{m}_{N,r} \tilde{H}_r^{-1} - \tilde{m}_r \tilde{H}_r^{-1}) \\
\to 0 \quad \text{in probability}
\end{equation}

because of (2.17), the fact that

\[ s_r^{-1} ||\tilde{m}_{N,r} \tilde{H}_r^{-1} - \tilde{m}_r \tilde{H}_r^{-1}|| \to 0 \quad \text{in probability} \]

and that \(\varphi'\) is continuous.

Combining (3.9), (3.8), (3.7) and (3.6) one has proved

\begin{equation}
(3.10) \quad s_r^{-1} |B_{1N,r} - B_{1a,r}| \to 0 \quad \text{in probability.}
\end{equation}

Now integrating by parts the \(B_{2a,r}\) term and adjusting for constants arising thereby one gets

\[
P[B_{2a,r} = \int_{-\infty}^{\infty} \varphi'(\tilde{H}_r(y)) \, d\tilde{m}_r(y) \, dZ_{a_r}(x)] = 1 \quad \forall r \geq 1.
\]

After some simple algebra one gets

\begin{equation}
(3.11) \quad B_{2a,r} = a_r^{-1} \sum_{s=1}^{a_r} \{ a_r^{-1} \sum_{j=1}^{a_r} c_j \int [u(y - X_i) - F_i(y)] \varphi'(\tilde{H}_r(y)) \, dF_j(y) \}
\quad \text{wp 1 for all } r \geq 1.
\end{equation}
Using \( \varphi(u) = \int_u^\infty \varphi'(v) \, dv \), we have similarly

\[
3.12 \quad B_{1a_r} = -a_r^{-1} \sum_{i \in r} c_i \left[ a_r^{-1} \sum_{j \in r} [u(y - X_i) - F_i(y)] \varphi' (\bar{H}_r(y)) \right] \, dF_j(y) \quad \text{wp 1}.
\]

Consequently

\[
3.13 \quad B_{a_r} = B_{1a_r} + B_{2a_r} = a_r^{-1} \sum_{i \in r} [a_r^{-1} \sum_{j \in r} (c_j - c_i) \left[ u(y - X_i) - F_i(y) \right] \varphi' (\bar{H}_r(y)) \right] \, dF_j(y) = a_r^{-1} \sum_{i \in r} l_r(x_i) = a_r^{-1} \tilde{S}_a_r
\]

where \( l_r(x) \) and \( \tilde{S}_a_r \) are defined by (1.19) and (1.20). Therefore in view of (3.10) and (3.2) it is enough to show that

\[
3.14 \quad \mathcal{L}(a_r^{-1} \tilde{S}_a_r) \to N(0, 1).
\]

But it is not hard to show that the summands \( s_r^{-1} a_r^{-1} l_r(x_i) \) satisfy Lindberg-Feller condition, thereby enabling us to conclude (3.14). Consequently (3.1) is proved for each fixed \( \varphi \in M \).

The fact about uniformity in \( \varphi \in M \) follows because all the four terms \( E_i \bar{H}_r \) and \( B_{i \times r} \), \( i = 1, 2 \), may be shown to satisfy Lipschitz condition in the norm \( \| \varphi' \|_1 \). This can be done in the same way as was done in [3].

**Note.** It may be noted that the condition (1.22) is similar to (2.16) of [4].

**Remark.** 1. Theorem 3.1 is true also for random rank-sign statistics. To see this we introduce

\[
3.15 \quad S_{n^+} = n^{-1} \sum_{i=1}^n c_i \varphi(R_i^n/(n + 1)) \, \text{sgn}(x_i)
\]

where

\[
3.16 \quad R_i^n = \sum_{j=1}^n u(|X_i| - |X_j|)
\]

Also let

\[
3.17 \quad \nu_n(x) = n^{-1} \sum_{i=1}^n c_i u(x - X_i) \, \text{sgn}(x_i)
\]

\[
\tilde{\nu}_n(x) = n^{-1} \sum_{i=1}^n c_i [\text{sgn}(x) F_i(x) - 2F_i(0) u(x)]
\]

\[
H_n^+(|x|) = n^{-1} \sum_{i=1}^n u(|x| - |X_i|)
\]

\[
\bar{H}_n^+(|x|) = n^{-1} \sum_{i=1}^n [F_i(|x|) - F_i(-|x|)]
\]

\[
L_n^+(x) = n^2 (\nu_n(x) - \tilde{\nu}_n(x))
\]

\[
Z_n^+(|x|) = n^2 (H_n^+(|x|) - \bar{H}_n^+(|x|)).
\]

