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LARGE DEVIATION PROBABILITIES FOR WEIGHTED SUMS[1]

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Taking as our point of departure the methods and results in a 1960 paper of Bahadur and Ranga Rao, we derive asymptotic representations of large deviation probabilities for weighted sums of independent, identically distributed random variables. The main theorem generalizes the Bahadur-Ranga Rao result in the absolutely continuous case. The method of proof closely parallels that of the 1960 paper, a major component of which was the use of Cramér's 1923 theorem on asymptotic expansions. For our result, we need an extension of Cramér's theorem to triangular arrays, and that extension is also developed in the paper. We then show that the main theorem implies a logarithmic result which generalizes a 1952 theorem of Chernoff and is of more precision but less generality than a 1969 result of Feller. Finally, we note that in the exponential case the theorem can be used to estimate large deviation probabilities for linear combinations of exponential order statistics.

0. Introduction. The principal result in this paper, Theorem 4.8, generalizes the theorem of Bahadur and Ranga Rao (1960) to the case of weighted sums of independent, identically distributed (i.i.d.) absolutely continuous random variables. Bahadur and Ranga Rao's work involved ordinary sums of i.i.d. random variables of all types. In Section 1, the problem is carefully stated and the preliminary formulas are worked out. Section 2 introduces some conditions on the random variables and the weights. In place of Bahadur and Ranga Rao's use of Theorem 25 in Cramér's book [6], we employ an extension of Cramér's theorem that is developed in Section 3. The large deviation theorem itself is stated and proved in Section 4. In Section 5, we derive from the theorem a logarithmic estimate of the sort first due to Chernoff (1952) and recently dealt with on a larger scale by Feller (1969). Section 6 contains an application of the theorem in the study of linear combinations of exponential order statistics.

1. Preliminaries. Our goal is to derive an asymptotic representation for $P(S_n > c\sum_{k=1}^{n} a_{nk})$, where $S_n = \sum_{k=1}^{n} a_{nk}X_k$ for a sequence $\{X_k: 1 \leqq k < \infty\}$ of i.i.d. nondegenerate random variables and a double array $\{a_{nk}: 1 \leqq k \leqq n, 1 \leqq n < \infty\}$ of nonnegative real numbers such that $\sum_{k=1}^{n} a_{nk}^2 = 1$. We assume that X_1 is normalized so that $E(X_1) = 0$ and $E(X_1^2) = 1$. We denote by $F(x)$ the distribution function (df) of X_1, and by $\phi(t)$ the moment-generating function (m.g.f.) of X_1, which we assume to exist in a nondegenerate interval $|t| < B$. We need an additional condition on $\phi(t)$ that is similar to the condition in italics on page 1015 of the Bahadur–Ranga Rao paper: for some numbers a, there exists

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a corresponding τ such that $\phi'(\tau)/\phi(\tau) = a$. This requirement will be satisfied, as in [1], in the situations of the following lemma and its corollary:

(1.1) LEMMA. *If $\phi(t) < \infty$ for all real t, $E(X_1) = \phi'(0) = 0$, $a > 0$, and $P(X_1 > a) > 0$, then there exists a number $\tau > 0$ such that $\phi'(\tau)/\phi(\tau) = a$.*

(1.2) COROLLARY. *If $\phi(t) < \infty$ for all real t, $E(X_1) = \phi'(0) = 0$, and $P(X_1 > a_0) > 0$ for some $a_0 > 0$, then the function $Q(t) = \phi'(t)/\phi(t)$ is a one-to-one (invertible) mapping at least of $[0, \tau_0]$ onto $[0, a_0]$, where $Q(\tau_0) = a_0$.*

Because X_1 is nondegenerate, we have:

(1.3) LEMMA. *If $\phi(t) < \infty$ for $|t| < B$ and $E(X_1) = 0$, then the function $Q(t)$ is a one-to-one (invertible) mapping from $[0, B)$ onto its range.*

Later in the paper we will need to know that $Q^{-1}(a)$ exists for certain positive values of a, and we will then make the necessary assumptions, as in Bahadur and Ranga Rao (1960).

Having made these preliminary comments, we define the random variables $Y_{nk} = a_{nk}X_k - ca_{nk}$ and observe that $P(S_n > c\sum_{k=1}^n a_{nk}) = P(\sum_{k=1}^n Y_{nk} > 0)$. The df of Y_{nk} is $H_{nk}(y) = F(ya_{nk}^{-1} + c)$, and the m.g.f. of Y_{nk} is $\phi_{nk}(h) = \exp(-hca_{nk})\phi(ha_{nk})$, where $\phi_{nk}(h)$ exists for $|h| < Ba_{nk}^{-1}$. Assuming, as we are, that each $a_{nk} \geqq 0$, we see that all $\phi_{nk}(h)$, $1 \leqq k \leqq n$, exist for $|h| < B\sigma_n^{-1}$, where $\sigma_n = \max\{a_{nk} : 1 \leqq k \leqq n\}$. We restrict h to this interval from now on. We define an "associated" df $\bar{H}_{nk}(y)$ by $d\bar{H}_{nk}(y) = [e^{hy}/\phi_{nk}(h)]\,dH_{nk}(y)$ for each h, $0 < h < B\sigma_n^{-1}$, and we denote by $\bar{Y}_{nk}$ a random variable distributed according to this df. We have the following formula, whose proof is identical in form with the proof of the analogous Lemma 2 on page 1017 of the Bahadur–Ranga Rao paper:

(1.4) LEMMA. *If $\bar{H}_n(y) = P(\sum_{k=1}^n \bar{Y}_{nk} \leqq y)$, then*

$$P(S_n > c\textstyle\sum_{k=1}^n a_{nk}) \\ = \exp(-hc\textstyle\sum_{k=1}^n a_{nk})[\prod_{k=1}^n \phi(ha_{nk})]h\int_0^\infty e^{-hy}[\bar{H}_n(y) - \bar{H}_n(0)]\,dy\,.$$

2. Conditions. To make use of the formula in Section 1, we need to know more about the quantity $\bar{H}_n(y) - \bar{H}_n(0)$, in particular how far away from 0 it stays as $n \to \infty$. We eventually find conditions under which the quantity remains bounded away from 0 for each $y > 0$ as $n \to \infty$ and under which it can be approximated by the corresponding normal probability $\Phi(y) - \Phi(0)$.

Recall that $\bar{H}_n(y) = P(\bar{S}_n \leqq y)$ where $\bar{S}_n = \sum_{k=1}^n \bar{Y}_{nk}$, and $\bar{Y}_{nk}$ has m.g.f. $\bar{\phi}_{nk}(t) = \int e^{ty}\,d\bar{H}_{nk}(y) = [1/\phi_{nk}(h)]\int e^{(t+h)y}\,dH_{nk}(y) = \phi_{nk}(t+h)/\phi_{nk}(h)$. Therefore $\bar{S}_n$ has m.g.f. $\bar{\phi}_n(t) = \prod_{k=1}^n [\phi_{nk}(t+h)/\phi_{nk}(h)]$, and the moment-generating properties of the m.g.f. imply that $E(\bar{S}_n) = \sum_{k=1}^n \bar{\phi}_{nk}'(0)$ and $\mathrm{Var}(\bar{S}_n) = \sum_{k=1}^n [\bar{\phi}_{nk}''(0) - (\bar{\phi}_{nk}'(0))^2]$. Upon computing these derivatives from the expression for $\bar{\phi}_{nk}(t)$, we obtain explicit expressions as follows:

(2.1) LEMMA. $E(\bar{S}_n) = \sum_{k=1}^n a_{nk}[\phi'(ha_{nk})/\phi(ha_{nk})] - c\sum_{k=1}^n a_{nk}$
$= \sum_{k=1}^n a_{nk}Q(ha_{nk}) - c\sum_{k=1}^n a_{nk}$,

and
$$\operatorname{Var}(\bar{S}_n) = \sum_{k=1}^n a_{nk}^2 \frac{\phi(ha_{nk})\phi''(ha_{nk}) - [\phi'(ha_{nk})]^2}{[\phi(ha_{nk})]^2} = \sum_{k=1}^n a_{nk}^2 Q'(ha_{nk}) .$$

In this section, we find some conditions under which we can say that $E(\bar{S}_n) = 0$ and that the variances $\operatorname{Var}(\bar{S}_n)$ are uniformly bounded away from 0 and ∞ for all n. This amounts, it turns out, to choosing a sequence $\{h_n : 1 \leqq n < \infty\}$ of h's to appear in the definition of the associated df's. As the next lemmas show, the existence of such a sequence having the desired effects on $E(\bar{S}_n)$ and $\operatorname{Var}(\bar{S}_n)$ is closely bound up with the invertibility of the function Q.

(2.2) LEMMA. *If $Q(t) = \phi'(t)/\phi(t)$ takes the positive real axis onto itself, then for every positive integer n there exists a solution $h = h_n$ of the equation $E(\bar{S}_n) = 0$, and the solution satisfies for all n the inequalities*
$$\sigma_n^{-1}Q^{-1}(n^{-1}\sigma_n^{-1}c \textstyle\sum_{k=1}^n a_{nk}) \leqq h_n \leqq \sigma_n^{-1}Q^{-1}(\sigma_n^{-1}c \sum_{k=1}^n a_{nk}) ,$$
where $\sigma_n = \max\{a_{nk} : 1 \leqq k \leqq n\}$.

PROOF. Consider the function $Q_n^*(h) = \sum_{k=1}^n a_{nk} Q(ha_{nk})$ so that $E(\bar{S}_n) = Q_n^*(h) - c \sum_{k=1}^n a_{nk}$. Then $Q_n^*(0) = 0$, $Q_n^*(h) \geqq \sigma_n Q(h\sigma_n)$, and Q_n^* is continuous on the positive real line. Since Q takes on all positive real values, there exists a value of h for which $Q(h\sigma_n) = \sigma_n^{-1} c \sum_{k=1}^n a_{nk}$. Continuity then implies the existence of an h_n such that $Q_n^*(h_n) = c \sum_{k=1}^n a_{nk}$. The bounds on h_n are obtained from the inequality $\sigma_n Q(h_n\sigma_n) \leqq Q_n^*(h_n) \leqq n\sigma_n Q(h_n\sigma_n)$, recalling that Q is monotonically increasing and that $Q_n^*(h_n) = c \sum_{k=1}^n a_{nk}$.

Note that Lemma 2.2 holds if Q takes a sufficiently large interval of the positive real line onto another sufficiently large interval. It is not necessary for Q to take the positive real axis onto itself. In the next lemma, we put some conditions on the a_{nk}'s in order to obtain more detailed knowledge of the behavior of the sequence of h_n's.

(2.3) LEMMA. *If there exist numbers α and θ, $0 < \alpha \leqq 1$, $0 < \theta \leqq 1$, such that Q assumes the value $c/\alpha\theta$ at some point and $\theta^{-1}Q^{-1}(c/\alpha\theta)$ lies in the domain of the* m.g.f. ϕ, *and at least αn of the a_{nk}'s exceed or equal $\theta\sigma_n$, then*
$$\sigma_n^{-1}Q^{-1}(c\alpha\theta^2) \leqq h_n \leqq \sigma_n^{-1}\theta^{-1}Q^{-1}(c/\alpha\theta) .$$

PROOF. If at least αn of the a_{nk}'s exceed or equal $\theta\sigma_n$, then $c \sum_{k=1}^n a_{nk} = Q_n^*(h_n) = \sum_{k=1}^n a_{nk} Q(h_n a_{nk}) \geqq \alpha n\theta\sigma_n Q(h_n\theta\sigma_n)$ so that $c \geqq \alpha\theta n\sigma_n(\sum_{k=1}^n a_{nk})^{-1}Q(h_n\theta\sigma_n)$. But $n\sigma_n(\sum_{k=1}^n a_{nk})^{-1} \geqq n\sigma_n(n\sigma_n)^{-1} = 1$, so that $c \geqq \alpha\theta Q(h_n\theta\sigma_n)$. Since Q is increasing, this means that $h_n\theta\sigma_n \leqq Q^{-1}(c/\alpha\theta)$, providing the upper bound. For the lower bound, $c \sum_{k=1}^n a_{nk} = \sum_{k=1}^n a_{nk} Q(h_n a_{nk}) \leqq n\sigma_n Q(h_n\sigma_n)$ implies that $c \leqq n\sigma_n(\sum_{k=1}^n a_{nk})^{-1}Q(h_n\sigma_n) \leqq (\alpha\theta^2)^{-1}Q(h_n\sigma_n)$, since $\sigma_n \sum_{k=1}^n a_{nk} \geqq \sum_{k=1}^n a_{nk}^2 \geqq \alpha n\theta^2\sigma_n^2$.

According to Lemma 2.3, if $\phi(t) < \infty$ for all t, then for any pair of values of α and θ, the conditions of the lemma are satisfied if the range of Q is large. This means that at least a fixed fraction of the terms in S_n must always contribute significantly to the sum, a sort of infinitesimal array situation. Combining Lemma 2.3 with Corollary 1.2, we obtain:

(2.4) COROLLARY. *If $\phi(t)$ exists for all $t > 0$, at least αn of the a_{nk}'s exceed or equal $\theta\sigma_n$ for $0 < \alpha \leqq 1$, $0 < \theta \leqq 1$, and $P(X_1 > c/\alpha\theta) > 0$, then there exist numbers $b_0 > 0$, $B_0 < \infty$, such that $b_0 \leqq h_n\sigma_n \leqq B_0$.*

(2.5) LEMMA. *Under conditions which guarantee the existence of numbers $b_0 > 0$ and $B_0 < \infty$, where $\phi(B_0) < \infty$, such that $b_0 \leqq h_n\sigma_n \leqq B_0$, there exist numbers $d_0^2 > 0$ and $D_0^2 < \infty$ such that $d_0^2 \leqq \mathrm{Var}(\bar{S}_n) \leqq D_0^2$ for all n.*

PROOF. Because X_1 is nondegenerate, $Q'(t) > 0$ for all values of t, for $Q'(t)$ is the variance of the associated random variable $\bar{X}_1$. It follows that $d_0^2 = \min\{Q'(z) : 0 \leqq z \leqq B_0\} > 0$, and so $\mathrm{Var}(\bar{S}_n) = \sum_{k=1}^n a_{nk}^2 Q'(h_n a_{nk}) \geqq d_0^2 \sum_{k=1}^n a_{nk}^2 = d_0^2$. Since the facts that $\phi(0) = 1$ and $\phi(B_0) < \infty$ imply that $D_0^2 = \max\{Q'(z): 0 \leqq z \leqq B_0\} < \infty$, we have also $\mathrm{Var}(\bar{S}_n) \leqq D_0^2$.

