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LOCALLY BEST INVARIANT TEST FOR SPHERICITY AND THE LIMITING DISTRIBUTIONS

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The locally best invariant test for testing sphericity in a multivariate normal distribution is obtained. It is different from the likelihood ratio criterion. Some other known results are remarked. The limiting distributions of the test statistics are investigated. In particular Pitman efficiency of the likelihood ratio test with respect to the locally best invariant test is shown to be one for normal alternatives.

1. Determination of the critical region. Let Ω_1 be a subset of p-dimensional Euclidean space. For testing $H: \Theta = \Theta_0$ against $K: \Theta \in \Omega_1$, the test ϕ_0 is called locally best invariant, if for any given other invariant test ϕ, there is a neighborhood $N(\Theta_0)$ of Θ_0, such that the power of ϕ_0 is not less than that of ϕ for all $\Theta \in N(\Theta_0) \cap \Omega_1$.

A sufficient condition for this in one-sided multivariate problems is given by Giri [3]. Let T be a maximal invariant statistic in m dimensional Euclidean space, having density $f(t, \Theta)$ for $\Theta = \operatorname{diag}(\theta_1, \cdots, \theta_p)$. The problem is to test $H: \Theta = \Theta_0$ against $K: \Theta \geqq \Theta_0$, where strict inequality holds at least for a component of Θ. Assume that for all $\Theta \in N(\Theta_0) \cap \Omega_1$,

$$(1.1) \qquad \frac{f(t, \Theta)}{f(t, \Theta_0)} = 1 + h(\Theta - \Theta_0)[g(\Theta, \Theta_0) + k(\Theta, \Theta_0)U(t)] + o(h(\Theta - \Theta_0)),$$

where the real valued functions $h > 0$, g, $k > 0$ are bounded and $h(\theta)$ is a continuous function of a norm $||\theta||$ in Euclidean space with $h(0) = 0$. Then the critical region $\{T \mid U(T) > c\}$ is locally best invariant. Giri [3] showed that for testing $H: \Sigma_1 = \Sigma_2$ against $K: \Sigma_1 - \Sigma_2 \geqq 0$ based on two Wishart matrices S_i having $W_p(\Sigma_i, n_i)$, the critical region $\{(S_1, S_2) \mid \operatorname{tr} S_2(S_1 + S_2)^{-1} < c\}$ is locally best invariant.

Let S be a Wishart matrix having $W_p(\Sigma, n)$. For testing sphericity $H: \Sigma = \sigma^2 I$ against $K: \Sigma \neq \sigma^2 I$ with unknown σ^2, the problem is invariant under the transformation $S \to c^2 HSH'$ for any orthogonal matrix H of order p and any positive number c^2. A maximal invariant statistic is given by $(l_1/l_p, \cdots, l_{p-1}/l_p)$, where l_i is the ith largest characteristic root of S.

We shall derive the distribution of this maximal invariant statistic in a form convenient for our discussion. By invariance, we may assume that $\Sigma = \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$ for $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p$. The joint density of $L = \operatorname{diag}(l_1, \cdots, l_p)$ is given by James [4]

$$(1.2) \qquad c \int_{O(p)} |\Lambda|^{-n/2} |L|^{(n-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}\Lambda^{-1}HLH') \prod_{i<j} (l_i - l_j)\, dL\, dH,$$

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where $c = \pi^{p^2/2}2^{-np/2}\{\Gamma_p(\frac{1}{2}p)\Gamma_p(\frac{1}{2}n)\}^{-1}$ and dH means Haar measure on the orthogonal group $O(p)$ of order p. Making the transformation $u_i = l_i/\lambda_p$ and then $v_i = u_i/u_p$ for $i = 1, 2, \cdots, p-1$, we can write the density of $V_1 = \text{diag}(v_1, \cdots, v_{p-1}, 1)$ and u_p as

$$(1.3) \qquad c \int_{O(p)} |\Lambda_1|^{-n/2} |V_1|^{(n-p-1)/2} u_p^{\frac{1}{2}np-1} \operatorname{etr}[-\tfrac{1}{2}u_p\{V_1 + (\Lambda_1^{-1} - I)HV_1H'\}] \times \prod_{i<j<p}(v_i - v_j) \prod_{i=1}^{p-1}(v_i - 1)\, dH\, du_p\, dV_1 ,$$

