NOTE ON THE TIGHTNESS OF THE METRIC ON THE SET OF COMPLETE SUB $\sigma$-ALGEBRAS OF A PROBABILITY SPACE

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The purpose of this note is to show that the usual metric on the set of complete sub $\sigma$-algebras of a probability space is very tight; a recent result from E. B. Boylan on equi-convergence of martingales follows and is thereby, we believe, better understood.

Given a probability space $(\Omega, \mathcal{F}, P)$, a metric can be introduced on the set $S$ of all complete sub $\sigma$-algebras $\mathcal{B}$ of $\mathcal{F}$ by letting (see [1])

$$d(\mathcal{B}_1, \mathcal{B}_2) = \sup_{B_1 \in \mathcal{B}_1} \inf_{B_2 \in \mathcal{B}_2} P(B_1 \triangle B_2) + \sup_{B_2 \in \mathcal{B}_2} \inf_{B_1 \in \mathcal{B}_1} P(B_1 \triangle B_2)$$

where $\triangle$ denotes symmetric difference of sets as usual; here a sub $\sigma$-algebra $\mathcal{B}$ of $\mathcal{F}$ is said to be complete (relatively to $\mathcal{F}$) if every set $A \in \mathcal{F}$ of probability zero belongs to $\mathcal{B}$. The following proposition is then the main result of this note.

PROPOSITION. Let $\mathcal{B}, \mathcal{B}'$ be two complete sub $\sigma$-algebras of $\mathcal{F}$ in $(\Omega, \mathcal{F}, P)$ such that $\mathcal{B} \subset \mathcal{B}'$. Then there exists a set $A \in \mathcal{B}$ such that

$$P(A') \leq 4d(\mathcal{B}, \mathcal{B}') \quad \text{and} \quad A \cap \mathcal{B}' = A \cap \mathcal{B}.$$ 

Conversely if $A \in \mathcal{B}'$ is such that $A \cap \mathcal{B}' = A \cap \mathcal{B}$, then $d(\mathcal{B}, \mathcal{B}') \leq P(A')$.

When $A \in \mathcal{F}$ and $\mathcal{B} \in S$, we denote by $A \cap \mathcal{B}$ the $\sigma$-algebra of subsets of $A$ which are of the from $A \cap B$ for a $B \in \mathcal{B}$; when $A \in \mathcal{B}$, then this class $A \cap \mathcal{B}$ is also the $\sigma$-algebra of subsets of $A$ belonging to $\mathcal{B}$.

Proof.

(a) When $\mathcal{B} \subset \mathcal{B}'$, the distance $d(\mathcal{B}, \mathcal{B}')$ can be characterized as the smallest positive real number $d$ for which the following implication holds

$$B' \in \mathcal{B}', \ P(\mathcal{B}') \leq \frac{1}{2} \ a.s. \implies P(B') \leq d,$$

where $P(\mathcal{B}')$ denotes the conditional expectation of the indicator function $1_{B'}$ of $B'$ with respect to $\mathcal{B}$. This is easily proved as follows.

Because $P(B \triangle B') = E[|P(\mathcal{B}') - 1_B|]$ when $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$, we have for every $B' \in \mathcal{B}'$

$$\inf_{B \in \mathcal{B}} P(B \triangle B') = \inf_{B \in \mathcal{B}} E[|P(\mathcal{B}') - 1_B|]$$

$$= E[\min\{P(\mathcal{B}') - 1_B, 1 - P(\mathcal{B}')\}]$$

the infimum being for instance achieved by the $\mathcal{B}$-set $\{P(\mathcal{B}') > \frac{1}{2}\}$. Hence

$$d(\mathcal{B}, \mathcal{B}') = \sup_{B' \in \mathcal{B}'} E[\min\{P(\mathcal{B}') - 1_B, 1 - P(\mathcal{B}')\}].$$

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Now we use the hypothesis $\mathcal{B} \subset \mathcal{B}'$ to assert that for every $B' \in \mathcal{B}'$, the set $B^* = B' \triangle \{P^\mathcal{B}(B') > \frac{1}{2}\}$ is also in $\mathcal{B}'$; since

$$P^\mathcal{B}(B^*) = \min[P^\mathcal{B}(B'), 1 - P^\mathcal{B}(B')] \leq \frac{1}{2}$$

it is not hard to see that

$$d(\mathcal{B}, \mathcal{B}') = \sup\{P(B''); B'' \in \mathcal{B}', P^\mathcal{B}(B'') \leq \frac{1}{2}\}$$

as announced.

(b) If $\mathcal{B}, \mathcal{B}' \in \mathcal{S}$ still verify $\mathcal{B} \subset \mathcal{B}'$, let $A$ be a $\mathcal{B}$-atom of $\mathcal{B}'$ [i.e. a set $A \in \mathcal{B}'$ such that $A \cap \mathcal{B}' = A \cap \mathcal{B}$] with largest possible conditional expectation $P^\mathcal{B}(A)$; the existence of such a set has been proved in [2], Theorem 1, page 258. Let us show now that $P(A^c) \leq 4d(\mathcal{B}, \mathcal{B}')$.