For the rest of the paper we assume that conditions on ϕ hold which guarantee the existence of a sequence of numbers h_n such that $\sum_{k=1}^n a_{nk} Q(h_n a_{nk}) = c \sum_{k=1}^n a_{nk}$, and that conditions on the a_{nk}'s hold which guarantee the existence of numbers $b_0 > 0$ and $B_0 < \infty$, where $\phi(B_0) < \infty$, such that $b_0 \leqq h_n\sigma_n \leqq B_0$.

3. Cramér's theorem. In this section, we extend Cramér's Theorem 26 of [6] on asymptotic expansions to the case of triangular arrays, which includes our situation of weighted sums. The proof of the extension closely parallels the development in Chapter VII of [6], with only minimal modifications being made to Cramér's work. A complete proof, including all the details, of Cramér's theorem for triangular arrays can be found in Chapter 4 of [2]. Here we shall give a complete statement of the theorem, but only the briefest outline of its proof.

Denoting β_{mnk} as the mth absolute moment of the random variable X_{nk}, we set $B_{mn} = n^{-1} \sum_{k=1}^n \beta_{mnk}$, $\rho_{mn} = n^{m/2}B_{mn}$, and then $T_{mn} = n^{\frac{1}{2}}/4\rho_{mn}^{3/m}$. The quantity $P_{mn}(-\Phi)$ denotes a certain linear combination of the first $3m$ derivatives of the normal probability df. $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$, and is discussed in detail on page 76 of [6] and on page 73 of [2]. The quantity $p_{3m-1,n}(x) = e^{x^2/2}P_{mn}(-\Phi)$ is a polynomial of degree $3m - 1$ in x.

(3.1) CRAMÉR'S THEOREM. *If $\{X_{nk} : 1 \leqq k \leqq n,\ 1 \leqq n < \infty\}$ is a triangular array of random variables such that*

(i) *$X_{1n}, \cdots, X_{nn}$ are independent for each n;*
(ii) *$E(X_{nk}) = 0$ for all k and n;*
(iii) *$\sum_{k=1}^n E(X_{nk}^2) = 1$ for all n;*
(iv) *$E(|X_{nk}|^{m_0}) = \beta_{m_0nk} < \infty$ for all k and n, for some $m_0 \geqq 3$;*
(v) *each X_{nk} has* df *$F_{nk}(x) = \alpha_{nk}F_{1nk}(x) + (1 - \alpha_{nk})F_{2nk}(x)$, where $0 < \alpha_{nk} \leqq 1$, $F_{1nk}(x)$ is absolutely continuous, and $F_{2nk}(x)$ has no absolutely continuous component;*
(vi) *each density $f_{1nk}(x) = F'_{1nk}(x)$ has finite total variation v_{1nk} on $(-\infty, \infty)$; and*
(vii) *if $\Omega_n = \{k : 1 \leqq k \leqq n,\ v_{1nk} \leqq (3^{\frac{1}{2}}/8)T_{m_0n}\}$, then every sequence $\{n_r : 1 \leqq$*

$r < \infty\}$ of positive integers contains a subsequence $\{n_p: 1 \leqq p < \infty\}$ such that either

$$\text{(A)} \quad \lim_{p\to\infty} (\log n_p)^{-1} \sum_{k \in \Omega_{n_p}} \alpha_{n_p k} = \infty\,, \qquad \textit{or}$$

$$\text{(B)} \quad \lim_{p\to\infty} (T^2_{m_0 n_p}/\log n_p) \sum_{k \in \Omega^c_{n_p}} (\alpha_{n_p k}/v^2_{1n_p k}) = \infty\,,$$

then, if $F_n(x) = P(S_n \leqq x)$ is the distribution function of $S_n = \sum_{k=1}^n X_{nk}$,

$$\begin{aligned} F_n(x) &= \Phi(x) + \textstyle\sum_{m=1}^{m_0-3} n^{-m/2} P_{mn}(-\Phi) + R_{m_0 n}(x) \\ &= \Phi(x) + \textstyle\sum_{m=1}^{m_0-3} n^{-m/2} p_{3m-1,n}(x) e^{-x^2/2} + R_{m_0 n}(x)\,, \end{aligned}$$

where $|R_{m_0 n}(x)| < M/T^{m_0-2}_{m_0 n}$ for M dependent on m_0 and the F_{nk}'s but independent of n and x.

PROOF. The argument on pages 71–78 of [6], including Lemmas 2, 3, and 4, goes through with only minor alterations, as does the proof of Theorem 26 until the estimation of the magnitude of Z on page 85. The argument results in the indicated expansion of $F_n(x)$ with all merit hinging on a bound for $R_{m_0 n}(x)$, given as

$$|R_{m_0 n}(x)| < \theta(m_0)\{T^{-m_0+2}_{m_0 n} + Z \log T_{m_0 n}\}\,,$$

where $Z = \sup\{|\prod_{k=1}^n \phi_{nk}(t)| : t > T_{m_0 n}\}$, for $\phi_{nk}(t)$ the characteristic function of $F_{nk}(x)$, and $\theta(m_0)$ is a quantity bounded by a number depending only on m_0. If ϕ_{1nk} and ϕ_{2nk} are the characteristic functions of F_{1nk} and F_{2nk}, respectively, then $\phi_{nk}(t) = \alpha_{nk}\phi_{1nk}(t) + (1 - \alpha_{nk})\phi_{2nk}(t)$. An integration by parts shows that $|\phi_{1nk}(t)| < v_{1nk}/|t|$, from which it follows that

$$|\phi_{nk}(t)| \leqq (\alpha_{nk} v_{1nk}/|t|) + (1 - \alpha_{nk})\,.$$

So, for $|t| \geqq 2v_{1nk}$, $|\phi_{nk}(t)| \leqq 1 - \frac{1}{2}\alpha_{nk}$ and by Lemma 1 of [6], we have for $|t| < 2v_{1nk}$ that

$$\begin{aligned} |\phi_{nk}(t)| &\leqq 1 - (\alpha_{nk} - \tfrac{1}{2}\alpha^2_{nk})(t^2/32v^2_{1nk}) \\ &\leqq 1 - (3\alpha_{nk} t^2/128 v^2_{1nk})\,. \end{aligned}$$

Therefore, for all $t > 0$, $|\phi_{nk}(t)| \leqq 1 - \alpha_{nk}\min(\frac{1}{2}, 3t^2/128v^2_{1nk})$. It follows for $t > T_{m_0 n}$ that

$$\begin{aligned} |\phi_{nk}(t)| &\leqq 1 - \alpha_{nk}\min(\tfrac{1}{2}, 3T^2_{m_0 n}/128v^2_{1nk}) \\ &\leqq \exp(-\alpha_{nk}\min(\tfrac{1}{2}, 3T^2_{m_0 n}/128v^2_{1nk}))\,. \end{aligned}$$

So
$$Z \leqq \exp(-\textstyle\sum_{k=1}^n \alpha_{nk}\min(\tfrac{1}{2}, 3T^2_{m_0 n}/128v^2_{1nk}))\,.$$

Noting that the set

$$\Omega_n = \{k: 1 \leqq k \leqq n,\ \min(\tfrac{1}{2}, 3T^2_{m_0 n}/128v^2_{1nk}) = \tfrac{1}{2}\}\,,$$

we can write that

$$\begin{aligned} Z &\leqq \exp\left(-\textstyle\sum_{k\Omega\in_n} \tfrac{1}{2}\alpha_{nk} - \sum_{k\in\Omega^c_n}(3\alpha_{nk}T^2_{m_0 n}/128v^2_{1nk})\right) \\ &\leqq (n^{-\frac{1}{2}})^{(\sum_{k\in\Omega_n}\alpha_{nk})/\log n + (3T^2_{m_0 n}/64\log n)\sum_{k\in\Omega^c_n}(\alpha_{nk}/v^2_{1nk})}\,. \end{aligned}$$

Since either condition (A) or condition (B) of (vii) of the theorem holds for some subsequence of every sequence of positive integers, we know that the exponent of $n^{-\frac{1}{2}}$ tends to ∞ as $n \to \infty$, for otherwise there would exist a sequence along which the limit exists and is finite. Hence for any fixed $A > 0$, there exists an n_A such that $n \geqq n_A$ implies that the exponent exceeds A, i.e., that $Z \leqq n^{-\frac{1}{2}A} \leqq M_0/T_{m_0n}^A$ because $\rho_{m_0n} \geqq 1$, by pages 71–72 of [6], implies that $T_{m_0n} \leqq n^{\frac{1}{2}}/4$. Taking $A = m_0 - 1$, the bound on $R_{m_0n}(x)$ becomes $|R_{m_0n}(x)| \leqq M/T_{m_0n}^{m_0-2}$.

