AN ANALOGUE, FOR SIGNED RANK STATISTICS, 
of JUREČKOVÁ'S ASYMPTOTIC LINEARITY 
THEOREM FOR RANK STATISTICS

BY CONSTANCE VAN EEDEN 

Université de Montréal

1. Introduction. The purpose of this note is to prove that if, for each 
ν = 1, 2, ⋯, X_{v,i}, ⋯, X_{v,n_v} are a random sample from a distribution symmetric 
around 0, then the signed-rank statistic 

\[ T_v(θ) = \sum_{i=1}^{n_v} p_{v,i} \phi \left( \frac{R_{X_{v,i}, v_{i} - \theta}}{\frac{1}{n_v} + 1} \right) \text{sgn} (X_{v,i} - q_{v,i} θ), \]

where \( R_{X_{v,i}, v_{i} - \theta} \) is the rank of \( |X_{v,i} - q_{v,i} θ| \) among \( |X_{v,1} - q_{v,1} θ|, \cdots, |X_{v,n_v} - q_{v,n_v} θ| \), is under certain conditions on the common distribution of the 
\( X_{v,i} \), on the constants \( p_{v,i}, q_{v,i} \), and on the function \( \phi \), asymptotically approximately 
a linear function of \( θ \) in the sense that 

\[ \lim_{n_v \to \infty} P[\sup_{|θ| \leq C} |T_v(θ) - T_v(0) + θK \sum_{i=1}^{n_v} p_{v,i} q_{v,i}| \geq \varepsilon θ(T_v(0))] = 0, \]

for every \( C > 0 \) and every \( ε > 0 \), where \( K \) is a constant depending on the common distribution of the 
\( X_{v,i} \) and on the function \( \phi \).

This result is related to a result of Jurečková [3]; she proves (1.1) for the 
special case where \( p_{v,i} \equiv 1 \) and \( \sum_{i=1}^{n_v} q_{v,i} \equiv 0 \) under different conditions on the 
sequence of vectors \( (q_{v,1}, \cdots, q_{v,n_v}) \).

An analogous result was proved by Jurečková [2] for the statistic 

\[ S_v(θ) = \sum_{i=1}^{n_v} C_{v,i} \phi \left( \frac{R_{X_{v,i}, d_{v,i} - θ}}{\frac{1}{n_v} + 1} \right), \]

where \( R_{X_{v,i}, d_{v,i} - θ} \) is the rank of \( X_{v,i} - d_{v,i} θ \) among \( X_{v,1} - d_{v,1} θ, \cdots, X_{v,n_v} - d_{v,n_v} θ \) and where, for each \( ν = 1, 2, \cdots, \) the \( X_{v,i} \) are independently and identically 
distributed.

For the proof of our result some lemmas are needed which are given in 
Section 2; one of these lemmas is a generalization of Theorem 5 of Lehmann [9]; 
two of the lemmas are analogous to Corollaries 1 and 2 of Lehmann [9]. The 
main result and their proofs are given in Section 3.

2. Some Lemmas. Let \( i_1, \cdots, i_n \) and \( j_1, \cdots, j_n \) each be a permutation of the numbers \( 1, \cdots, n \) and let \( ε_1, \cdots, ε_n, δ_1, \cdots, δ_n \) each be +1 or −1 such that 
\((i_1, ε_1, j_1, δ_1, \cdots, i_n, ε_n, j_n, δ_n)\) satisfies

**CONDITION A.**

**CONDITION A.**

**CONDITION A.**

Received July 31, 1970; revised February 2, 1971.

1 This paper was written while the author was visiting the University of Rennes. It was 
partially supported by the National Research Council of Canada under Grant A 3114.

791
For fixed $M(1 \leq M \leq n)$ define
\begin{equation}
 a_{M,1} > a_{M,2} > \cdots > a_{M,K_M}
\end{equation}
as the ordered values of those $i_k$ among $i_{n-M+1}, i_{n-M+2}, \ldots, i_n$ for which $\varepsilon_k = +1$ and
\begin{equation}
 b_{M,1} > b_{M,2} > \cdots > b_{M,L_M}
\end{equation}
as the ordered values of those $j_k$ among $j_{n-M+1}, j_{n-M+2}, \ldots, j_n$ for which $\delta_k = +1$.
Obviously, by Condition $A_{n-1}$, $K_M \leq L_M$, further $K_M \leq M$. Further define
\begin{equation}
 c_{M,1} > c_{M,2} > \cdots > c_{M,M-K_M}
\end{equation}
as the ordered values of those $i_k$ among $i_{n-M+1}, i_{n-M+2}, \ldots, i_n$ for which $\varepsilon_k = -1$ and
\begin{equation}
 d_{M,1} > d_{M,2} > \cdots > d_{M,M-L_M}
\end{equation}
as the ordered values of those $j_k$ among $j_{n-M+1}, j_{n-M+2}, \ldots, j_n$ for which $\delta_k = -1$.