where $\Lambda_1 = \Lambda/\lambda_p$. Now the null hypothesis is $H\colon \Lambda_1 = I$ and $K\colon \Lambda_1 \geqq I$. The domain of integration of u_p and V_1 in (1.3) is $v_1 > v_2 > \cdots > v_{p-1} > 1$ and $0 < u_p < \infty$. Integration by dH first and then du_p, noting that the zonal polynomial $C_\kappa(Z)$ corresponding to the partition κ of k is a kth degree symmetric homogeneous polynomial with respect to the characteristic roots of Z, we finally obtain the joint density of $v_i = l_i/l_p$ $i = 1, 2, \cdots, p-1$ as

$$(1.4) \qquad f(V_1, \Lambda_1) = c|\Lambda_1|^{-n/2}|V_1|^{(n-p-1)/2}(\operatorname{tr} V_1/2)^{-np/2} \prod_{i<j<p}(v_i - v_j) \times \prod_{i=1}^{p-1}(v_i - 1) \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{\Gamma(\frac{1}{2}np + k)}{k!\,(\operatorname{tr} V_1)^k} \frac{C_\kappa(I - \Lambda_1^{-1})C_\kappa(V_1)}{C_\kappa(I)} .$$

Hence

$$(1.5) \qquad \frac{f(V_1, \Lambda_1)}{f(V_1, I)} = |\Lambda_1|^{-n/2} \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{\Gamma(\frac{1}{2}np + k)}{\Gamma(\frac{1}{2}np)} \cdot \frac{C_\kappa(I - \Lambda_1^{-1})C_\kappa(V_1)}{k!\,(\operatorname{tr} V_1)^k C_\kappa(I)} .$$

From James [5], the exact form of zonal polynomials of degree less than seven is known. For $k = 0, 1, 2$,

$$(1.6) \qquad \begin{aligned} C_{(0)}(Z) &= 1 , \qquad C_{(1)}(Z) = \operatorname{tr} Z , \\ C_{(2)}(Z) &= \tfrac{1}{3}\{(\operatorname{tr} Z)^2 + 2 \operatorname{tr} Z^2\} , \\ C_{(1^2)}(Z) &= \tfrac{2}{3}\{(\operatorname{tr} Z)^2 - \operatorname{tr} Z^2\} , \end{aligned}$$

which can also be found in Fujikoshi [2]. The right-hand side of (1.5), together with (1.6) yields

$$(1.7) \qquad \frac{f(V_1, \Lambda_1)}{f(V_1, I)} = 1 - \frac{n}{4} t_2 - \frac{n^2}{8} t_1^2 + \frac{n(np+2)}{24(p-1)(p+2)} [3(p+1)(t_1^2 - 6t_2) + \{-6t_1^2 + 6pt_2\} \operatorname{tr} V_1^2/(\operatorname{tr} V_1)^2] + O(t_1^3) ,$$

where $t_j = \operatorname{tr}(\Lambda_1^{-1} - I)^j$ for $j = 1, 2, 3$. By the Schwarz inequality, $(\operatorname{tr} A)^2 \leqq p(\operatorname{tr} A^2)$ for any $p \times p$ matrix A, so that the coefficient of $\operatorname{tr} V_1^2/(\operatorname{tr} V_1)^2$ in (1.7) is positive under K. By (1.1) we can conclude that the critical region $\{V_1 \mid \operatorname{tr} V_1^2/(\operatorname{tr} V_1)^2 > c_\alpha\}$ is locally best invariant. Hence we have

THEOREM 1. *Let S have the Wishart distribution $W_p(\Sigma, n)$. For testing sphericity $H\colon \Sigma = \sigma^2 I$ against $K\colon \Sigma \neq \sigma^2 I$, where σ^2 is unknown, the locally best invariant critical region is given by $\{S \mid \operatorname{tr} S^2/(\operatorname{tr} S)^2 > c_\alpha\}$.*

It may be remarked that from (41) and (59) in Constantine [1], Pillai's criteria $\operatorname{tr} S_h(S_e + S_h)^{-1}$ for multivariate linear hypothesis and $\sum_{i=1}^{p_1} r_i^2$ for testing independence between two sets of variates, where r_i is the sample canonical correlation, are locally best invariant. These results are obtained by Schwartz [9].