To apply this extension of Cramér's theorem to the large deviations problem, we have to require conditions which imply that $\rho_{m_0n} \leqq \Delta_0 < \infty$, where Δ_0 is a constant. This means that T_{m_0n} is of the order of $n^{\frac{1}{2}}$, in particular that $(4\Delta_0^{3/m_0})^{-1}n^{\frac{1}{2}} \leqq T_{m_0n} \leqq 4^{-1}n^{\frac{1}{2}}$. The bound on the remainder term can then be written as $|R_{m_0n}(x)| \leqq M/n^{(m_0-2)/2}$.

Of the seven conditions in Cramér's theorem, perhaps condition (vii) is the most "unreasonable." In view of that observation, the following fundamental lemma turns out to be crucial.

(3.2) LEMMA. *If $\{x_{nk}: 1 \leqq k \leqq n, 1 \leqq n < \infty\}$ is a triangular array of real numbers for which there exists a $\Delta_0^* < \infty$ with $n \sum_{k=1}^n x_{nk}^4 \leqq \Delta_0^*$ for all n, $\{\alpha_{nk}: 1 \leqq k \leqq n, 1 \leqq n < \infty\}$ is a triangular array of numbers between 0 and 1 such that $\sum_{k=1}^n \alpha_{nk} x_{nk}^2 \geqq \gamma > 0$ for all n and some number γ, and, for each n, Ω_n is a subset of $\{k: 1 \leqq k \leqq n\}$, then every sequence $\{n_r: 1 \leqq r < \infty\}$ of positive integers contains a subsequence $\{n_p: 1 \leqq p < \infty\}$ such that either*

$$\text{(A}'\text{)} \quad \lim_{p\to\infty} (\log n_p)^{-1} \textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} = \infty\,, \qquad \textit{or}$$

$$\text{(B}'\text{)} \quad \lim_{p\to\infty} (n_p/\log n_p) \textstyle\sum_{k\in\Omega_{n_p}^c} \alpha_{n_pk} x_{n_pk}^2 = \infty\,.$$

PROOF. By the Schwarz inequality, we have that

$$\textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} x_{n_pk}^2 \leqq (\sum_{k\in\Omega_{n_p}} \alpha_{n_pk}^2)^{\frac{1}{2}} (\sum_{k\in\Omega_{n_p}} x_{n_pk}^4)^{\frac{1}{2}}$$

so that

$$\begin{aligned}\textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} &\geqq \textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk}^2 \geqq (\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} x_{n_pk}^2)^2 (\sum_{k\in\Omega_{n_p}} x_{n_pk}^4)^{-1} \\ &\geqq n_p(\Delta_0^*)^{-1} (\textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} x_{n_pk}^2)^2\,.\end{aligned}$$

Therefore

$$(\log n_p)^{-1} \textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} \geqq (\Delta_0^*)^{-1} (n_p/\log n_p) (\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} x_{n_pk}^2)^2\,.$$

Now, if condition (A′) holds for no subsequence $\{n_p\}$, then $(\log n_p)^{-1} \sum_{k\in\Omega_{n_p}} \alpha_{n_pk} \leqq M < \infty$ for some number M, for each subsequence. We would then have that

$$\textstyle\sum_{k\in\Omega_{n_p}} \alpha_{n_pk} x_{n_pk}^2 \leqq (M\Delta_0^* n_p^{-1} \log n_p)^{\frac{1}{2}}\,.$$

It follows that

$$(n_p/\log n_p) \textstyle\sum_{k\in\Omega_n^c} \alpha_{n_pk} x_{n_pk}^2 \geqq (n_p/\log n_p)[\gamma - (M\Delta_0^* n_p^{-1} \log n_p)^{\frac{1}{2}}] \to \infty$$

as $p \to \infty$. So if (A′) does not hold, then (B′) must hold.

The next lemma, together with the result of Corollary 2.4, insures that the forthcoming theorem on large deviations is not vacuous. Its objective is to show that, when the conditions of Corollary 2.4 hold, the conditions of Lemma 3.2 hold also.

(3.3) LEMMA. *If there exist numbers* α *and* θ, $0 < \alpha \leqq 1$, $0 < \theta \leqq 1$, *such that at least* αn *of the* a_{nk}*'s exceed or equal* $\theta\sigma_n$, *then there exists a* $\Delta_0^* < \infty$ *such that* $n \sum_{k=1}^n a_{nk}^4 \leqq \Delta_0^*$.

PROOF. $1 = \sum_{k=1}^n a_{nk}^2 \geqq \alpha n \theta^2 \sigma_n^{\ 2}$ so that $n\sigma_n^{\ 2} \leqq (\alpha\theta^2)^{-1}$. Hence $n \sum_{k=1}^n a_{nk}^4 \leqq n n \sigma_n^{\ 4} = (n\sigma_n^{\ 2})^2 \leqq (\alpha^2\theta^4)^{-1} = \Delta_0^*$.

4. The large deviation theorem. Our point of departure in proving the large deviation theorem for weighted sums of i.i.d. random variables is the expression in Lemma 1.4 for $P(S_n > c \sum_{k=1}^n a_{nk})$, where $h = h_n$ is the unique solution of $\sum_{k=1}^n a_{nk} Q(h a_{nk}) = c \sum_{k=1}^n a_{nk}$. In order to use our form of Cramér's theorem, we make the additional assumption that the df F of X_1 is absolutely continuous with density $f = F'$.