**Lemma 2.1.** If $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_n$, then
\begin{align}
 b_{M,l} &\leq a_{M,l} & l &= 1, \ldots, L_M \\
 c_{M,l} &\leq d_{M,l} & l &= 1, \ldots, M-K_M \\
 M &= 1, \ldots, n.
\end{align}

**Proof.** The proof will be given in four parts.
(i) The lemma is true for $M = 1$ and any $n \geq 1$. To prove this, notice that by Condition $A_{n-1}$ it is sufficient to prove that
\begin{equation}
 j_n \leq i_n \quad \text{if } \delta_n = 1 \\
 j_n \geq i_n \quad \text{if } \varepsilon_n = -1.
\end{equation}
This can be seen as follows.
\begin{align}
 j_n &= (\text{# of } j_k \leq j_n) = n - (\text{# of } j_k > j_n) \\
 i_n &= (\text{# of } i_k \leq i_n) = n - (\text{# of } i_k > i_n).
\end{align}
By Condition $A_{n-2}$
\begin{equation}
 (\text{# of } j_k \leq j_n) \leq (\text{# of } i_k \leq i_n) \quad \text{if } \delta_n = 1
\end{equation}
and by Condition $A_{n-3}$
\begin{equation}
 (\text{# of } j_k > j_n) \leq (\text{# of } i_k > i_n) \quad \text{if } \varepsilon_n = -1.
\end{equation}
(ii) If the lemma is true for some $(n, M)$ then the lemma is true for $(n + 1, M)$. To see this consider, for some $n \geq 1$, $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ satisfying Condition $A_{n+1}$. From $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ derive $(i_k', \varepsilon_k, j_k', \delta_k)_{k=2}^{n+2}$, satisfying Condition $A_{n+1}$, as follows. Let
\begin{equation}
r_k = \text{rank of } i_k \quad \text{among } (i_l, i_k) \\
s_k = \text{rank of } j_k \quad \text{among } (j_l, j_k) \quad k = 2, \ldots, n + 1
\end{equation}
and let
\begin{align}
i'_{k} &= i_k - (r_k - 1) \\
j'_{k} &= j_k - (s_k - 1) \quad k = 2, \ldots, n + 1.
\end{align}
Then \(i'_k, \ldots, i'_{n+1}\) and \(j'_k, \ldots, j'_{n+1}\) are each permutations of the numbers \(1, \ldots, n\) and from

\[
(2.12) \quad i_k < i_l \iff i'_k < i'_l \\
\quad j_k < j_l \iff j'_k < j'_l \\
\quad k, l = 2, \ldots, n + 1
\]

it then follows that \([i'_k, \varepsilon_k, j'_k, \delta_k]_{k=2}^{n+1}\) satisfies condition \(A_n\).

For fixed \(M \leq n\) let \(a'_{M,l}, b'_{M,l}, c'_{M,l}, d'_{M,l}, L_{M'}\) and \(K_{M'}\) be defined, as in (2.2) — (2.4), for \((i'_k, \varepsilon_k, j'_k, \delta_k)_{k=2}^{n+1-\alpha}\) and let \(a_{M,l}, b_{M,l}, c_{M,l}, d_{M,l}, K_M\) and \(L_M\) be so defined for \((i_k, \varepsilon_k, j_k, \delta_k)_{k=2}^{n+1-M}\), then \(L_M = L'_{M'}\) and \(K_M = K'_{M'}\). Assuming the lemma to be true for \((n, M)\) we have

\[
(2.13) \quad b'_{M,l} \leq d'_{M,l} \quad l = 1, \ldots, L_M \\
\quad c'_{M,l} \leq d'_{M,l} \quad l = 1, \ldots, M - K_M.
\]

Now let \(l_0\) be the number of \(b_{M,l} > j_l\), then by (2.11)

\[
(2.14) \quad b'_{M,l} = b^{-1}_{M,l} \quad l = 1, \ldots, l_0 \\
\quad = b_{M,l} \quad l = l_0 + 1, \ldots, L_M.
\]

Let \(k_0\) be the number of \(a_{M,l} > i_l\), then by (2.11)

\[
(2.15) \quad a'_{M,l} = a^{-1}_{M,l} \quad l = 1, \ldots, k_0 \\
\quad = a_{M,l} \quad l = k_0 + 1, \ldots, K_M.
\]

Further, by Condition \(A_{n+1,2}\), \(l_0 \leq k_0\). From (2.13) — (2.15) it then follows that

\[
(2.16) \quad b_{M,l} \leq a_{M,l} \quad l = 1, \ldots, L_M.
\]

The proof that

\[
(2.17) \quad c_{M,l} \leq d_{M,l} \quad l = 1, \ldots, M - K_M
\]

is analogous, using Condition \(A_{n+1,3}\).

(iii) If the lemma is true for some \(n \geq 2\) with \(M = n - 1\), then the lemma is true for the same \(n\) with \(M = n\). This can be seen as follows. Assuming the lemma to be true for \(M = n - 1\) we have

\[
(2.18) \quad b_{n-1,l} \leq a_{n-1,l} \quad l = 1, \ldots, L_{n-1} \\
\quad c_{n-1,l} \leq d_{n-1,l} \quad l = 1, \ldots, n - 1 - K_{n-1}
\]

and it will be proved that

\[
(2.19.1) \quad b_{n,l} \leq a_{n,l} \quad l = 1, \ldots, L_n \\
(2.19.2) \quad c_{n,l} \leq d_{n,l} \quad l = 1, \ldots, n - K_n
\]

The following three cases can be distinguished

(a) \(
\delta_1 = \varepsilon_1 = -1. \quad \text{Then} \quad L_n = L_{n-1}, \quad K_n = K_{n-1}, \quad b_{n,l} = b_{n-1,l}(l = 1, \ldots, L_n) \quad \text{and} \quad a_{n,l} = a_{n-1,l}(l = 1, \ldots, K_n), \quad \text{so that} \quad (2.19.1) \quad \text{is obvious. Further} \quad (a_{n,l}, l = 1, \ldots, K_n, c_{n,l}, l = 1, \ldots, n - K_n) \quad \text{and} \quad (b_{n,l}, l = 1, \ldots, L_n, d_{n,l}, l = 1, \ldots,
\)
\( n - L_n \) are each permutations of the numbers \( 1, \ldots, n \) so that (2.19.ii)) follows from (2.19.i)).