By Theorem 2 of [2], we may find a set $C \in \mathcal{B}'$ such that

$$P^\mathcal{B}(C) \leq \frac{1}{2} \leq P^\mathcal{B}(C) + P^\mathcal{B}(A).$$

Then we let

$$D = C[P^\mathcal{B}(A) \leq \frac{1}{2}] + A[\frac{1}{2} < P^\mathcal{B}(A) \leq \frac{1}{2}] + A^c[P^\mathcal{B}(A) > \frac{1}{2}].$$

This set $D$ belongs to $\mathcal{B}'$ and is such that

$$\frac{1}{2}[1 - P^\mathcal{B}(A)] \leq P^\mathcal{B}(D) \leq \frac{1}{2} \quad \text{a.s.}$$

as is easily checked on each of the three sets $\{P^\mathcal{B}A \leq \frac{1}{2}\}$, $\{\frac{1}{2} < P^\mathcal{B}A \leq \frac{1}{2}\}$ and $\{\frac{1}{2} < P^\mathcal{B}(A)\}$ on which respectively $P^\mathcal{B}(D) = P^\mathcal{B}(C)$, $P^\mathcal{B}(A)$, or $1 - P^\mathcal{B}(A)$. The properties of this set $D$ imply with the aid of (a) that

$$d(\mathcal{B}, \mathcal{B}') \geq P(D) \geq \frac{1}{4}[1 - P(A)].$$

The direct part of the proposition is thus proved. The converse is immediate; indeed if $A \in \mathcal{B}'$ is such that $A \cap \mathcal{B}' = A \cap \mathcal{B}$, then for every $B' \in \mathcal{B}'$ there exists a $B \in \mathcal{B}$ for which $AB' = AB$ and then $P(B \triangle B') \leq P(A^c)$; hence $d(\mathcal{B}, \mathcal{B}') \leq P(A^c)$. [Q.E.D]

The following easy corollary has an interest only for equi-integrable sets of functions (for which $\delta_h(a) \downarrow 0$ as $a \nearrow \infty$).

**Corollary.** Let $H$ be a set of real integrable functions defined on a probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{B}, \mathcal{B}'$ be two sub-$\sigma$-algebras of $\mathcal{A}$ such that $\mathcal{B} \subset \mathcal{B}'$. Then the following inequality holds

$$\sup_{f \in H} ||E^\mathcal{B}(f) - E^\mathcal{B}'(f)||_1 \leq 16d(\mathcal{B}, \mathcal{B}') + 4\delta_h(a)$$

for every real $a > 0$, provided one lets

$$\delta_h(a) = \sup_{f \in H} \int_{|f| > a} |f| \, dP.$$

**Proof.** Let $A$ be a set with the properties stated in the preceding proposition and let $f$ be a $\mathcal{B}'$-integrable function. Then it is easily checked (see Lemma 2, page 257 of [2]) that $E^\mathcal{B}(f1_A) = fP^\mathcal{B}(A)$ a.s. on $A$; hence
\[ ||E^{\mathcal{A}}(f1_{\mathcal{A}}) - f1_{\mathcal{A}}||_1 \leq \int_{\Omega} E^{\mathcal{A}}(f1_{\mathcal{A}}) \, dP + \int_{\Omega} f[(1 - P^{\mathcal{A}})(A)] \, dP \]
\[ \leq \int_{\Omega} E^{\mathcal{A}}(f)(1_{\mathcal{A}} + P^{\mathcal{A}}(A^c)) \, dP \]
\[ = 2 \int_{\Omega} |f| P^{\mathcal{A}}(A^c) \, dP. \]

On the other hand
\[ ||E^{\mathcal{A}}(f1_{\mathcal{A}}) - f1_{\mathcal{A}}||_1 \leq 2 ||f1_{\mathcal{A}}||_1 = 2 \int_{\Omega} |f| 1_{\mathcal{A}} \, dP. \]

The addition of the two inequalities gives that
\[ ||E^{\mathcal{A}}(f) - f||_1 \leq 2 \int_{\Omega} |f|(P^{\mathcal{A}}(A^c) + 1_{\mathcal{A}}) \, dP \]
for every \( \mathcal{B}' \)-integrable function \( f \); the inequality remains valid for every integrable \( f \), if \( E^{\mathcal{A}}(f) \) is substituted for \( f \) in the first member and then
\[ ||E^{\mathcal{A}}f - E^{\mathcal{A}'}(f)||_1 \leq 2 \int_{\Omega} |f|(P^{\mathcal{A}'}(A^c) + 1_{\mathcal{A}}) \, dP \]
\[ \leq 4aP(A^c) + 4 \int_{\Omega} |f| \, dP \]
\[ \leq 16ad(\mathcal{A}, \mathcal{B}') + 4 \int_{\Omega} |f| \, dP. \]

By taking the supremum of the extreme members over \( H \), we obtain the formula of the corollary. \[ \square \]

Boylan has recently proved [1] that for any equi-integrable subset \( H \) of \( L'(\Omega, \mathcal{A}, P) \) and for any monotone sequence \((\mathcal{B}_{n}, n \in N)\) of sub \( \sigma \)-algebras of \( \mathcal{A} \) the \( L' \)-convergence \( \lim_{n \to \infty} E^{\mathcal{B}_{n}}(f) = E^{\mathcal{B}_{\infty}}(f) \) holds uniformly on \( H \), provided \( \mathcal{B}_{\infty} \) denotes the limiting \( \sigma \)-algebra of the increasing or decreasing sequence \((\mathcal{B}_{n}, n \in N)\) and provided \( d(\mathcal{B}_{n}, \mathcal{B}_{m}) \to 0 \) as \( n \uparrow \infty \). This theorem can be readily obtained from the preceding corollary, since by this result
\[ \sup_{H} ||E^{\mathcal{B}_{n}}(f) - E^{\mathcal{B}_{\infty}}(f)|| \leq 16ad(\mathcal{B}_{n}, \mathcal{B}_{\infty}) + 4\delta_{H}(a) \]
\[ \to 0 \]
when \( n \not\to \infty \) and then \( a \not\to \infty \).}

REFERENCES