In view of these definitions and that of \( W_n \) in (2.1), we have, if \( d_i = c_i \) in (2.1)

\[
L_{N_r}^+(x) = W_{N_r}(x) \, \text{sgn}(x) - 2W_{N_r}(0) u(x)
\]

and, if \( d_i = 1 \) in (2.1),

\[
Z_{N_r}^+(|x|) = W_{N_r(|x|)} - W_{N_r}(-(|x|))
\]
so that all the conclusions of Corollary 2.1 remain valid also for \( L_{x_r}^+ \) and \( Z_{x_r}^+ \) process. Therefore we state the following theorem, the proof of which is precisely similar to that of Theorem 3.1 in view of the above remarks and hence omitted.

**Theorem 3.2.** Assume (1.3), (1.6), (1.7), (1.8), (1.9) and (1.10) and (1.11) are true. Then existence of a constant \( k^+ \) such that

\[
(3.20) \quad \lim \sup \max_{1 \leq i \leq n_r} |a_i|^2/v_r^2 \leq k^+ < \infty
\]

entails

\[
(3.21) \quad \lim \mathcal{L}(v_r^{-1}N_r^{+}(S_{N_r}^+ - \mu_{N_r}^+)) = N(0, 1)
\]

where

\[
\mu_{N_r}^+ = N_r^{-1} \sum_{j=1}^{N_r} c_j \phi(H_{N_r}(y))) \text{sgn}(y) \, dF_j(y)
\]

\[
v_r^2 = a_r^{-1} \sum_{i=1}^{N_r} \text{Var}(l_{i+}(X_i))
\]

with

\[
(3.22) \quad l_{i+}^r(x) = a_r^{-1} \sum_{j=1}^{N_r} [(c_j - c_i \text{sgn}(x))(\{ u(|y| - |x|) - G_i(|y|) \})
\]

\[
\times \text{sgn}(y)\phi'(H_r^+(|y|)) \, dF_j(y)
\]

where

\[
(3.23) \quad G_i(|y|) = F_i(|y|) - F_i(-|y|).
\]

**Remark 2.** It is not hard to see that in both these theorem \( s_r^2 \) and \( v_r^2 \) may be replaced by

\[
s_{N_r}^+ = N_r^{-1} \sum_{i=1}^{N_r} \text{Var}(l_{i+}(X_i))
\]

and

\[
v_{N_r}^2 = N_r^{-1} \sum_{i=1}^{N_r} \text{Var}(l_{i+}(X_i))
\]

respectively, which is slight improvement over having asymptotic variance entirely devoid of \( N_r \).

**Remark 3.** Conditions (1.6) and (1.7) are equivalent to saying that there exists a sequence \( n_r \to \infty \) such that

\[
N_r/n_r \to 1 \quad \text{in probability.}
\]

**Remark 4.** All the above and following results remain valid if \( r \) is taken to be in set of positive numbers instead of in the set of positive integers.

**Remark 5.** Finally all the above results remain valid if \( \{d_i\} \), \( \{X_i\} \) and \( \{c_i\} \) are replaced by \( \{d_i^r\} \), \( \{X_i^r\} \) and \( \{c_i^r\} \) respectively so long as dependence on \( r \) is via \( a_r \) or \( b_r \) only and not via \( N_r \).

**4. Asymptotic uniform linearity (AUL) of random rank statistics in regression parameter.** In this section we take

\[
(4.1) \quad F_i(y) = F(y + tx_i)
\]

or equivalently \( X_i = X_i' - tx_i \), where \( X_i' \) are i.i.d. \( F, F \) a cdf, \( \{x_i\} \) some real numbers and \( t \) is the regression parameter of interest.
\(S_n(t), W_n(t, \cdot), L_n(t, \cdot)\) etc. will denote the corresponding statistics and processes \(S_n, W_n(\cdot), L_n(\cdot)\) etc. based on the above \([X_i]\). Thus, for example, \(S_n(t)\) is the rank statistic based on the ranks of \([x_i - t x_i]\), \(1 \leq i \leq n\) etc.

Our problem here is to show that \(T_{N, x}(t) = S_{N, x}(t N^{-1} \sigma_{N, x}^{-1})\), when suitably normalized, is asymptotically uniformly linear (hence continuous) (AUL) in \(t\) in a bounded set in probability.