(b) \( \delta_l = -1, \varepsilon_l = 1 \). Then \( L_n = L_{n-1}, K_n = K_{n-1} + 1, b_{n,l} = b_{n-1,l}(l = 1, \ldots, L_n) \) and \( c_{n,l} = c_{n-1,l}(l = 1, \ldots, n - K_n) \). To prove (2.19.i)) let \( k_0 \) be the number of \( a_{n-1,l}(l = 1, \ldots, K_{n-1}) \) larger than \( i_l \), then

\[
\begin{align*}
a_{n,l} &= a_{n-1,l} & l &= 1, \ldots, k_0 \\
&= i_l & l &= k_0 + 1 \\
&= a_{n-1,l-1} & l &= k_0 + 2, \ldots, K_n.
\end{align*}
\]

If \( L_n \leq k_0 \leq K_{n-1} \) then (2.19.i)) is immediate. If \( 0 \leq k_0 < L_n = L_{n-1} \), then (2.19.i)) is immediate for \( l = 1, \ldots, k_0 \). Further

\[
(2.20) \quad b_{n,k_0+1} = b_{n-1,k_0+1} \leq a_{n-1,k_0+1} \leq i_{k_0} = a_{n,k_0+1}
\]

and for \( l = k_0 + 2, \ldots, L_n \)

\[
(2.21) \quad b_{n,l} = b_{n-1,l} \leq a_{n-1,l} = a_{n,l+1} \leq a_{n,l}.
\]

The proof of (2.19.iii)) is analogous.

(c) \( \delta_l = \varepsilon_l = 1 \). Then \( L_n = L_{n-1} + 1, K_n = K_{n-1} + 1, c_{n,l} = c_{n-1,l}(l = 1, \ldots, n - K_n) \) and \( d_{n,l} = d_{n-1,l}(l = 1, \ldots, n - L_n) \) so that (2.19.ii)) is obvious. Further (see (a)) (2.19.i)) follows from (2.19.iii)).

(iv) The lemma now follows by induction on \( M \). According to part 1 of the proof, the lemma is true for \( M = 1 \) and any \( n \geq 1 \). Let \( M_0 \) be an integer \( \geq 1 \) and assume the lemma is true for \( M = M_0 \) and any \( n \geq M_0 \), then it will be proved that the lemma is true for \( M = M_0 + 1 \) and any \( n \geq M_0 + 1 \). This can be seen as follows. According to the induction hypothesis the lemma is true for \( n = M_0 + 1 \) and \( M = M_0 \); according to part 3 of the proof this implies the truth for \( n = M_0 + 1 \) and \( M = M_0 + 1 \); according to part 2 of the proof this implies the truth for \( M = M_0 + 1 \) and any \( n \geq M_0 + 1 \). □

In Lemma 2.1 it was shown that Condition \( A_n \) is sufficient for (2.5) to hold for each \( M = 1, \ldots, n \). For (2.5) to hold for a particular value of \( M \) it is obviously sufficient that \((i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n} \) satisfies

\[
\text{CONDITION } A_{n,M} \begin{cases}
\text{For each } k \geq n - M + 1 \\
1. \quad \delta_k = 1 \Rightarrow \varepsilon_k = 1 \\
2. \quad \text{for each } l \leq k - 1 \quad (\delta_k = 1, j_l < j_k) \Rightarrow i_l < i_k \\
3. \quad \text{for each } l \leq k - 1 \quad (\varepsilon_k = -1, j_l > j_k) \Rightarrow i_l > i_k.
\end{cases}
\]

Further, if \((i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n} \) satisfies Condition \( A_{n,M} \) for \( M = M_0 \) then \((i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n} \) also satisfies Condition \( A_{n,M} \) for all \( M \leq M_0 \), which proves the following lemma.

**LEMMA 2.2.** If \((i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n} \) satisfies Condition \( A_{n,M} \) for \( M = M_0 \), then

\[
(2.23) \quad \begin{align*}
a_{M,l} &\leq b_{M,l} & l &= 1, \ldots, L_M \\
c_{M,l} &\leq d_{M,l} & l &= 1, \ldots, M - K_M & 1 \leq M \leq M_0.
\end{align*}
\]
**Lemma 2.3.** If \( h \) is nondecreasing and nonnegative and if \((i_k, \varepsilon_k, \delta_k, \beta_k)_{k=1}^n\) satisfies Condition \( A_{n,M} \) for \( M = M_0 \), then

\[
\begin{align*}
\sum_{i=1}^{n} h(i) &\geq \sum_{i=1}^{n} h(i) \\
\sum_{i=1}^{n} h(i) &\leq \sum_{i=1}^{n} h(i)
\end{align*}
\]

\( 1 \leq M \leq M_0 \).