For testing $H: \Sigma = \Sigma_0$ against $K: \Sigma \geqq \Sigma_0$, based on a Wishart matrix S, having $W_p(\Sigma, n)$, the expression of the density function of the characteristic roots of S given in (1.2) is not appropriate for our discussion. It can also be expressed by James [4] and Pillai [8] as

$$(1.8) \qquad c|L|^{(n-p-1)/2}|\Sigma|^{-\frac{1}{2}n} \operatorname{etr}(-\tfrac{1}{2}L) \prod_{i<j} (l_i - l_j) \times \sum_{k=0}^{\infty} \sum_{(\kappa)} C_\kappa(\tfrac{1}{2}(I - \Sigma^{-1}))C_\kappa(L)/\{C_\kappa(I)k!\},$$

from which we can see that the critical region $\{S \mid \operatorname{tr} S\Sigma_0^{-1} \geqq c_\alpha\}$ is locally best invariant.
[Incidentally, the criteria in this section are also proposed recently by John [6], based on a different formulation.]

2. Limiting distributions. We shall investigate the asymptotic distribution of the test statistic $\operatorname{tr} S^2/(\operatorname{tr} S)^2$ obtained in Theorem 1 both under K and under the sequence of alternatives $K_n: \Sigma = \sigma^2(I + a_n\Theta)$, where $\Theta = \operatorname{diag}(\theta_1, \cdots, \theta_p)$ and $\lim_{n\to\infty} a_n = 0$. Put

$$(2.1) \qquad T = \left(\frac{1}{n} \Sigma^{-\frac{1}{2}} S \Sigma^{-\frac{1}{2}} - I\right) n^{\frac{1}{2}}.$$

Then for any symmetric matrix $A(p \times p)$, it is easy to see

$$(2.2) \qquad \lim_{n\to\infty} E[\operatorname{etr}(iuAT)] = \exp(-u^2 \operatorname{tr} A^2),$$

which implies the following lemmas due to Nagao [7].

LEMMA 1. *The statistic* $\operatorname{tr} AT$ *has asymptotically normal distribution*, $N(0, 2\operatorname{tr} A^2)$.

LEMMA 2. *Let T_{ij} be the (i, j) element of T. The $p(p+1)/2$ random variables T_{ij} $(i \leqq j)$ are stochastically independent as $n \to \infty$. T_{ii} converges in law to $N(0, 2)$ and T_{ij} $(i < j)$ converges in law to $N(0, 1)$.*

Putting $s_j = \operatorname{tr} \Sigma^j$ and expressing the test statistic in terms of T, yields

$$(2.3) \qquad \begin{aligned} n^{\frac{1}{2}}\{\operatorname{tr} S^2/(\operatorname{tr} S)^2 - s_2 s_1^{-2}\} &= 2s_1^{-2} \operatorname{tr}(\Sigma^2 - s_2 s_1^{-1}\Sigma)T \\ &\quad + n^{-\frac{1}{2}} s_1^{-2}\{3 s_2 s_1^{-2} (\operatorname{tr} \Sigma T)^2 \\ &\quad - 4 s_1^{-1} \operatorname{tr} \Sigma^2 T \operatorname{tr} \Sigma T + \operatorname{tr}(\Sigma T)^2\} + O_p(n^{-1}), \end{aligned}$$

which immediately implies, by Lemma 1, the first statement of Theorem 2. Under $K_n: \Sigma = \sigma^2(I + a_n\Theta)$, we can rewrite (2.3) when $\lim_{n\to\infty} n^{\frac{1}{2}} a_n = \infty$,

$$(2.4) \qquad n^{\frac{1}{2}} a_n^{-1}\{\operatorname{tr} S^2/(\operatorname{tr} S)^2 - s_2 s_1^{-2}\} = 2p^{-2}[\operatorname{tr}\{\Theta - p^{-1}(\operatorname{tr} \Theta)I\}T] + o_p(1).$$