The random variables Y_{nk}, defined in Section 1, have densities $h_{nk}(y) = a_{nk}^{-1} f(y a_{nk}^{-1} + c)$. The associated random variables $\bar{Y}_{nk}$ have densities

$$\begin{aligned} \bar{h}_{nk}(y) &= [e^{h_n y}/\phi_{nk}(h_n)] h_{nk}(y) \\ &= [e^{h_n y + h_n c a_{nk}}/a_{nk}\phi(h_n a_{nk})] f(y a_{nk}^{-1} + c) . \end{aligned}$$

With $\bar{\sigma}_n = (\operatorname{Var} \bar{S}_n)^{\frac{1}{2}}$, where $\bar{S}_n = \sum_{k=1}^n \bar{Y}_{nk}$, we have that $\bar{H}_n(y) = P(\bar{S}_n \leqq y) = P(\sum_{k=1}^n X_{nk} \leqq y\bar{\sigma}_n^{\ -1}) = F_n(y\bar{\sigma}_n^{\ -1})$, in view of the definition of h_n, where $X_{nk} = [\bar{Y}_{nk} - E(\bar{Y}_{nk})]\bar{\sigma}_n^{\ -1}$ is the element of the array called for in Cramér's theorem, and F_n denotes the df of the row sum $\sum_{k=1}^n X_{nk}$. It will suffice for our present purposes to take, in Section 3, $m_0 = 4$, and we proceed to verify that the conditions of Theorem 3.1 hold. Elementary calculations, especially changes of variable and expansion of binomial powers, yield:

(4.1) LEMMA. *The fourth absolute moment of* X_{nk} *is given by*

$$\beta_{4nk} = a_{nk}^4 \bar{\sigma}_n^{\ -4} G(h_n a_{nk}) ,$$

where

$$\begin{aligned} G(t) = &-3[\phi'(t)/\phi(t)]^4 + 6[\phi'(t)/\phi(t)]^2[\phi''(t)/\phi(t)] \\ &- 4[\phi'(t)/\phi(t)][\phi^{(3)}(t)/\phi(t)] + [\phi^{(4)}(t)/\phi(t)] . \end{aligned}$$

The definition of ρ_{mn} at the beginning of Section 3 gives:

(4.2) COROLLARY. $\rho_{4n} = n\bar{\sigma}_n^{\ -4} \sum_{k=1}^n a_{nk}^4 G(h_n a_{nk})$.

PROOF. $\rho_{4n} = n^2 B_{4n} = n \sum_{k=1}^n \beta_{4nk}$.

In view of Lemmas 2.4 and 3.3, which give credence to the conditions, we have

(4.3) LEMMA. *Under conditions which guarantee the existence of a* $b_0 > 0$ *and a* $B_0 < \infty$, *where* $\phi(B_0) < \infty$, *such that* $b_0 \leqq h_n \sigma_n \leqq B_0$, *if there exists a* $\Delta_0^* < \infty$ *with* $n \sum_{k=1}^n a_{nk}^4 \leqq \Delta_0^*$ *for all positive integers* n, *then there exists a* $\Delta_0 < \infty$ *with* $\rho_{4n} \leqq \Delta_0$ *for all positive integers* n.

PROOF. Because β_{4nk} is a fourth moment of a nondegenerate distribution, we know that $G(h_n a_{nk}) > 0$ when $a_{nk} > 0$. Now $G(t)$ is continuous on the closed interval $0 \leqq t \leqq B_0$, and therefore it attains a maximum value $G_0 > 0$ on that interval. Corollary 4.2 implies then that $\rho_{4n} \leqq G_0 d_0^{-4} n \sum_{k=1}^{n} a_{nk}^4 \leqq G_0 d_0^{-4} \Delta_0^* = \Delta_0$, where $\bar{\sigma}_n^{-4} \leqq d_0^{-4} < \infty$ due to Lemma 2.5.

As noted after the proof of Cramér's theorem, Lemma 4.3 shows that the remainder term in the asymptotic expansion is bounded by a constant times n^{-1} for $m_0 = 4$. The next two lemmas deal with condition (vi) of Theorem 3.1, on the total variations of the density functions.

(4.4) LEMMA. *The total variation of the density function $f_{nk}(x)$ of the random variable $X_{nk} = [\bar{Y}_{nk} - E(\bar{Y}_{nk})]\bar{\sigma}_n^{-1}$ is*

$$v_{nk} = [\bar{\sigma}_n / a_{nk} \phi(h_n a_{nk})] v_{nk}^* ,$$

where v_{nk}^* is the total variation of $f_{nk}^*(z) = e^{h_n a_{nk} z} f(z)$, where f is the density of X_1.

PROOF. Recalling that $E(\bar{Y}_{nk}) = a_{nk} Q(h_n a_{nk}) - c a_{nk}$ from the proof of Lemma 2.1, we have, using the expression for $\bar{h}_{nk}(y)$ at the beginning of this section,

$$\begin{aligned} f_{nk}(x) &= (d/dx) P(\bar{Y}_{nk} - E(\bar{Y}_{nk}) \leqq x\bar{\sigma}_n) \\ &= [\bar{\sigma}_n / a_{nk} \phi(h_n a_{nk})] \exp[(x\bar{\sigma}_n / a_{nk}) + Q(h_n a_{nk})] f((x\bar{\sigma}_n / a_{nk}) + Q(h_n a_{nk})) . \end{aligned}$$

The result follows if we set $z = (x\bar{\sigma}_n / a_{nk}) + Q(h_n a_{nk})$.

(4.5) LEMMA. *If there exist numbers $b_0 > 0$ and $B_0 < \infty$, where $\phi(B_0) < \infty$, such that $b_0 \leqq h_n \sigma_n \leqq B_0$, and the functions $f_\lambda(z) = e^{\lambda z} f(z)$ are of uniformly bounded total variations $V_\lambda^* \leqq V^*$ for $0 \leqq \lambda \leqq B_0$, then there exist numbers $v_0 > 0$ and $V_0 < \infty$ such that $v_0 a_{nk}^{-1} \leqq v_{nk} \leqq V_0 a_{nk}^{-1}$.*

PROOF. Since $\phi(\lambda) = \int_{-\infty}^{\infty} e^{\lambda z} f(z)\, dz < \infty$ at least for $0 \leqq \lambda \leqq B_0$, we know that $\lim_{z\to\infty} e^{\lambda z} f(z) = 0$ for those λ's. Therefore the total variations $V_\lambda^* \geqq e^{\lambda a} f(a) \geqq f(a) > 0$ for some value of $a > 0$ by the assumptions of Corollaries 1.2 or 2.4, and so we have that $f(a) \leqq V_\lambda^* \leqq V^*$ for $0 \leqq \lambda \leqq B_0$. If we set $\lambda = h_n a_{nk}$, as $0 \leqq h_n a_{nk} \leqq h_n \sigma_n \leqq B_0$, we see that $f(a) \leqq v_{nk}^* \leqq V^*$. Using the result of Lemma 4.4, $\bar{\sigma}_n f(a)[a_{nk} \phi(h_n a_{nk})]^{-1} \leqq v_{nk} \leqq \bar{\sigma}_n V^* [a_{nk} \phi(h_n a_{nk})]^{-1}$. But we already know that $d_0 \leqq \bar{\sigma}_n \leqq D_0$ and $1 = \phi(0) \leqq \phi(h_n a_{nk}) \leqq \phi(B_0)$, so we can take $v_0 = d_0 f(a)/\phi(B_0)$ and $V_0 = D_0 V^*$ in the statement of the lemma.

The condition that the total variations V_λ^* be uniformly bounded in the closed interval $0 \leqq \lambda \leqq B_0$ holds if f is an exponential, uniform, normal, or other density "usually encountered in practice." It holds, in fact, whenever f has finitely many "peaks." It may hold even when f has infinitely many peaks, provided their heights are successively small enough and are spaced far enough apart.

(4.6) COROLLARY. *Under the conditions of Lemma 4.5, for any sequence $\{n_p\}$ of positive integers,*

$$\lim_{p\to\infty} (T_{4n_p}^2 / \log n_p) \sum_{k \in \Omega_{n_p}^c} v_{n_p k}^{-2} = \infty$$

if and only if $\lim_{p\to\infty} (T_{4n_p}^2 / \log n_p) \sum_{k \in \Omega_{n_p}^c} a_{n_p k}^2 = \infty$.