**Proof.** Because \( h \) is nondecreasing, it follows from Lemma 2.2 that for \( 1 \leq M \leq M_0 \)

\( \begin{align*}
1. & \quad h(b_{M, i}) \leq h(a_{M, i}) \quad l = 1, \ldots, L_M \\
2. & \quad h(c_{M, i}) \leq h(d_{M, i}) \quad l = 1, \ldots, M - K_M \\
\end{align*} \)

From (2.25.1) and the fact that \( h \) is nonnegative it follows that, for \( 1 \leq M \leq M_0 \),

\( \begin{align*}
\sum_{i=1}^{n} h(i) &\leq \sum_{i=1}^{n} h(i) \\
\sum_{i=1}^{n} h(i) &\leq \sum_{i=1}^{n} h(i)
\end{align*} \)

From (2.25.2) and the fact that \( h \) is nonnegative it follows that for \( 1 \leq M \leq M_0 \),

\( \begin{align*}
\sum_{i=1}^{n} h(i) &\leq \sum_{i=1}^{n} h(i) \\
\sum_{i=1}^{n} h(i) &\leq \sum_{i=1}^{n} h(i)
\end{align*} \)

**Remark.** In the two special cases, where \( \delta_k = 1 \) for all \( k \) or \( \varepsilon_k = -1 \) for all \( k \), Lemma 2.1 reduces to Theorem 5 of Lehmann [9]. Further, in each of these special cases, Lemma 2.3 is analogous to Corollary 1 of Lehmann [9].

**Lemma 2.4.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be \( n \) numbers satisfying

\( 0 \leq \alpha_1 \leq \cdots \leq \alpha_n \),

let \( h \) be nondecreasing and nonnegative and let \((i_k, \varepsilon_k, \delta_k, \beta_k)_{k=1}^n\) satisfy

\( \begin{align*}
1. & \quad (\delta_k = 1, \alpha_k > 0) \Rightarrow \varepsilon_k = 1 \\
2. & \quad (\delta_k = 1, \alpha_k > 0, i < k, j_i < j_k) \Rightarrow i_i < i_k \\
3. & \quad (\varepsilon_k = -1, \alpha_k > 0, i < k, j_i < j_k) \Rightarrow i_i > i_k \\
\end{align*} \)

then

\( \sum_{k=1}^{n} \alpha_k \varepsilon_k h(i_k) \leq \sum_{k=1}^{n} \alpha_k \delta_k h(j_k) \).

**Proof.** The following proof is analogous to Lehmann's proof of his Corollary 2 in [9].

(2.30) is obviously true if \( \alpha_k = 0 \) for all \( k = 1, \ldots, n \), so in the following it will be supposed that \( \alpha_k > 0 \) for at least one \( k \). Further, since \( h \) is nonnegative,

\( \sum_{i=1}^{n} h(i) \geq 0 \) and \( \sum_{i=1}^{n} h(i) = 0 \) if and only if \( h(i) = 0 \)

for all \( i = 1, \ldots, n \),

in which case (2.30) is obvious. In the following it will be supposed that \( \sum_{i=1}^{n} h(i) > 0 \).

Let \( 0 \leq \beta_1 < \beta_2 < \cdots < \beta_x \) be the different values of \( \alpha_1, \ldots, \alpha_n \) and let
\( n_t(t = 1, \cdots, T) \) be the number of \( \alpha_t \) equal \( \beta_t \). Further let \( N_t = \sum_{t=1}^{T} n_t(t = 1, \cdots, T) \) and \( N_0 = 0 \). Consider the random variables \( X \) and \( Y \) each taking the values \((-\beta_T, -\beta_{T-1}, \cdots, -\beta_1, \beta_1, \cdots, \beta_{T-1}, \beta_T)\) with

\[
(2.31) \quad \begin{align*}
1. \quad P(X \leq -\beta_s) &= \frac{\sum_{i=N_{s-1}+1; \delta_i < 0} h(i)}{\sum_{i=1}^{n} h(l)} \\
2. \quad P(X \leq \beta_s) &= 1 - \frac{\sum_{i=N_s; \delta_i \geq 0} h(i)}{\sum_{i=1}^{n} h(l)} \quad s = 1, \cdots, T
\end{align*}
\]

and

\[
(2.32) \quad \begin{align*}
1. \quad P(Y \leq -\beta_s) &= \frac{\sum_{i=N_{s-1}+1; \delta_i < 0} h(i)}{\sum_{i=1}^{n} h(l)} \\
2. \quad P(Y \leq \beta_s) &= 1 - \frac{\sum_{i=N_s; \delta_i \geq 0} h(i)}{\sum_{i=1}^{n} h(l)} \quad s = 1, \cdots, T,
\end{align*}
\]

where, if \( \beta_1 = 0 \), \( P(X \leq 0) \) and \( P(Y \leq 0) \) are defined by (2.31.2) and (2.32.2) respectively.

If \( \beta_1 > 0 \), condition (2.29) reduces to Condition \( A_n \) and from Lemma 2.3 it then follows that

\[
(2.33) \quad P(X \leq x) \leq P(Y \leq x) \quad \text{for all } x.
\]

If \( \beta_1 = 0 \), condition (2.29) is Condition \( A_{n,M} \) for \( M = N_T - N_1 = n - n_1 \), so that in this case (2.24) holds for \( M \leq n - n_1 \), which proves (2.33). From (2.33) it follows that

\[
(2.34) \quad \mathcal{E} X \geq \mathcal{E} Y,
\]

which is equivalent to

\[
(2.35) \quad \sum_{s=1}^{T} \beta_s \sum_{i=N_{s-1}+1; \delta_i < 0} h(i) \geq \sum_{s=1}^{T} \beta_s \sum_{i=N_{s-1}+1; \delta_i \geq 0} h(i),
\]

which is equivalent to

\[
(2.36) \quad \sum_{s=1}^{T} \alpha_s \delta_i h(i) \geq \sum_{s=1}^{T} \alpha_s \delta_i h(i).
\]

3. Main Results. Let, for each \( \nu = 1, 2, \cdots, X_{\nu,1}, \cdots, X_{\nu,n_\nu} \) be independently and identically distributed random variables with common distribution function \( F(x) \) satisfying

\[
(3.1) \quad \begin{align*}
1. \quad F(x) \ \text{has an absolutely continuous density } f(x) \\
2. \quad \int_{0}^{1} \varphi_p^2(u) \, du < \infty, \quad \text{where } \varphi_p(u) = \frac{f'(F^{-1}(u))}{f(F^{-1}(u))} \quad (0 \leq u \leq 1)
\end{align*}
\]

and where \( f' \) is the derivative of \( f \)

\[
3. \quad f(x) = f(-x) \quad \text{for all } x.
\]

Let \( \varphi(u)(0 \leq u \leq 1) \) be a function satisfying
(3.2) 1. $\phi(u)$ can be written as the sum of two functions $\phi_u(u)$ and $\phi_y(u)$ where $\phi_u(u)$ is nondecreasing and nonnegative and $\phi_y(u)$ is non-increasing and nonpositive.