The solution consists of first proving that the statistics \(S_{N, x}(t a_r^{-1} \sigma_{a_r}^{-1})\) are AUL and then using this and (1.11) one gets the desired result. Note that the rv's that define \(T_{N, x}(t)\) are \([X_i - t x_i N, x]\), where

\[
x_{in} = (x_i / n^1 \sigma_{n, x},)
\]

and therefore are not independent whereas the rv's that define \(S_{N, x}(t a_r^{-1} \sigma_{a_r}^{-1})\) are independent and this is the reason to proceed to solve the problem in the above mentioned fashion.

In what follows \(\sup_{t, x} |g(t, x)|\) or \(|g|_a\) will stand for sup being taken over all \(|t| \leq a\) and \(-\infty \leq x \leq +\infty\) of a function \(g(\cdot, \cdot)\), for a \(0 < a < \infty\) fixed. Furthermore let

\[
W_{rn}(t, y) = W_n(t a_r^{-1} \sigma_{a_r}^{-1}, y)
\]

\[
L_{rn}(t, y) = L_n(t a_r^{-1} \sigma_{a_r}^{-1}, y)
\]

etc.

Note that \(W_{rn}(0, y) = W_n(0, y) = W_n(y)\) as defined by (2.1) when \(F_i \equiv F\). Similar statements may be applied to \(L_{rn}(0, y), Z_{rn}(0, y)\) etc.

We have the

**Lemma 4.1.** If \(F\) is absolutely continuous with a bounded and uniformly continuous pdf \(f\) and if \([d]\) and \([x]\) satisfy a condition like (1.8) then \(\forall \varepsilon > 0\) and any \(0 < a < \infty\) fixed,

\[
\lim_{n \to \infty} P[\sup_{t, x} |W_{rn}(t, x) - W_n(0, x)| > \varepsilon \sigma_{n, d}] = 0.
\]

Consequently

\[
\mathcal{L}(|W_{rn}|_a) \to \mathcal{L}(|W_d|^0).
\]

**Proof.** If \(\sigma_{n, d}^2\) and \(\sigma_{n, x}^2\) were bounded in addition to the above assumptions then the proof is given in the Appendix of [6]. In order to carry that proof through under the current assumptions one needs to make the following modifications in Lemmas A1 through A6 of [6]:

(i) Premultiply everything by \(\sigma_{n, d}^{-1}\) and replace \([c]\) by \([d]\).

(ii) Replace \(\delta_{n}(t)\) by \(t x_{in}\).

(iii) Replace \(\lambda_{n}(\varepsilon')\) by \(\varepsilon'|x_{in}|\).

Recall the definition of \([x_{in}]\) from (4.2) above. After these modifications all the details of Appendix [6] will go through in view of the fact that \(\sigma_{n, d}^{-1} \sum |d, x_{in}| \sigma_{n, d}^{-1} \leq 1\) and that the processes \(\{\sigma_{n, d}^{-1} W_n(t, x); -\infty \leq x \leq +\infty\}\) weakly converge to a continuous Gaussian process for each fixed \(t\) which follows from Lemma 2.1
above and which is basic to the method of proof in [6]. We leave out the
details. □

Note. In [6] Lemma A1 pdf “f” is supposed to be uniformly continuous but
not absolutely continuous as appears there.

**Lemma 4.2.** In addition to the conditions of Lemma 4.1, let \{d_i\} satisfy (1.11). Then under (1.6) and (1.7) \(\forall \varepsilon > 0\) and any \(0 < a < \infty\) fixed,

\[
\lim \sup \left[ \sup_{t} \sup_{x} \left| \sigma_{r}^{-1} W_{r,N_r}(t, x) - \sigma_{a}^{-1} W_{a,r}(t, x) \right| > 2\varepsilon \right] = 0.
\]

Consequently in view of (4.3) and (4.4) \(\forall \varepsilon > 0\) and any \(0 < a < \infty\) fixed,

\[
\lim \sup \left[ \sup_{t} \sup_{x} \left| W_{r,N_r}(t, x) - W_{a,N_r}(0, x) \right| > \varepsilon \sigma_{r,N_r} \right] = 0
\]

and

\[
\mathcal{L}(\sigma_{r}^{-1} || W_{r,N} ||) \rightarrow \mathcal{L}(|| W_{a} ||).
\]

Furthermore in view of (4.6) and (2.6) when adapted to the i.i.d. case, we have \(\forall \varepsilon > 0\)

\[
\lim \sup \left[ \sup_{t} \left| W_{r,N_r}(t, x) - W_{a,N_r}(t, y) \right| > \varepsilon \sigma_{r,N_r} \right] = 0.
\]

