The Annals of Mathematical Statistics
1972, Vol. 43, No. 4, 1204-1212

APPLICATION OF THE SKOROKHOD REPRESENTATION THEOREM TO RATES OF CONVERGENCE FOR LINEAR COMBINATIONS OF ORDER STATISTICS

BY WALTER ROSENKRANTZ[1] AND NEVILLE E. O'REILLY

Courant Institute of Mathematical Sciences and Rutgers University

Rates of convergence for linear combinations of order statistics are obtained. The work is in the spirit of those authors who have used in one form or another the weak convergence of the sample empirical process to a tied-down Wiener process, except that the Skorokhod embedding is explicitly used to obtain a rate of convergence via control on the tail-behavior of the stopping times. The paper concludes with a remark on the limitations of the technique as far as getting the best possible rate is concerned.

1. Introduction. Suppose that there are given $U_{1n}, U_{2n}, \cdots, U_{nn}$ ordered observations from a uniform distribution on (0,1). Set $U_{0n} = 0$, $U_{(n+1)n} = 1$ and let F_n^{-1} be a version of the inverse of the empirical distribution function of the U_{jn}'s defined by

$$(1.1)\qquad \begin{aligned} F_n^{-1}(t) &= U_{(j-1)n}\,, \qquad (j-1)/(n+1) \leqq t < j/(n+1);\ j = 1, 2, \cdots, (n+1)\,. \\ &= 1\,, \qquad t = 1\,. \end{aligned}$$

Our purpose is to consider rates of convergence for the asymptotic normality of statistics, appropriately normalized, of the form

$$(1.2)\qquad T_n = n^{-1} \textstyle\sum_{j=1}^{n} C_{jn} H(X_{jn})$$

where the C_{jn}'s are specified constants, H is a real valued Borel measurable function on the real line and $X_{1n}, X_{2n}, \cdots, X_{nn}$ are the order statistics of a sample of size n from a continuous distribution F. It is convenient to represent T_n as

$$(1.3)\qquad T_n = \textstyle\int_0^1 h[F_n^{-1}(t)]\, d\nu_n(t)\,,$$

where h is the composition of H with F^{-1} i.e. $h(t) = H[F^{-1}(t)]$, and ν_n is a discrete signed measure defined by

$$(1.4)\qquad \nu_n[j/(n+1)] = n^{-1} C_{jn}\,, \qquad j = 1, 2, \cdots, n\,.$$

If ν is a signed measure we denote its total variation by $|\nu|$. That is $|\nu| = \nu^+ + \nu^-$ where ν^+, ν^- are the components appearing in the Jordan-Hahn decomposition of ν. Pyke in [2] notes that "in most applications the sequence of measure ν_n converges suitably to a finite Lebesgue Stieltjes signed measure ν in such a way as to allow one to replace ν_n by the limiting measure ν." Initially then, we

Received August 21, 1971; revised January 20, 1972.

[1] Research supported in part by N.S.F. Grant GP-11460.

consider not T_n but the normalized statistic

$$V_n = n^{\frac{1}{2}} \int_0^1 \{h[F_n^{-1}(t)] - h(t)\}\, d\nu(t) = \int_0^1 A_n(t) D_n(t)\, d\nu(t) \tag{1.5}$$

where

$$A_n(t) = \frac{h(F_n^{-1}(t)) - h(t)}{F_n^{-1}(t) - t}, \qquad 0 \leqq t \leqq 1 \tag{1.6}$$

and

$$D_n(t) = n^{\frac{1}{2}}(F_n^{-1}(t) - t), \qquad 0 \leqq t \leqq 1. \tag{1.7}$$

The method depends on using the Skorokhod embedding theorem as in Rosenkrantz [4] to get a speed for the convergence to zero in probability of the maximum distance between a version of the process D_n and a tied down Wiener process $W_0(t)$ defined on a common probability space. In Section 2 preliminary results for a special version of D_n are established in Lemmas 2.1, 2.2 and 2.3. The results on rates are given in Section 3. Theorem 3.1 applies to the statistic V_n, while Theorem 3.2 gives a result for T_n in the case that the weights are given by a "scoring" function. Section 4 is a comment on the "right" rate and the limitations of the method used.