2. $\int_0^1 \phi_i^*(u) \, du < \infty (i = 1, 2)$ and $\int_0^1 \phi_i^*(u) \, du > 0$.

Let $p_{\nu, i}$, $\ldots$, $p_{\nu, n_{\nu}}$ and $q_{\nu, i}$, $\ldots$, $q_{\nu, n_{\nu}}$ be vectors of constants satisfying

(3.3) 1. $\sum_{i=1}^{n_{\nu}} p^2_{\nu, i} > 0$

2. $\lim_{\nu \to \infty} \frac{\max_{1 \leq i \leq n_{\nu}} p^2_{\nu, i}}{\sum_{i=1}^{n_{\nu}} p^2_{\nu, i}} = 0$.

(3.4) 1. $\sum_{i=1}^{n_{\nu}} q^2_{\nu, i} \leq M$ for some positive number $M$ independent of $\nu$

2. $\lim_{\nu \to \infty} \max_{1 \leq i \leq n_{\nu}} q^2_{\nu, i} = 0$.

and, for each $\nu = 1, 2, \ldots$, either

(3.5) 1. $p_{\nu, i} q_{\nu, i} \geq 0$ for all $i = 1, \ldots, n_{\nu}$

2. $(|p_{\nu, i}| - |p_{\nu, i'}|)(|q_{\nu, i}| - |q_{\nu, i'}|) \geq 0$ for all $i, i' = 1, \ldots, n_{\nu}$

or,

(3.6) 1. $p_{\nu, i} q_{\nu, i} \leq 0$ for all $i = 1, \ldots, n_{\nu}$

2. $(|p_{\nu, i}| - |p_{\nu, i'}|)(|q_{\nu, i}| - |q_{\nu, i'}|) \geq 0$ for all $i, i' = 1, \ldots, n_{\nu}$.

Let $R_{|X_{\nu, i} - q_{\nu, i}\theta|}$ be the rank of $|X_{\nu, i} - q_{\nu, i}\theta|$ among $|X_{\nu, 1} - q_{\nu, 1}\theta|, \ldots, |X_{\nu, n_{\nu}} - q_{\nu, n_{\nu}}\theta|$, let

(3.7) $\text{sgn } u = 1$ if $u > 0$

$= -1$ if $u < 0$

and let

(3.8) $T_\nu(\theta) = \sum_{i=1}^{n_{\nu}} p_{\nu, i} \phi\left(\frac{R_{|X_{\nu, i} - q_{\nu, i}\theta|}}{n_{\nu} + 1}\right) \text{sgn } (X_{\nu, i} - q_{\nu, i}\theta)$.

**Theorem 3.1.** If $F(x)$ is continuous, if $\phi(u)$ is nondecreasing and nonnegative then, for each $\nu = 1, 2, \ldots$, $T_\nu(\theta)$ is with probability one a nonincreasing step function of $\theta$ if (3.5) holds and a nondecreasing step function of $\theta$ if (3.6) holds.

**Proof.** In the proof the index $\nu$ will be omitted. The proof will be given for the case that (3.5) holds. The result for the case that (3.6) holds is then obvious.

If $F(x)$ continuous, $T(\theta)$ is, with probability one, not well defined only for those values of $\theta$ satisfying $\theta = -(X_i/q_i)$ for some $i$ with $q_i \neq 0$ and for those values of $\theta$ satisfying $|X_i - q_i\theta| = |X_{i'} - q_{i'}\theta|$ for some pair $(i, i')$ with $q_i \neq 0$ or $q_{i'} \neq 0$. These values of $\theta$ where $T(\theta)$ is not well defined, define a finite number of intervals for $\theta$ within each of which $T(\theta)$ is independent of $\theta$.

Now consider two values $\theta_1$ and $\theta_2$ of $\theta$ for which $T(\theta)$ is well defined and let $\theta_1 < \theta_2$. Then it will be proved that $T(\theta_1) \leq T(\theta_2)$. Without loss of generality the $X_i$ can be numbered in such a way that $|p_1| \leq \cdots \leq |p_n|$. Then, by (3.5.2),
\[ |q_1| \leq \cdots \leq |q_n|. \] Write \( T(\theta) \) as

\[
T(\theta) = \sum_{k=1}^{n} |p_k| \phi \left( \frac{R_{|X_k - q_k \theta|}}{n+1} \right) \text{sgn } p_k (X_k - q_k \theta),
\]

where, for \( p_k = 0 \), \( \text{sgn } p_k (X_k - q_k \theta) \) is defined as 1.