**Proof.** Let \(\varepsilon_r = 2\varepsilon a_r \sigma_{a,N_r} \sigma_{r,N_r} \sigma_{a}^{-1} \). From (2.1) we have

\[
\left| W_{r,N_r}(t, x) - W_{a,N_r}(t, x) \right| \leq \left| (N_r^{-1} - a_r^{-1})a_r^{-1}a_r^{-1} V_{r,N_r}(t, x) \right|
\]

\[
+ \left| V_{r,N_r}(t, x) - V_{a,N_r}(t, x) \right| a_r^{-1}.
\]

Let us drop first suffix \(r\) in the sequel to facilitate writing. Thus e.g. \(V_{r,N} \equiv V_{r,N_r} \) etc. Now

\[
P[\left| V_{r,N} - V_{a,N} \right| > 2\varepsilon_r] \leq P[\max \left| V_j - V_{a,j} \right| > 2\varepsilon_r] + P(A_r)
\]

where “max” is over \(a_r \leq j \leq b_r\).

By simply imitating the proof of Lemma 3.21 of [1] one can conclude that

\[
P[\max \left| V_j - V_{a,j} \right| > 2\varepsilon_r] \leq P[\left| V_{a,r} - V_{a,j} \right| > \varepsilon_r] (1 - \eta_r')^{-1}
\]

where \(\eta_r' = \max P[\left| V_{a,r} - V_{j} \right| > \varepsilon_r].\)

In carrying out the details to prove (4.11) one needs to remember that the rv’s \(\left| V_j - V_{a,j} \right|\) and \(\left| V_{a,r} - V_{j} \right|\) are independent for each \(a_r \leq j \leq b_r\). We

will leave out the detail. Now

\[
a_r^{-1} \sigma_{r,a}^{-1} (V_{a,r}(t, x) - V_{a,a}(t, x)) = \lambda_r^{-1} (\sigma_{r,a}^{-1} a_r^{-1}) \sigma_{a,r}^{-1} W_{a,r} - \sigma_{a}^{-1} W_{a,a}
\]

\[
= k_r \sigma_{a,r}^{-1} W_{a,r}(t, x) - \sigma_{a}^{-1} W_{a,a}(t, x)
\]

where \(\lambda_r = (a_r/b_r)^{-1}, k_r = \lambda_r^{-1} (\sigma_{r,a}^{-1} a_r^{-1}). \) Therefore

\[
a_r^{-1} \sigma_{r,a}^{-1} \left| V_{a,r}(t, x) - V_{a,a}(t, x) \right|
\]

\[
\leq k_r \sigma_{a,r}^{-1} \left| W_{a,r}(t, x) - W_{a,a}(0, x) \right| + \sigma_{a}^{-1} \left| W_{a,a}(t, x) - W_{a,a}(0, x) \right|
\]

\[
+ \sigma_{r,a}^{-1} \left| W_{a,a}(0, x) - W_{a,a}(0, x) \right| + |\lambda_r^{-1} - 1| \sigma_{r}^{-1} \left| W_{a,a}(0, x) \right|.
\]

Using (1.11) it is easy to show that if \(T \equiv (W_{a,a}(0, x) - W_{a,a}(0, x)), then
Var \((T_r(x))\) → 0 for each fixed \(x\). Moreover using (2.2) when adapted to the i.i.d. case one concludes that

\[
\sup_{|x-y| \leq \delta} |T_r(x) - T_r(y)| \to 0 \quad \text{in probability}
\]

when \(r \to \infty\) and then \(\delta \to 0\).

Consequently \(\sup_{x} \sigma_{r}^{-2}||W_{kr}(0, x) - W_{kr}(0, x)|| \to 0\) in probability. Furthermore observe that the sequence \(\sup_{x} \sigma_{r}^{-2}||W_{kr}(0, x)||\) of rv’s is bounded in probability. Combining these facts with (4.3) and that \(k_{r} \to 1, |\beta_{r}^{k} - 1| \to 0\), one concludes, in view of (4.12), that

\[
\sigma_{r}^{-1}c^{-1}||V_{kr} - V_{kr}||_{a} \to 0 \quad \text{in probability.}
\]

Similarly one can show that \(\eta_{r} \to 0\). Consequently in view of (4.11), (4.10) and (1.7)

\[
a_{r}^{-1}\sigma_{r}^{-2}||V_{N_{r}} - V_{kr}||_{a} \to 0 \quad \text{in probability.}
\]

(4.14) and (4.4) imply that

\[
\mathbb{P}(a_{r}^{-1}\sigma_{r}^{-2}||V_{N_{r}}||_{a}) \to \mathbb{P}(||W_{kr}||)
\]

and hence the rv’s \(a_{r}^{-1}\sigma_{r}^{-2}||V_{N_{r}}||_{a}\) are bounded in probability in the limit. Using this and (1.7) with (4.9) we conclude that

\[
\sigma_{r}^{-2}||W_{rN_{r}} - W_{rkr}||_{a} \to 0 \quad \text{in probability}
\]

which in turn implies that the rv’s \(\sigma_{r}^{-2}||W_{rN_{r}}||_{a}\) are bounded in probability in the limit because of (4.4).