The corollary above brings us into contact with the last remaining unverified condition of Cramér's theorem. To show that Theorem 3.1 applies to the present situation, we combine Lemma 3.2 with Lemma 4.3 and Corollary 4.6, and observe that T_{4n} behaves like $n^{\frac{1}{2}}$ when $\rho_{4n} \leqq \Delta_0 < \infty$. Taking note of Lemmas 2.2 and 3.3, we derive the following corollary of Theorem 3.1:

(4.7) COROLLARY. *If there exist numbers $b_0 > 0$ and $B_0 < \infty$, where $\phi(B_0) < \infty$, such that $b_0 \leqq h_n \sigma_n \leqq B_0$ for all n, and a $\Delta_0^* < \infty$ such that $n \sum_{k=1}^n a_{nk}^4 \leqq \Delta_0^*$ for all n, and the functions $f_\lambda(z) = e^{\lambda z} f(z)$ are of uniformly bounded total variations $V_\lambda^* \leqq V^*$ for $0 \leqq \lambda \leqq B_0$, then*

$$\begin{aligned}\bar{H}_n(x\bar{\sigma}_n) &= \Phi(x) + n^{-\frac{1}{2}} P_{1n}(-\Phi) + R_{4n}(x) \\ &= \Phi(x) + n^{-\frac{1}{2}} p_{2n}(x) e^{-x^2/2} + R_{4n}(x)\,,\end{aligned}$$

where $p_{2n}(x)$ is a polynomial of degree 2 in x, and $|R_{4n}(x)| \leqq Mn^{-1}$, where M is a constant independent of n and x.

PROOF. We apply Theorem 3.1 with $m_0 = 4$, in accordance with Lemma 4.1. Taking $X_{nk} = [\bar{Y}_{nk} - E(\bar{Y}_{nk})]\bar{\sigma}_n^{-1}$, we see that $E(X_{nk}) = 0$ and $\sum_{k=1}^n E(X_{nk}^2) = 1$, since $\bar{\sigma}_n^2 = \operatorname{Var}(\bar{S}_n) = \operatorname{Var}(\sum_{k=1}^n \bar{Y}_{nk})$. Lemmas 4.4 and 4.5 give the total variations of the density functions of the X_{nk}'s, and, according to Lemma 4.3, we know that T_{4n} is of the order of $n^{\frac{1}{2}}$. Therefore Corollary 4.6 and Lemma 3.2 combine to yield condition (vii) of Theorem 3.1, with each $\alpha_{nk} = 1$. The conclusion then follows from the fact that $\bar{H}_n(x\bar{\sigma}_n) = P(\bar{S}_n \leqq x\bar{\sigma}_n) = P(\sum_{k=1}^n X_{nk} \leqq x)$.

We are finally ready to state and prove the main theorem:

(4.8) THEOREM. *Under the conditions of Corollary 4.7,*

$$\begin{aligned}&P(S_n > c \textstyle\sum_{k=1}^n a_{nk}) \\ &\quad = (2\pi)^{-\frac{1}{2}}(\bar{\sigma}_n h_n)^{-1} \exp(-h_n c \textstyle\sum_{k=1}^n a_{nk})(\prod_{k=1}^n \phi(h_n a_{nk}))(1 + O(\sigma_n))\,,\end{aligned}$$

where $O(\sigma_n) \to 0$ as $n \to \infty$.

PROOF. In Corollary 4.7, we set $y = x\bar{\sigma}_n$ to obtain

$$\begin{aligned}\bar{H}_n(y) &= \Phi(y\bar{\sigma}_n^{-1}) + n^{-\frac{1}{2}} p_{2n}(y\bar{\sigma}_n^{-1}) \exp(-y^2/2\bar{\sigma}_n^2) + R_{4n}(y\bar{\sigma}_n^{-1}) \\ &= K_{4n}^*(y\bar{\sigma}_n^{-1}) + R_{4n}(y\bar{\sigma}_n^{-1})\,,\end{aligned}$$

where $|R_{4n}(y\bar{\sigma}_n^{-1})| \leqq Mn^{-1}$ uniformly in y. Then, looking at Lemma 1.4, we write

$$\begin{aligned}I_n &= h_n \textstyle\int_0^\infty e^{-h_n y}[\bar{H}_n(y) - \bar{H}_n(0)]\, dy \\ &= h_n \textstyle\int_0^\infty e^{-h_n y}[K_{4n}^*(y\bar{\sigma}_n^{-1}) - K_{4n}^*(0)]\, dy + O(n^{-1})\,.\end{aligned}$$

Using the properties of the quantity $P_{mn}(-\Phi)$, as in [6] or [1], we can find the "characteristic function" of $K_{4n}^*(y\bar{\sigma}_n^{-1})$.

$$\begin{aligned}\gamma_{4n}^*(t) &= \textstyle\int_{-\infty}^\infty e^{ity}\, dK_{4n}^*(y) = \sum_{j=0}^1 n^{-j/2} \int_{-\infty}^\infty e^{ity}\, dP_{jn}(-\Phi) \\ &= \textstyle\sum_{j=0}^1 n^{-j/2} P_{jn}(it) e^{-t^2/2}\end{aligned}$$

is the Fourier transform of $K_{4n}^*(y)$, so what we want is

$$\gamma_{4n}^*(t\bar{\sigma}_n) = \textstyle\sum_{j=0}^1 n^{-j/2} P_{jn}(it\bar{\sigma}_n) \exp[-\frac{1}{2}t^2\bar{\sigma}_n^2]\,.$$

We define
$$f_n(y) = e^{-h_n y} \quad \text{for} \quad y \geqq 0$$
$$= 0 \quad \text{for} \quad y < 0 .$$