Also we have

$$(2.5)\qquad \frac{1}{2}np^2\left\{\frac{\operatorname{tr} S^2}{(\operatorname{tr} S)^2}-\frac{1}{p}\right\}=\frac{1}{2}\left\{\operatorname{tr} T^2-\frac{1}{p}(\operatorname{tr} T)^2\right\}+o_p(1)\,,\quad \text{when } \lim_{n\to\infty} n^{\frac{1}{2}}a_n=0\,,$$

$$=\frac{1}{2}\left[\operatorname{tr}(T+\Theta)^2-\frac{1}{p}\{\operatorname{tr}(T+\Theta)\}^2\right]+o_p(1)\,,\quad \text{when } \lim_{n\to\infty} n^{\frac{1}{2}}a_n=1\,.$$

Thus we can conclude

THEOREM 2. *Under K, the asymptotic distribution of $n^{\frac{1}{2}}\{\operatorname{tr} S^2/(\operatorname{tr} S)^2 - s_2 s_1^{-2}\}$ is $N(0, 8s_1^{-4}\operatorname{tr}(\Sigma^2 - s_2 s_1^{-1}\Sigma)^2)$, where $s_j = \operatorname{tr}\Sigma^j$. Under $K_n: \Sigma = \sigma^2(I + a_n\Theta)$, with $\lim_{n\to\infty} a_n = 0$, the limiting distribution of $n^{\frac{1}{2}}a_n^{-1}\{\operatorname{tr} S^2/(\operatorname{tr} S)^2 - s_2 s_1^{-2}\}$ is $N(0, 8p^{-4}\operatorname{tr}(\Theta - p^{-1}(\operatorname{tr}\Theta)I)^2)$, when $\lim_{n\to\infty} n^{\frac{1}{2}}a_n = \infty$. When $\lim_{n\to\infty} n^{\frac{1}{2}}a_n = 1$, the statistic $Z_n = \frac{1}{2}np^2\{\operatorname{tr} S^2/(\operatorname{tr} S)^2 - p^{-1}\}$ has asymptotically noncentral χ^2-distribution with $f = p(p-1)/2 + p - 1$ degrees of freedom and noncentrality parameter $\frac{1}{4}\{\operatorname{tr}\Theta^2 - p^{-1}(\operatorname{tr}\Theta)^2\}$. When $\lim_{n\to\infty} n^{\frac{1}{2}}a_n = 0$, the statistic Z_n has asymptotically χ^2-distribution with f degrees of freedom.*

We can also express the likelihood ratio statistic in terms of T as

$$(2.6)\qquad -n^{\frac{1}{2}}\log\{|S|/(p^{-1}\operatorname{tr} S)^p\} + n^{\frac{1}{2}}\log\{|\Sigma|/(p^{-1}\operatorname{tr}\Sigma)^p\}$$
$$= \operatorname{tr}(p^{-1}s_1^{-1}\Sigma - I)T + \frac{1}{2n^{\frac{1}{2}}}\{\operatorname{tr} T^2 - p s_1^{-2}(\operatorname{tr}\Sigma T)^2\} + O_p(n^{-1})\,,$$

which yields, by the similar argument,

THEOREM 3. *Under K, the likelihood ratio statistic given by the left-hand side of* (2.6) *has asymptotically $N(0, 2\operatorname{tr}(p^{-1}s_1^{-1}\Sigma - I)^2)$. Under K_n, when $\lim_{n\to\infty} n^{\frac{1}{2}}a_n = \infty$, $a_n^{-1}\times$ L.H.S. of* (2.6) *has asymptotically $N(0, 2\operatorname{tr}(\Theta - p^{-1}(\operatorname{tr}\Theta)I)^2)$. When $\lim_{n\to\infty} n^{\frac{1}{2}}a_n = 1$, $-n\log\{|S|/(p^{-1}\operatorname{tr} S)^p\}$ has asymptotically noncentral χ^2-distribution and when $\lim_{n\to\infty} n^{\frac{1}{2}}a_n = 0$, it has asymptotically χ^2-distribution, with the same parameters as in Theorem* 2.

Asymptotic expansion of the non-null distribution of the likelihood ratio statistic was derived under K by Sugiura [10] and under K_n with $a_n = n^{-\frac{1}{2}}$ or n^{-1} by Nagao [7], using a different technique. Hence Theorem 3 is not all new. Finally from Theorem 2 and Theorem 3, it may be remarked that the Pitman efficiency of the likelihood ratio test with respect to the locally best invariant test is equal to 1 for normal alternatives.

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