Again using this, (4.16) and (1.11) it is easy to conclude that

\[
||\sigma_{r}^{-1}W_{rN_{r}} - W_{rkr}||_{a} \to 0 \quad \text{in probability.}
\]

If we put \(d_{i} = c_{i}\) in \(W_{kr}\), we have \(W_{kr}(t, x) = L_{kr}(t, x) \forall t, x\ wp 1\) and if we put \(d_{i} = 1\) in \(W_{kr}\), we have \(W_{kr}(t, x) = Z_{kr}(t, x) \forall t, x\ wp 1\). Using these relations with Lemma 4.2 we have

**Corollary 4.1.** \(\forall \varepsilon > 0\) and any \(0 < \theta < \infty\) fixed

\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} P[\sup_{t, x} ||L_{kr}(t, x) - L_{rkr}(t, x)|| > \varepsilon \sigma_{rkr}] = 0,
\]

\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} P[\sup_{t, x} ||Z_{kr}(t, x) - Z_{rkr}(t, x)|| > \varepsilon] = 0,
\]

\[
\lim_{t, x} P[\sup_{t, x} ||L_{kr}(t, x) - Z_{rkr}(0, x)|| > \varepsilon \sigma_{rkr}] = 0,
\]

\[
\lim_{t, x} P[\sup_{t, x} ||Z_{kr}(t, x) - Z_{kr}(0, x)|| > \varepsilon] = 0.
\]

Moreover \(\exists b < \infty \exists a_{r} > 0\)

\[
P(||Z_{rkr}||_{a} > b) \to 0.
\]

Also \(\forall \varepsilon > 0\)

\[
\lim_{r \to \infty} \sup_{t, x} \sup_{0 < u \leq 1} |L_{rkr}(t, H_{rkr}(t, u)) - L_{rkr}(t, H_{rkr}(t, u))| > \varepsilon \sigma_{rkr} = 0.
\]

We now state our
THEOREM 4.1. Let \( \{X_i\} \) be i.i.d. rv's with an absolutely continuous cdf \( F \) and uniformly continuous and bounded pdf \( f \). Let \( \{c_i\} \) and \( \{x_i\} \) satisfy (1.11) and (1.8). Let \( \{b_i\} \) and \( \{a_i\} \) satisfy (1.6) and (1.7). Furthermore assume \( f \) is such that

\[
I(f) = \sum_{i=1}^{\infty} \left( f^\prime/f \right)^2 \, dF < \infty.
\]

Moreover assume that \( \varphi \in \mathcal{C}_0 \) where

\[
\mathcal{C}_0 = \{ \varphi : \varphi \in \mathcal{C} ; \int |\varphi'|^2 < \infty ; ||\varphi''|| \leq K \}.
\]

Then \( \forall \varepsilon > 0 \) and any \( 0 < a < \infty \)

\[
\lim P[\sup_{t \leq a} N_r^{-1} S_{N_r}(ta^{-1}a_{\varepsilon, r}^{-1}) - S_{N_r}(0) - tN_r^{-1} \tilde{b}_n(\varphi, f)] > \varepsilon \sigma_{N_r, \varepsilon, r} = 0
\]

where

\[
\sigma_{\varepsilon, r}^2 = \sum_{i=1}^{N_r} (x_i - \tilde{x}_n)(c_i - \bar{c}_n)\sigma_{\varepsilon, r}^{-1}
\]

with \( \tilde{x}_n = \frac{1}{N_r} \sum_{i=1}^{N_r} x_i \), \( \bar{c}_n = \frac{1}{N_r} \sum_{i=1}^{N_r} c_i \) and \( \varphi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u)) \).