Now apply Lemma 2.4 with, for \( k = 1, \ldots, n \)

\[
\alpha_k = |p_k|, \quad \varepsilon_k = \text{sgn } p_k (X_k - q_k \theta), \quad \delta_k = \text{sgn } p_k (X_k - q_k \theta),
\]

\[
i_k = R_{|X_k - q_k \theta|}, \quad j_k = R_{|X_k - q_k \theta|}.
\]

Then \( T(\theta) \geq T(\theta) \) if (2.29) is satisfied. That (2.29) is satisfied can be seen from the following steps (a), (b) and (c).

(a) (2.29.1) is identical with

\[ \{p_k(X_k - q_k \theta) > 0, p_k \neq 0\} \Rightarrow p_k(X_k - q_k \theta) > 0 \]

which follows immediately from (3.5.1) and

\[ p_k(X_k - q_k \theta) = p_k(X_k - q_k \theta) + p_k q_k (\theta - \theta). \]

(b) (2.29.2) is identical with

\[ \{p_k(X_k - q_k \theta) > 0, p_k \neq 0, l < k, |X_l - q_l \theta| < |X_k - q_k \theta|\}\]

\[ \Rightarrow |X_l - q_l \theta| < |X_k - q_k \theta|. \]

This can be seen as follows. We have

\[ -\frac{p_k}{|p_k|} (X_k - q_k \theta) < X_l - q_l \theta < \frac{p_k}{|p_k|} (X_k - q_k \theta) \]

so that, using (3.5),

\[
X_l - q_l \theta < \frac{p_k}{|p_k|} (X_k - q_k \theta) + (\theta - \theta)\left( q_l - \frac{p_k}{|p_k|} q_k \right) = \frac{p_k}{|p_k|} (X_k - q_k \theta) + (\theta - \theta)q_l - |q_k| \leq \frac{p_k}{|p_k|} (X_k - q_k \theta).
\]

Also

\[
X_l - q_l \theta > -\frac{p_k}{|p_k|} (X_k - q_k \theta) + (\theta - \theta)\left( q_l + \frac{p_k}{|p_k|} q_k \right) = -\frac{p_k}{|p_k|} (X_k - q_k \theta) + (\theta - \theta)(q_l + |q_k|) \geq -\frac{p_k}{|p_k|} (X_k - q_k \theta),
\]

so that \( |X_l - q_l \theta| \leq |X_k - q_k \theta|. \)
(c) (2.29.3) is identical with
\[ p_k(X_k - q_k \theta_k) < 0, p_k \neq 0, l < k, |X_l - q_l \theta_l| > |X_k - q_k \theta_k| \]
\[ \implies |X_l - q_l \theta_l| > |X_k - q_k \theta_k|. \]
The proof of this is analogous to that for (2.29.2). \[ \square \]
A special case of Theorem 3.1 with \( \psi(u) = u \) and \( p_{v,i} = q_{v,i} (i = 1, \ldots, n_v) \)
was proved by Koul ([5], Lemma 2.2).

**THEOREM 3.2.** If (3.1)–(3.4) and (3.5) or (3.6) are satisfied then
\[ (3.11) \lim_{v \to \infty} P[\sup_{|\theta| \leq C} |T_v(\theta) - T_v(0) + \theta K \sum_{l=1}^{n_v} p_{v,i} q_{v,i}| > \varepsilon(T_v(0))] = 0, \]
where \( K = \frac{1}{b} \psi(u) \varphi(u)^2 (u + 1/2) du. \)

**PROOF.** The index \( v \) will be omitted in the proof. It is sufficient to prove the theorem for the case where \( \psi(u) = 0 \) for all \( u \). Further the proof will be given for the case where (3.5) holds; the result for the case where (3.6) holds is then obvious.

The proof is analogous to the proof of Jurečková of her Theorem 3.1 in [2]. As in her case it can be supposed without loss of generality that \( \sum_{j=1}^{n_v} p_{v,j} = 1 \) and it can be seen, using the result of Hájek and Šidák ([1], Theorem V. 1.7) that it is sufficient to prove
\[ \lim_{v \to \infty} P[\sup_{|\theta| \leq C} |T(\theta) - T(0) + \theta K \sum_{j=1}^{n_v} p_{v,j} q_{v,j}| > \varepsilon] = 0. \]
As in Jurečková’s proof and using the results of Hájek and Šidák ([1], section VI. 2.5) it can be proved that for any fixed set of points \( \theta_1, \ldots, \theta_r \)
\[ \lim_{v \to \infty} P[|T(\theta_i) - T(0) + \theta_i K \sum_{j=1}^{n_v} p_{v,j} q_{v,j}| \leq \varepsilon \quad \text{for all} \quad i = 1, \ldots, r] = 1. \]
Further, for a fixed \( C > 0 \), choosing \( \theta_1, \ldots, \theta_r \) with
\[ -C = \theta_1 < \theta_2 < \cdots < \theta_{r-1} < \theta_r = C \]
and
\[ |K| |\theta_{i+1} - \theta_i| \leq \frac{1}{2} \varepsilon M^{-1} \]
where \( M \) is the constant in (3.4), it can be seen, as in Jurečková’s proof [2] and using Theorem 3.1 above, that
\[ |T(\theta_i) - T(0) + \theta_i K \sum_{j=1}^{n_v} p_{v,j} q_{v,j}| \leq \frac{1}{2} \varepsilon \quad \text{for all} \quad i = 1, \ldots, r \]
\[ \implies \sup_{|\theta| \leq C} |T(\theta) - T(0) + \theta K \sum_{j=1}^{n_v} p_{v,j} q_{v,j}| \leq \varepsilon. \] \[ \square \]

