SHORT PROOFS OF TWO CONVERGENCE THEOREMS
FOR CONDITIONAL EXPECTATIONS

BY D. LANDERS AND L. ROGGE

University of Cologne

In this paper there are given new proofs of two convergence theorems
for conditional expectations, concerning convergence in measure and
convergence almost everywhere of a sequence of conditional expectations
$P_n \circ f$, of a bounded function $f$, given a $\sigma$-field $\mathcal{F}_0$, with respect to varying
probability measures $P_n$.

We shall give new proofs of two convergence theorems for conditional expec-
tations which were proven in [1]. These new proofs are shorter and more
transparent.

If $\mu \mid \mathcal{F}$ is a measure, $\sigma$-finite on the $\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$, and $f : X \to \mathbb{R}$ is $\mu$-
integrable, then $\mu \circ f$ denotes the $\mu \mid \mathcal{F}_0 \circ \mathcal{F}$-equivalence class of all conditional
expectations of $f$, relative $\mu \mid \mathcal{F}$, given $\mathcal{F}_0$. A sequence of equivalence classes
of conditional expectations converges a.e. or in measure if every choice of ver-
sions converges a.e. or in measure. A sequence $P_n \mid \mathcal{F}$, $n \in \mathbb{N}$, of probability
measures converges uniformly to $P_0 \mid \mathcal{F}$, if $\sup \{ |P_n(A) - P(A)| : A \in \mathcal{F} \} \to 0$.

**Theorem 1.** Let $\mu \mid \mathcal{F}$ be a measure which is $\sigma$-finite on the $\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$
and dominates the probability measures $P_n \mid \mathcal{F}$, $n \in \mathbb{N} \cup \{0\}$. Let $h_n$ be a density
of $P_n \mid \mathcal{F}$ with respect to $\mu \mid \mathcal{F}$ and $h_{0, n}$ a density of $P_n \mid \mathcal{F}_0$ with respect to $\mu \mid \mathcal{F}_0$,
$n \in \mathbb{N} \cup \{0\}$. Assume that

(i) $h_0 \leq \liminf_{n \to \infty} h_n$ $\mu$-a.e.,

(ii) $h_{0, n} = \lim_{n \to \infty} h_{0, n}$ $\mu$-a.e.

Then $(P_n \circ f)_{n \in \mathbb{N}}$ converges $P_0 \circ f$ to $P_0 \circ f$ for each $\mathcal{F}$-measurable bounded function $f$.

**Proof.** We have $(*)$: $P_n \circ f \circ h_n = \mu \circ f \circ h_n$ for all $n \in \mathbb{N} \cup \{0\}$. As $h_{0, n} \in
\mu \circ h_n$, (ii) implies $\mu \circ (h_n - h_0) \to 0 \mu$-a.e. Since $(h_n - h_0) \leq h_0$, (i) implies
$\mu \circ (h_n - h_0) \to 0 \mu$-a.e. Therefore we obtain $\mu \circ h_n - h_0 = \mu \circ (h_n - h_0) +
2 \mu \circ (h_n - h_0) \to 0 \mu$-a.e. As $f$ is bounded this implies $\mu \circ f \circ h_n \to \mu \circ f \circ h_0 \mu$-a.e.
Hence by $(*)$ $P_n \circ f \circ h_n \to P_0 \circ f \circ h_0 \mu$-a.e. whence $P_0(A : (\mu \circ f \circ h_0)(x) > 0) = 1$
and (ii) imply the assertion.

We remark that condition (i) is slightly weaker than condition (i) of Theorem 1
of [1].

Using $\mu \circ h_n - h_0 \to 0 \mu$-a.e. we can also obtain the following result of [1]:
If $\mu \mid \mathcal{F}$ is a probability measure admitting a regular conditional probability,
given $\mathcal{T}_0$, then there exist regular conditional probabilities $p_n | \mathcal{T} \times X$, relative $P_n$, given $\mathcal{T}_0$, $n \in \mathbb{N} \cup \{0\}$, such that $\sup \{|p_n(A, x) - p_0(A, x)| : A \in \mathcal{T}\} \to 0$ $P_0$-a.e.

**Theorem 2.** Let $P_n | \mathcal{T}$, $n \in \mathbb{N}$, be probability measures, converging uniformly to the probability measure $P_0 | \mathcal{T}$, and let $\mathcal{T}_0 \subset \mathcal{T}$ be a a-field. Then $(P_n \circ f)_{n \in \mathbb{N}}$ converges to $P_0 \circ f$ in $P_0$-measure for every $\mathcal{T}$-measurable bounded function $f$.

**Proof.** Let $\mu = \sum \{2^{-n}P_n : n \in \mathbb{N}\}$ and $h_n$ be a density of $P_n | \mathcal{T}$ with respect to $\mu | \mathcal{T}$, $n \in \mathbb{N} \cup \{0\}$. Since $P_n | \mathcal{T}$ converges uniformly to $P_0 | \mathcal{T}$, we obtain $\mu(|\mu \circ h_n - h_0|) \to 0$ in $\mu$-measure, whence $\mu \circ h_n \to \mu \circ h_0$ and $\mu \circ fh_n \to \mu \circ fh_0$ in $\mu$-measure. According to relation (**) of Theorem 1 this implies $P_n \circ f \to P_0 \circ f$ in $\mu$-measure. As $P_0(x : (\mu \circ h_n)(x) > 0) = 1$ and $\mu \circ h_n \to \mu \circ h_0$ in $\mu$-measure, we obtain the assertion.

In Lemma 2 of [1] it is proved that $\mu \circ h_n \to \mu \circ h_0$ $P_0$-a.e. and $\mu \circ h_n \to \mu \circ h_0$ $P_0$-a.e. imply $P_n \circ f \to P_0 \circ f$ $P_0$-a.e. This follows directly from relation (**) of Theorem 1.

**References**