PROOF. In view of Corollary 4.1 the proof of this theorem is analogous to that of Theorem 2.1, Lemma 2.3 and Theorem 2.2 of [7], where a result like (4.25) was proved for nonrandom sample size case, to which, results like Corollary 4.1, for a nonrandom sample size case, were basic. However there is a slight difference in our present assumptions and those of the above mentioned results. In Theorem 2.1 of [7] we assume \( F \) to be strictly increasing and \( \sigma_{\varepsilon, r}^2 \) and \( \sigma_{\varepsilon, r}^2 \) to be bounded. Boundedness of \( \sigma_{\varepsilon, r}^2 \) and \( \sigma_{\varepsilon, r}^2 \) were used in obtaining an analogue of Corollary 4.1 for nonrandom sample size case but we have seen it here that Corollary 4.1 is true without these assumptions. Next, strict monotonicity of \( F \) was used in showing that \( \sup_{t \leq a} |R_n(t)| \to 0 \) in probability, where \( R_n(t) \) is either \( R_n \) of Theorem 3.1 above when adapted to present situation or \( R_n(t) \) of Theorem 2.1 of [7]. We will show here that under \( \varphi \in \mathcal{C}_0 \)

\[
\sigma_{N_r, \varepsilon, r}^{-1} \sup_{t \leq a} |R_{2N_r}(t)| \to 0 \quad \text{in probability.}
\]

Let \( g_r(u, z) = \{\varphi(u + a^{-1}z) - \varphi(u) - za_{\varepsilon, r}^{-1}\varphi'(u)\} \). Now in view of (4.21) and that fact that \( (N_r a_{\varepsilon, r}^{-1}) \to 1 \) in probability, we have that \( \forall \varepsilon < 0 \exists \beta \equiv \beta(\varepsilon) \) and \( r_0 \equiv r(\varepsilon) \rightarrow r_0 \rightarrow \)

\[
P[H_{N_r}(t, y) = H_{N_r}(t, y) + za_{\varepsilon, r}^{-1} \forall |t| \leq a; \quad -\infty \leq y \leq +\infty \quad \text{and} \quad |z| \leq \beta] \geq 1 - \varepsilon.
\]

Hence with probability at least \( 1 - \varepsilon \) for \( r \geq r_0 \) we have

\[
R_{2N_r}(t) = N_r^{-1} \sum_{i=1}^{N_r} c_i \sum_{i=1}^{N_r} g_i(\tilde{H}_{N_r}(t, y), z) \, dF(y + tx_i) = N_r^{-1} \sum_{i=1}^{N_r} c_i Z_{ir} (\text{say}) \leq \sigma_{N_r, \varepsilon, r} \left( N_r^{-1} \sum_{i=1}^{N_r} Z_{ir}^2 \right)^{1/2} \quad \text{(by Cauchy Schwarz)}.
\]
But again by Cauchy-Schwarz inequality

\[ N_r^{-1} \sum_{i=1}^{N_r} Z_r^2 \leq N_r^{-1} \sum_{i=1}^{N_r} \| N_r^{-1} g_r(\tilde{H}_{N_r}(t, y), z) \|^2 dF(y + tx_r) \]
\[ = \int [N_r^{-1} g_r(\tilde{H}_{N_r}(t, y), z)]^2 d\tilde{H}_{N_r}(t, y) \]
\[ = \int [N_r^{-1} g_r(u, z)]^2 du \]

so that with probability at least \(1 - \varepsilon\) we have, for \(r \geq r_0\),

\[ \sigma_{N_r)^{-1}}^{-1} |\tilde{R}_{2N_r}(t)| \leq \| \int [N_r^{-1} g_r(u, z)]^2 du \| \]
\[ \leq b[\int [\varphi_r^2(u)] du ](N_r^{-1} a_r^{-1}) \]

where

\[ \varphi_r(u) = \sup_{|z| \leq b} \left| \frac{\varphi_r(u + za_r^{-1}) - \varphi_r(u)}{za_r^{-1}} - \varphi'(u) \right| \to 0 \]
\[ \text{for almost all } u. \]

Furthermore since \(\int |\varphi'|^2 < \infty\) and since \(\int |\varphi|^2 < \infty\), we have

\[ \int [\varphi_r^2(u)] du \to 0 \]
\[ \text{also.} \]

Hence we have

\[ \sup_{r} \sigma_{N_r)^{-1}}^{-1} |\tilde{R}_{2N_r}(t)| \to 0 \]
\[ \text{in probability.} \]

The rest of the proof is precisely similar to that of the nonrandom sample size case, as given in [7]. \(\square\)

Now an obvious implication of Theorem 4.1 above and conditions (1.6), (1.7) and (1.11) is

**Theorem 4.2.** Under the conditions of Theorem 4.1 we have \(\forall \varepsilon > 0\) and any \(0 < a < \infty\) fixed,

\[ \lim P[\sup_{|z| \leq b} N_r^{-1} |T_{N_r}(t) - T_{N_r}(0)| > tN_r^{-1} b_{N_r}(\varphi, f)] > \varepsilon \sigma_{N_r)^{-1}}^{-1} \]
\[ |T_{N_r}(t) - T_{N_r}(0)| = 0 \]
\[ \text{where} \]
\[ b_{N_r}(\varphi, f) = \sigma_{N_r)^{-1}}^{-1} \tilde{b}_{N_r}(\varphi, f) \]

and \(T_{N_r}(t) = S_{N_r}(tn^{-1}a_{N_r}^{-1})\).