The conditions on the \( p_{v,i} \) and \( q_{v,i} \) in Theorem 3.2 can be weakened as follows (see also Jurečková [2], Remark, page 1897). First, it can be assumed, without loss of generality, that \( q_{v,i} \geq 0 \) for all \( i = 1, \ldots, n_v \) or that \( q_{v,i} \leq 0 \) for all \( i = 1, \ldots, n_v \). This can be seen as follows. Let \( p_{v,i} \) and \( q_{v,i} (i = 1, \ldots, n_v) \) satisfy (3.3) and (3.4) and suppose \( q_{v,i} < 0 \) for at least one \( i \). Let \( A_v \) be the set of values of \( i \) with \( q_{v,i} < 0 \) and define, for \( i = 1, \ldots, n_v \),
\[ (3.12) \quad p_{v,i}^* = p_{v,i}, \quad q_{v,i}^* = q_{v,i}, \quad i \in A_v \]
\[ Y_{v,i} = X_{v,i} \quad i \notin A_v \]
\[ = -p_{v,i}, \quad i \in A_v \]
\[ = -q_{v,i}, \quad i \in A_v \]
\[ = -X_{v,i}, \quad i \in A_v. \]
then

$$T_{i}(\theta) = \sum_{i=1}^{n_{\nu}} p_{\nu,i}^{*} \left( \frac{R_{\nu,i}^{*} - q_{\nu,i}^{*} \theta}{n_{\nu} + 1} \right) \text{sgn} \left( Y_{\nu,i} - q_{\nu,i}^{*} \theta \right),$$

where $Y_{\nu,1}, \ldots, Y_{\nu,n_{\nu}}$ are independent random variables with common distribution function $F(x)$ satisfying (3.1), where the $p_{\nu,i}^{*}$ and $q_{\nu,i}^{*}$ satisfy (3.3) and (3.4) and where $q_{\nu,i}^{*} \geq 0$ for all $i = 1, \ldots, n_{\nu}$.

Further, if $q_{\nu,i}$ has the same sign for all $i = 1, \ldots, n_{\nu}$, it is possible to find a sequence of pairs of vectors $(p_{\nu,i}^{(1)}, \ldots, p_{\nu,i}^{(l)})$ $(l = 1, 2)$ such that

(3.13)

1. $p_{\nu,i} = \sum_{i=1}^{2} p_{\nu,i}^{(l)}$ \quad $i = 1, \ldots, n_{\nu}$
2. $p_{\nu,i}^{(1)} q_{\nu,i} \geq 0$ \quad $i = 1, \ldots, n_{\nu}$
   \hspace{1cm} $\quad p_{\nu,i}^{(2)} q_{\nu,i} \leq 0$ \quad $i = 1, \ldots, n_{\nu}$
3. $(|p_{\nu,i}^{(l)}| - |p_{\nu,i}^{(l')}|)(|q_{\nu,i}| - |q_{\nu,i'}|) \geq 0$
   \hspace{1cm} $l = 1, 2$ and \hspace{0.5cm} $i, i' = 1, \ldots, n_{\nu}.$

That this is possible can be seen as follows. Assume $q_{\nu,i} \geq 0$ for all $i = 1, \ldots, n_{\nu}$. For every pair of vectors $(p_{\nu,1}, \ldots, p_{\nu,n_{\nu}})$, $(q_{\nu,1}, \ldots, q_{\nu,n_{\nu}})$ one can find $(\alpha_{\nu,1}, \ldots, \alpha_{\nu,n_{\nu}})$ and $(\beta_{\nu,1}, \ldots, \beta_{\nu,n_{\nu}})$ such that $p_{\nu,i} = \alpha_{\nu,i} + \beta_{\nu,i}$ and

$$\begin{align*}
(\alpha_{\nu,i} - \alpha_{\nu,i})(|q_{\nu,i}| - |q_{\nu,i'}|) & \geq 0 \\
(\beta_{\nu,i} - \beta_{\nu,i})(|q_{\nu,i}| - |q_{\nu,i'}|) & \leq 0
\end{align*}$$

$i = 1, \ldots, n_{\nu}.$

Further one can find $\gamma_{\nu} \geq 0$ such that $\alpha_{\nu,i} + \gamma_{\nu} \geq 0$ and $\beta_{\nu,i} - \gamma_{\nu} \leq 0$ for all $i = 1, \ldots, n_{\nu}$. By taking $p_{\nu,i}^{(1)} = \alpha_{\nu,i} + \gamma_{\nu}$, $p_{\nu,i}^{(2)} = \beta_{\nu,i} - \gamma_{\nu}$ one has found $(p_{\nu,i}^{(1)}, \ldots, p_{\nu,i}^{(l)})$, $l = 1, 2$ such that (3.13) is satisfied.

Further, if $p_{\nu,1}, \ldots, p_{\nu,\nu}$ satisfies $\sum_{i=1}^{\nu} p_{\nu,i}^{2} > 0$ for each $\nu$ (Condition 3.3.1) then, for each $\nu$, there exists an $l (l = 1, 2)$ such that $\sum_{i=1}^{\nu} [p_{\nu,i}^{(l)}]^{2} > 0$. Also, if $p_{\nu,i}$ is written as $\sum_{i=1}^{\nu} p_{\nu,i}^{(l)}$, $T_{i}(\theta)$ can be written as the sum of two statistics and (3.11) remains true if it is true for each of these two statistics and

(3.14)

$$\sum_{l=1}^{2} \left[ \sum_{i=1}^{\nu} [p_{\nu,i}^{(l)}]^{2} \right] \leq M_{l} \left[ \sum_{i=1}^{\nu} p_{\nu,i}^{2} \right],$$

for some positive constant $M_{l}$ independent of $\nu$. Further (3.11) is true for each of these two statistics if (3.1), (3.2) and (3.4) are satisfied and $p_{\nu,i}^{(l)} (l = 1, 2)$ satisfy (3.13) and