Then $g_n(t) = \int_{-\infty}^{\infty} e^{ity} f_n(y)\, dy = (h_n - it)^{-1}$ so that, by integration by parts and Parseval's formula,
$$I_n^* = h_n \int_0^\infty e^{-h_n y}[K_{4n}^*(y\bar{\sigma}_n^{-1}) - K_{4n}^*(0)]\, dy = (2\pi)^{-1} \int_{-\infty}^{\infty} \bar{g}_n(t)\gamma_{4n}^*(t\bar{\sigma}_n)\, dt .$$
It follows that
$$\begin{aligned}(2\pi)^{\frac{1}{2}} h_n I_n &= (2\pi)^{\frac{1}{2}} h_n I_n^* + (2\pi)^{\frac{1}{2}} h_n O(n^{-1}) \\ &= (2\pi)^{-\frac{1}{2}} h_n \int_{-\infty}^{\infty} \bar{g}_n(t)\gamma_{4n}^*(t\bar{\sigma}_n)\, dt + O(n^{-1}h_n) \\ &= \bar{\sigma}_n^{-1} \int_{-\infty}^{\infty} [1 + is(h_n\bar{\sigma}_n)^{-1}]^{-1}[\textstyle\sum_{j=0}^{1} n^{-j/2} P_{jn}(is)]\, d\Phi(s) + O(n^{-1}h_n) .\end{aligned}$$
Now $[1 + is(h_n\bar{\sigma}_n)^{-1}]^{-1} = 1 + sw_n(s)(h_n\bar{\sigma}_n)^{-1}$, where $|w_n(s)|$ is bounded in n and s. We can then define
$$\mu(n, r, q) = \bar{\sigma}_n^{-r} \int_{-\infty}^{\infty} (is)^r P_{qn}(is)\, d\Phi(s) .$$
Then $\mu(n. r, q) = 0$ if $r + q$ is odd, because the odd moments of the normal distribution vanish and $P_{qn}(is)$ is a polynomial of degree $3q$ in (is), containing only odd (respectively, even) powers if q is odd (respectively, even), according to the arguments on page 74 of [6] and page 60 of [2]. Since Φ has finite moments of all orders, we can continue from the above that
$$\begin{aligned}(2\pi)^{\frac{1}{2}} h_n I_n &= \bar{\sigma}_n^{-1} \int_{-\infty}^{\infty} P_{0n}(is)\, d\Phi(s) + \bar{\sigma}_n^{-1} \int_{-\infty}^{\infty} sw_n(s)(h_n\bar{\sigma}_n)^{-1} P_{0n}(is)\, d\Phi(s) \\ &\quad + \bar{\sigma}_n^{-1} \int_{-\infty}^{\infty} n^{-\frac{1}{2}} P_{1n}(is)\, d\Phi(s) \\ &\quad + \bar{\sigma}_n^{-1} \int_{-\infty}^{\infty} n^{-\frac{1}{2}} sw_n(s)(h_n\bar{\sigma}_n)^{-1} P_{1n}(is)\, d\Phi(s) + O(n^{-1}h_n) \\ &= \bar{\sigma}_n^{-1} + O(h_n^{-1}) + 0 + O(n^{-\frac{1}{2}}h_n^{-1}) + O(n^{-1}h_n) ,\end{aligned}$$
because $P_{0n}(is) = 1$, the second integral involves the first absolute moment of Φ, the third integral involves the third moment which is 0, and the fourth involves the fourth moment because $P_{1n}(is)$ is a polynomial of degree 3 in s. From the assumption that $b_0 \leqq h_n\sigma_n \leqq B_0$ and the fact that $\sigma_n \geqq n^{-\frac{1}{2}}$, it follows that $O(h_n^{-1}) = O(\sigma_n)$, $O(n^{-\frac{1}{2}}h_n^{-1}) \leqq O(\sigma_n)$ and $O(n^{-1}h_n) \leqq O(\sigma_n)$. Therefore,
$$(2\pi)^{\frac{1}{2}} h_n I_n = \bar{\sigma}_n^{-1} + O(\sigma_n) = \bar{\sigma}_n^{-1}(1 + O(\sigma_n)) ,$$
in view of Lemma 2.5. It follows that
$$I_n = (2\pi)^{-\frac{1}{2}}(\bar{\sigma}_n h_n)^{-1}(1 + O(\sigma_n)) .$$
The theorem then follows from Lemma 1.4, with $O(\sigma_n) \to 0$ as $n \to \infty$ because $\Delta_0^* \geqq n \sum_{k=1}^{n} a_{nk}^2 \geqq n\sigma_n^4$ implies that $\sigma_n \leqq (\Delta_0^* n^{-1})^{\frac{1}{4}} \to 0$ as $n \to \infty$.

Our theorem on large deviation probabilities for the weighted sums $S_n = \sum_{k=1}^{n} a_{nk} X_k$ reduces to case 1 of the Bahadur–Ranga Rao theorem when each $a_{nk} = n^{-\frac{1}{2}}$.

(4.9) BAHADUR–RANGA RAO THEOREM. *There exist positive numbers ρ and b, with $0 < \rho < 1$, such that*
$$P(\textstyle\sum_{k=1}^{n} X_k \geqq nc) = (2\pi n)^{-\frac{1}{2}} \rho^n b(1 + o(1)) .$$

PROOF. When each $a_{nk} = n^{-\frac{1}{2}}$, we have $S_n = \sum_{k=1}^n n^{-\frac{1}{2}} X_k = n^{-\frac{1}{2}} \sum_{k=1}^n X_k$ and $c \sum_{k=1}^n a_{nk} = cn^{\frac{1}{2}}$, so that in the absolutely continuous case, $P(S_n > c \sum_{k=1}^n a_{nk}) = P(\sum_{k=1}^n X_k \geqq nc)$. The condition $E(\bar{S}_n) = 0$ of Lemma 2.2 becomes the assertion that

$$\begin{aligned} 0 &= \textstyle\sum_{k=1}^n a_{nk}[\phi'(ha_{nk})/\phi(ha_{nk})] - c \sum_{k=1}^n a_{nk} \\ &= n^{\frac{1}{2}}[\phi'(hn^{-\frac{1}{2}})/\phi(hn^{-\frac{1}{2}})] - cn^{\frac{1}{2}} \end{aligned}$$

or that $\phi'(hn^{-\frac{1}{2}})/\phi(hn^{-\frac{1}{2}}) = c$. Taking $h_n = \tau n^{\frac{1}{2}}$, where τ is the value of t at which $\psi(t) = e^{-ct}\phi(t)$ is minimized with minimum ρ (such a τ is required to exist under the Bahadur–Ranga Rao conditions), we have, in our theorem, that

$$\begin{aligned} \exp(-h_n c \textstyle\sum_{k=1}^n a_{nk}) \prod_{k=1}^n \phi(h_n a_{nk}) &= e^{-c\tau n} \textstyle\prod_{k=1}^n \phi(\tau) \\ &= (e^{-ct}\phi(\tau))^n = \rho^n . \end{aligned}$$

Finally, from Lemma 2.1, we see that

$$\bar{\sigma}_n = (\operatorname{Var}(\bar{S}_n))^{\frac{1}{2}} = [\phi(\tau)\phi''(\tau) - (\phi'(\tau))^2]/(\phi(\tau))^2 = (b\tau)^{-1} .$$

Substituting these values into Theorem 4.8, and noting that $O(n^{-\frac{1}{2}}) = o(1)$, the reduction follows.

5. The logarithmic form. The theorem of the previous section leads to logarithmic results of the sort studied extensively by Sethuraman (1964) and (1970), for example, and more recently by Feller (1968) and (1969). We extend the simplest of these theorems, that due to Chernoff (1952), to the case of weighted sums. Feller's results deal with the more general case of triangular arrays, but the generality apparently does not permit as detailed knowledge of the sequence $\{h_n \colon 1 \leqq n < \infty\}$ as is available in our more restricted situation. Related problems have been studied by M. Stone (1969), in the discrete case, and Sievers (1969), from the viewpoint of a sequence of moment-generating functions.

The logarithmic result of this section, derived here as a consequence of Theorem 4.8, can actually be proved under more general conditions. In the general case, there is no assumption of absolute continuity and therefore no requirements on the total variations of the density functions that necessitate Lemmas 4.4, 4.5, and 4.6. In that generality, details of the formulation and proof can be found in Chapter 3 of [2], as well as in a forthcoming article. Here we give only the following consequence of the theorem of Section 4:

(5.1) COROLLARY. *If there exist positive numbers α and θ, $0 < \alpha \leqq 1$, $0 < \theta \leqq 1$, such that $Q = \phi'/\phi$ assumes the value $c(\alpha\theta)^{-1}$ at some point and $B_0 = \theta^{-1}Q^{-1}(c/\alpha\theta)$ lies in the domain of ϕ, and at least αn of the a_{nk}'s exceed or equal $\theta\sigma_n$, and the functions $f_\lambda(z) = e^{\lambda z}f(z)$ have uniformly bounded total variations $V_\lambda^* \leqq V^*$ for $0 \leqq \lambda \leqq B_0$, then there exist two positive numbers $\beta_1 \leqq \beta_2$, whose values can be precisely determined, such that*

$$-\beta_2 \leqq \sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1} \log P(S_n > c \sum_{k=1}^n a_{nk}) \leqq -\beta_1$$

for all sufficiently large n.