Because of relationships between \(L_{N_r}^+, Z_{N_r}^+, \text{etc.} \) with \(W_{N_r}\) as indicated in Remark 3 at the end of Section 3, in proving Theorem 4.2 we also have essentially proved

**Theorem 4.3.** In addition to the assumption of Theorem 4.1 assume \(F\) to be symmetric about zero. Then \(\forall \varepsilon > 0\) and any \(0 < a < \infty\),

\[ \lim P[\sup_{|z| \leq b} N_r^{-1} |T_{N_r}(t) - T_{N_r}(0)| > N_r^{-1} b_{N_r}(\varphi, f)] > \varepsilon \sigma_{N_r)^{-1}}^{-1} \]
\[ |T_{N_r}(t) - T_{N_r}(0)| = 0 \]
\[ \text{where} \]
\[ T_{N_r}(t) \text{ is the rank sign statistic based on the ranks of } \{ |X_i' - tx_{in}|; 1 \leq i \leq n \} \]
\[ \text{and signs of } \{ X_i' - tx_{in}; 1 \leq i \leq n \} \text{ or equivalently} \]
\[ T_{N_r}(t) = S_{N_r}(tn^{-1}a_{N_r}^{-1}) \]
where $S_n^+(t)$ is $S_n^+$ as defined by (3.18) and (3.19) for rv's \( \{X_i' - tx_i; 1 \leq i \leq n\} \) and where

\[
(4.30) \quad b^+(\varphi, f) = -2n^{-1} \sigma_{nx}^{-1} \sum_{i=1}^{n} c_i x_i \int_{0}^{\infty} f(x) \varphi'(G(x)) \, dF(x)
\]

and $G(x) = 2F(x) - 1$, $x \geq 0$.

5. An application. Suppose $\{X_i'\}$ are independent rv's $\omega$

\[
(5.1) \quad P[X_i' \leq y] = F(y + tx_i), \quad i \geq 1
\]

where $t$ is the regression parameter of interest. Given a number $d > 0$ and $0 < \alpha < 1$, we give here a class of confidence interval of prescribed length $2d$ which achieves prescribed coverage probability $1 - 2\alpha$ as $d \to 0$, using rank sign statistics.

Assume $\varphi \in \mathcal{C}_0$ and $\varphi \downarrow$ so that $S_n^+(t)$ is $\downarrow$ in $t$ also see e.g. [6]. Let $z_a = k_a^+$ be $\omega \Phi(z_a) = 1 - \alpha$, $\Phi$ the $N(0, 1)$ cdf. Define

\[
(5.2) \quad \sigma_n = \sigma_{nx} \sigma_{\varphi} \quad L_n = \inf \{t; \sigma_n^{-1} n^t S_n^+(t) \leq z_a\} \\
U_n = \sup \{t; \sigma_n^{-1} n^t S_n^+(t) \geq -z_a\}.
\]

Also define for $d > 0$

\[
(5.3) \quad N_d = \min \{n \geq 2; \sigma_{nx}(U_n - L_n) \leq 2d\}.
\]

From the methods in [10] and nonrandom sample size version of Theorem 4.3 above, when $c_i = x_i$, one can conclude that

\[
(5.4) \quad n^t \sigma_{nx}(U_n - L_n) \to \{2z_a/\sigma_{\varphi}^{-1} b^+(\varphi, f)\} \quad \text{in probability}
\]

where

\[
(5.5) \quad b^+(\varphi, f) = 2 \int_{0}^{\infty} f(x) \varphi'(G(x)) \, dF(x).
\]

Consequently

\[
(5.6) \quad \sigma_{nx}(U_n - L_n) \to 0 \quad \text{in probability}.
\]

Using (5.6) it is easy to show that

\[
(5.7) \quad N_d/n_d \to 1 \quad \text{in probability \quad as \quad } d \to 0
\]

where

\[
(5.8) \quad n_d^t = \{2z_a/\sigma_{\varphi}^{-1} b^+(\varphi, f)\} d^{-1}.
\]

Thus conditions (1.6) and (1.7) are satisfied with $r = 1/d$, $a_r \equiv a_d = n_d - 1$ and $b_r \equiv b_d = n_a + 1$, say, and $N_r \equiv N_d$.