(3.15)

1. for at least one $l$

$$\sum_{i=1}^{\nu} [p_{\nu,i}^{(l)}]^{2} > 0$$

for each $\nu$

2. for an $l$ for which 1. is not satisfied

$$\sum_{i=1}^{\nu} [p_{\nu,i}^{(l)}]^{2} = 0$$

for each $\nu$

3. for each $l$ for which 1. is satisfied

$$\lim_{\nu \to \infty} \max_{1 \leq i \leq \nu} \frac{[p_{\nu,i}^{(l)}]}{\sum_{i=1}^{\nu} [p_{\nu,i}^{(l)}]} = 0.$$
This proves the following theorem.

**Theorem 3.3.** If (3.1), (3.2) and (3.4) are satisfied, if there exist \( p_{v,i}^{(1)}, \ldots, p_{v,i}^{(n)} \) \((i = 1, 2)\) such that (3.13), (3.14) and (3.15) are satisfied, then (3.11) holds.

This theorem is related to a theorem of Jurečková [3]. She proves (3.11) for the case where \( p_{v,i} = 1 (i = 1, \ldots, n) \) and \( \sum_{i=1}^{n} q_{v,i} = 0 \) under the conditions (3.1), (3.2) and (3.4). Jurečková’s result [3] is not a special case of Theorem 3.3, as can be seen from the following two examples. Let, for \( n \) even, \( p_{v,i} = 1, i = 1, \ldots, n, q_{v,i} = n^{-i}, i = 1, \ldots, \frac{1}{2} n, \) and \( q_{v,i} = -n^{-i}, i = \frac{1}{2} n + 1, \ldots, n. \)

Then the conditions of Jurečková [3] are satisfied. That the conditions of Theorem 3.3 are also satisfied can be seen as follows. By (3.12) \( T_v(\theta) \) can be written as

\[
T_v(\theta) = \sum_{i=1}^{n} p_{v,i}^{*} \frac{R_{1}^{y_{v,i} - q_{v,i}^{*} \theta}}{n_{v} + 1} \text{sgn} (Y_{v,i} - q_{v,i}^{*} \theta),
\]

where \( p_{v,i}^{*} = 1, i = 1, \ldots, \frac{1}{2} n, p_{v,i}^{*} = -1, i = \frac{1}{2} n + 1, \ldots, n, q_{v,i}^{*} = n^{-i}(i = 1, \ldots, n). \) Further \( p_{v,i}^{*} \) can be written as \( \sum_{i=1}^{n} p_{v,i}^{(i)} \) satisfying (3.13) by choosing \( p_{v,i}^{(i)} = 1, i = 1, \ldots, n, \) and \( p_{v,i}^{(i)} = 0, i = 1, \ldots, \frac{1}{2} n, \) \( p_{v,i}^{(i)} = -2, i = \frac{1}{2} n + 1, \ldots, n. \)

Then \( \sum_{i=1}^{n} (p_{v,i}^{(i)})^2 = n, \sum_{i=1}^{n} (p_{v,i}^{(i)})^2 = 2n, \) and \( \sum_{i=1}^{n} p_{v,i}^{*} = n, \) so that (3.14) and (3.15) are satisfied. However, if one takes e.g. \( p_{v,i} = 1, i = 1, \ldots, n, \) and \( q_{v,i} = \left(\frac{1}{2}(i + 1)(-1)^{i+1}\right)/n^3, i = 1, \ldots, n, \) then the conditions of Jurečková [3] are satisfied but those of Theorem 3.3 are not. This can be seen as follows. By (3.12), \( p_{v,i} = (-1)^{i+1}, q_{v,i} = \left(\frac{1}{2}(i + 1)/n^3, i = 1, \ldots, n, \right) \) and, for any \( p_{v,i}^{(i)} \) and \( p_{v,i}^{(i)} \) satisfying (3.13), \( \sum_{i=1}^{n} (p_{v,i}^{(i)})^2 \) and \( \sum_{i=1}^{n} (p_{v,i}^{(i)})^2 \) are of the order \( n^2, \) whereas \( \sum_{i=1}^{n} p_{v,i}^{*} = n, \) so that (3.14) is not satisfied.

A special case of Theorem 3.3 with \( p_{v,i} = q_{v,i} = n^{-i} \) was used by Kraft and van Eeden ([6] and [7]) to find the asymptotic properties of linearized estimates based on signed ranks for the one-sample location problem.

An extension of Theorem 3.3 to the \( p \)-variate case, where \( R_{x_{v},i}^{y_{v},i} \theta \) is replaced by \( R_{x_{v},i}^{y_{v},i - q_{v,i}^{*} \theta} \) with \( p_{v,i} = q_{v,i,j} \) for some \( j \) and all \( i = 1, \ldots, n, \) is given in [8]; it is used there to find the asymptotic properties of linearized estimates based on signed ranks for the general linear hypothesis.

Koul [5] proves a theorem analogous to Theorem 3.2 for the \( p \)-variate case with \( \phi(u) = u \) and conditions on \( F \) that are stronger than (3.1).

Jurečková also treated in [3] the \( p \)-variate case with \( p_{v,i} = 1, i = 1, \ldots, n, \) and \( \sum_{i=1}^{n} q_{v,i,j} = 0 \) for all \( i = 1, \ldots, n, \) and all \( j = 1, \ldots, p. \)

REFERENCES