PROOF. Lemmas 2.3 and 3.3 guarantee the remaining conditions of Theorem 4.8. From the theorem, we can write

$$\begin{aligned}\sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\log P(S_n > c\sum_{k=1}^n a_{nk})\\ &= -h_n c\sigma_n + \sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\sum_{k=1}^n \log\phi(h_n a_{nk})\\ &\quad - \sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\log((2\pi)^{\frac{1}{2}}\bar{\sigma}_n h_n) + \sigma_n(\sum_{k=1}^n a_{nk})^{-1}\log(1+O(\sigma_n)).\end{aligned}$$

The last two terms can be easily disposed of: by Lemma 2.5,

$$\sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\log((2\pi)^{\frac{1}{2}}\bar{\sigma}_n h_n) \leqq \sigma_n\log((2\pi)^{\frac{1}{2}}D_0 B_0\sigma_n^{-1}) \to 0 \qquad \text{as} \quad n\to\infty,$$

since $\sum_{k=1}^n a_{nk} \geqq \sum_{k=1}^n a_{nk}^2 = 1$, and

$$\sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\log(1+O(\sigma_n)) \leqq \log(1+O(\sigma_n)) \to 0 \qquad \text{as } n\to\infty,$$

since $\sum_{k=1}^n a_{nk} \geqq \sigma_n$. It remains to show that

$$\beta_1^* \leqq h_n c\sigma_n - \sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\sum_{k=1}^n\log\phi(h_n a_{nk}) \leqq \beta_2^*,$$

where β_1^* and β_2^* are positive numbers whose values can be determined. Consider now the function $L(t) = tQ(t) - \log\phi(t)$. Its derivative $L'(t) = tQ'(t) > 0$ for all $t > 0$ by an argument used in the proof of Lemma 2.5. Since $L(0) = 0$, $L(t) \geqq 0$ for $t \geqq 0$ and increases in t. Using these facts, we write by Lemma 2.2.

$$\begin{aligned}h_n c\sigma_n - \sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\sum_{k=1}^n\log\phi(h_n a_{nk})\\ &= \sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\sum_{k=1}^n L(h_n a_{nk}) \leqq \sigma_n(\sigma_n^{-1})^{-1}nL(B_0)\\ &\leqq n\sigma_n^2 L(B_0) \leqq (\alpha\theta^2)^{-1}L(B_0) = \beta_2^*,\end{aligned}$$

because the conditions imply that $n\sigma_n^2 \leqq (\alpha\theta^2)^{-1}$, as in the proof of Lemma 3.3. The lower bound is obtained by:

$$\begin{aligned}\sigma_n(\textstyle\sum_{k=1}^n a_{nk})^{-1}\sum_{k=1}^n L(h_n a_{nk}) &\geqq \sigma_n(n\sigma_n)^{-1}\alpha nL(\theta h_n\sigma_n)\\ &\geqq \alpha L(\theta b_0) = \beta_1^*.\end{aligned}$$

Note that in the i.i.d. situation, Corollary 5.1 reduces to Chernoff's original theorem. We have each $a_{nk} = \sigma_n = n^{-\frac{1}{2}}$, $\sum_{k=1}^n a_{nk} = n^{\frac{1}{2}}$, $h_n = \tau n^{\frac{1}{2}}$ and $\rho = e^{-c\tau}\phi(\tau)$, where $Q(\tau) = c$, as in the proof of Corollary 4.9. It follows that $b_0 = B_0 = \tau$, $\alpha = \theta = 1$, and $L(\tau) = \tau Q(\tau) - \log\phi(\tau) = \tau c - \log\phi(\tau) = -\log\rho$. The assertion of the corollary becomes

$$\lim_{n\to\infty} n^{-1}\log P(\textstyle\sum_{k=1}^n X_k \geqq nc) = -\log\rho,$$

which is Chernoff's theorem.

6. The exponential case and order statistics. In the special case when the random variable X_1 is exponentially distributed with mean 0, variance 1, and m.g.f. $\phi(t) = (1-t)^{-1}e^{-t}$, it then turns out that $c(c+1)^{-1} \leqq h_n\sigma_n < 1$ always. If we require the existence of a number $\theta_0 < 1$ such that $h_n\sigma_n \leqq \theta_0$, a bound which holds under the condition of Lemma 3.3, the remaining conditions of Theorem 4.8 are satisfied for the exponential density. Details can be found in Chapter 5 of [2]. It is possible to then use Theorem 4.8 to derive a large deviation theorem for linear combinations of exponential order statistics. If $V_{1n}, \cdots, V_{nn}$ are the increasing order statistics of a sample of size n, then Chernoff, Gastwirth, and Johns (1967) showed that linear combinations $\sum_{j=1}^{n} c_{jn} V_{jn}$, when properly normalized, converge in distribution to the normal; they used the fact, proved in Rényi (1953), that exponential order statistics can be expressed as particular weighted sums of i.i.d. exponential random variables. More precisely, there exists a sequence $\{Z_n : 1 \leqq n < \infty\}$ of i.i.d. exponential random variables with density $f(x) = e^{-x}$ for $x > 0$, such that, for every positive integer n and each j, $1 \leqq j \leqq n$, $V_{jn} = \sum_{k=1}^{j} (n - k + 1)^{-1} Z_k$. We then have the following lemma:

(6.1) LEMMA. *There exist nonnegative numbers* $\{a_{nk} : 1 \leqq k \leqq n\}$, *depending on the coefficients* $\{c_{jn} : 1 \leqq j \leqq n\}$, *such that* $\sum_{k=1}^{n} a_{nk}^2 = 1$, *and a number* c_n, *depending on the* c_{jn}*'s and* λ_n, *so that* $P(\sum_{j=1}^{n} c_{jn} V_{jn} > \lambda_n) = P(\sum_{k=1}^{n} a_{nk}(Z_k - 1) > c_n)$.

PROOF. Defining $d_{nk} = (n - k + 1)^{-1} \sum_{j=k}^{n} c_{jn}$, noting that $E(Z_k) = 1$, and interchanging the order of summation, we have

$$\begin{aligned} P(\textstyle\sum_{j=1}^{n} c_{jn} V_{jn} > \lambda_n) &= P(\textstyle\sum_{k=1}^{n} d_{nk} Z_k > \lambda_n) \\ &= P(\textstyle\sum_{k=1}^{n} a_{nk}(Z_k - 1) > c_n), \end{aligned}$$

where $a_{nk}^2 = d_{nk}^2 (\sum_{k=1}^{n} d_{nk}^2)^{-1}$ and $c_n = (\lambda_n - \sum_{k=1}^{n} d_{nk})(\sum_{k=1}^{n} d_{nk}^2)^{-\frac{1}{2}}$.

If we set $X_k = Z_k - 1$ and $c_n = c \sum_{k=1}^{n} a_{nk}$, equivalently $\lambda_n = (c + 1) \sum_{k=1}^{n} d_{nk}$, then $P(\sum_{j=1}^{n} c_{jn} V_{jn} > \lambda_n)$ becomes $P(S_n > c \sum_{k=1}^{n} a_{nk})$ where $S_n = \sum_{k=1}^{n} a_{nk} X_k$. We can then apply Theorem 4.8 to find the asymptotic representation of the large deviation probability for $\sum_{j=1}^{n} c_{jn} V_{jn}$.

It is much simpler to represent large deviation probabilities for single order statistics than for proper linear combinations. The asymptotic representation for order statistics from an arbitrary distribution, not necessarily the exponential, can be found in [3] and follows directly from the original Bahadur–Ranga Rao theorem.

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