THEOREM 5.1. If $F$ is symmetric about zero and is absolutely continuous with a bounded and uniformly continuous density $f$, $\{x_i\}$ satisfy (1.11) and (1.8) and $\varphi \in \mathcal{C}_0$ and is $\downarrow$, then

\[
(5.9) \quad \lim_{d \to 0} \mathcal{L}_t(\sigma_{n_d^t} N_d^t(U_{N_d} - t)) = N(-z_a B^{-1}, B^{-1})
\]

\[
(5.10) \quad \lim_{d \to 0} \mathcal{L}_t(\sigma_{n_d^t} N_d^t(L_{N_d} - t)) = N(z_a B^{-1}, B^{-1})
\]
and
\begin{equation}
\lim_{d \to 0} \mathcal{L}_t(\sigma_{N_d}^{-1} N_d^{-1}(U_{N_d} - t), \sigma_{N_d}^{-1} N_d^{-1}(L_{N_d} - t)) = \text{Bivariate degenerate normal distribution with means } -z_a B^{-1}, z_a B^{-1} \text{ and variances } \sigma^{-2} \text{ where } B \equiv B(\varphi, f) = \sigma^{-1} b(\varphi, f) \text{ and where } \mathcal{L}_t(X) \text{ is law of an rv } X \text{ when } t \text{ is the true parameter}. \text{ Consequently, if } P_t \text{ is probability when } t \text{ is the true parameter, then}
\end{equation}
\begin{equation}
P_t[L_{N_d} \leq t \leq U_{N_d}] \to 1 - 2\alpha.
\end{equation}

\textbf{Proof.} Without any further mention, results of Section 4 will be applied with } \epsilon_i = x_i.

\text{In view of (5.7) and results of [9] we conclude that}
\begin{equation}
\mathcal{L}(\sigma_{N_d}^{-1} N_d^{-1} S_{N_d}^{-1}(0)) \to N(0, 1).
\end{equation}

\text{Now from (4.30) and (5.5) we have } b_d(\varphi, f) = -\alpha b(\varphi, f) \text{ so that}
\begin{equation}
\sigma_{N_d}^{-1} b_d(\varphi, f) = -b(\varphi, f) \text{ wp 1}.
\end{equation}

\text{Since conditions (1.6) and (1.7) are satisfied by } N_d \equiv N_r (r = 1/d) \text{ we can conclude from Theorem 4.3, in view of our assumptions, that}
\begin{equation}
\lim_{d \to 0} \mathcal{L}_d(\sigma_{N_d}^{-1} N_d^{-1} S_{N_d}(0)) = \mathcal{L}_d(\sigma_{N_d}^{-1} N_d^{-1} S_{N_d}(0) - tB(\varphi, f)).
\end{equation}

\text{Since } \varphi \text{ is } \frac{1}{d}, \text{ both } U_n \text{ and } L_n \text{ are well defined and we have in view of (5.2) and (4.29) that}
\begin{equation}
\lim_{d \to 0} P_t[\sigma_{N_d}^{-1} N_d^{-1}(U_{N_d} - t) \leq y]
\end{equation}
\begin{equation}
= \lim_{d \to 0} P_d[\sigma_{N_d}^{-1} N_d^{-1} U_{N_d} \leq y]
= \lim_{d \to 0} P_d[\sigma_{N_d}^{-1} N_d^{-1} S_{N_d}(y\sigma_{N_d}^{-1} N_d^{-1}) \leq -z_a]
= \lim_{d \to 0} P_d[\sigma_{N_d}^{-1} N_d^{-1} T_{N_d}(y) \leq -z_a]
\end{equation}

\text{which in view of (5.15) and (5.13) implies (5.9). (5.10) and (5.11) follow similarly. We have used translation invariance of } \{U_{N_d}\} \text{ in arriving at (5.16).}

\textbf{Remark.} \text{ We could have concluded (5.14) from Theorem 3.1 above after putting } F_i \equiv F \text{ and assuming that } \lim sup_{n \to \infty} \max_{1 \leq i \leq n} x_i^2/n_0 \leq k < \infty. \text{ But since the result is available without this boundedness condition in [9], we prefer not to use that. It is easy to see that this boundedness condition implies (1.11) and (1.8).}

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\textbf{REFERENCES}