2. Preliminary results. For any two functions $x(\cdot)$ and $y(\cdot)$ on [0, 1] let

$$d(x, y) = \sup(|x(t) - y(t)|;\ 0 \leqq t \leqq 1). \tag{2.1}$$

The rate of convergence will depend on the choice of a nonnegative sequence $\varepsilon_1, \varepsilon_2, \cdots$ decreasing to zero for which $P(d(D_n, W_0) \geqq \varepsilon_n) \to 0$, as $n \to \infty$, at a speed which can be determined. In Section 4 it is shown that a necessary condition on ε_n is that

$$\lim_{n\to\infty} \varepsilon_n n^{\frac{1}{4}} = \infty. \tag{2.2}$$

To get a version of D_n for which $P(d(D_n, W_0) \geqq \varepsilon_n)$ tends to zero for a suitable choice of ε_n we use the Skorokhod embedding.

Let $Y_k = \sum_{j=1}^k X_j$, where $X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables with

$$P(X_1 \geqq x) = \exp(-x), \qquad x \geqq 0.$$

It is well known (see e.g. [1] page 285) that the random vectors $(U_{1n}, U_{2n}, \cdots, U_{nn})$ and $(Y_1/Y_{n+1}, Y_2/Y_{n+2}, \cdots, Y_n/Y_{n+1})$ have the same distribution. If $S_{n+1}(t)$ is the "random broken line" defined by

$$\begin{aligned} S_{n+1}(t) &= (Y_j - j)/(n+1)^{\frac{1}{2}}, \qquad j/(n+1) \leqq t < (j+1)/(n+1) \\ &= [Y_{n+1} - (n+1)]/(n+1)^{\frac{1}{2}}, \qquad t = 1 \end{aligned} \tag{2.3}$$

where $j = 0, 1, 2, \cdots, n$ and $Z_0 \equiv 0$, then

$$D_n(t) \cong \frac{n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}}{Y_{n+1}}(S_{n+1}(t) - tS_{n+1}(1)) + n^{\frac{1}{2}}(e_{n+1}(t) - t) \tag{2.4}$$

where $\cong$ means that the two processes have the same distribution and $e_{n+1}(t) = (n+1)^{-1}[(n+1)t]$.

There exists a probability space with a Brownian motion $W(t)$ and a sequence $\tau_1, \tau_2, \cdots$ of nonnegative, independent and identically distributed random variables with the following properties:

(2.5) (a) the sequence $\{(Y_k - k)/n^{\frac{1}{2}}\}$, $k \geqq 1$ is distributed as the sequence $\{W(\sum_{j=1}^{k} \tau_j/n)\}$.

(b) $E(\tau_1) = 1$, $E|\tau_1|^r \leqq 4r\Gamma(r)E(|X_1 - 1|^{2r})$, $r \geqq 1$.

D_n can be represented on the same space as $W(t)$ since

$$D_n(t) \cong \frac{n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}}{W((n+1)^{-1}\sum_{j=1}^{n+1}\tau_j)}(\bar{S}_{n+1}(t) - t\bar{S}_{n+1}(1)) + n^{\frac{1}{2}}(e_{n+1}(t) - t) \tag{2.6}$$

where $\bar{S}_{n+1}(t)$ is defined in (2.3) but with $W(\sum_{k=1}^{j}\tau_k/n+1)$ replacing $(Y_j - j)/(n+1)^{\frac{1}{2}}$ there.

Rosenkrantz [4] showed that for a version of D_n almost like that in (2.6)

$$P(d(D_n, W_0) \geqq 12(\log n)n^{-\frac{1}{8}}) = O(n^{-\frac{1}{8}}) , \tag{2.7}$$

where $W_0(t) = W(t) - tW(1)$. Slight modifications to the methods in [4] will yield (2.7) for D_n exactly as in (2.6). However, this result can be improved if in place of the estimate (27) in Rosenkrantz [3] one uses

$$P(\max_{1 \leqq k \leqq (n+1)} |\textstyle\sum_{j=1}^{k}(\tau_j - 1)| \geqq (n+1)^{\frac{1}{2}} \log n) = O(n^{-1}) . \tag{2.8}$$

This gives in place of Lemma 6 of [3], with $\varepsilon_n = 2(\log n)n^{-\frac{1}{4}}$ and $\delta_n = n^{-\frac{1}{2}} \log n$,

$$P(\sup_{0 \leqq t \leqq 1} |W_{n+1}(t) - \bar{S}_{n+1}(t)| \geqq \varepsilon_n) = O(n^{-1}) , \tag{2.9}$$

where

$$\begin{aligned} W_{n+1}(t) &= W(j/(n+1)) , \qquad & j/(n+1) \leqq t < (j+1)/(n+1) \\ &= W(1) , & t = 1 . \end{aligned}$$

Lemma 8 of [4] can now be sharpened in an obvious way to get

LEMMA 2.1. *There exists a probability space with a Brownian motion $W(t)$ and $\tau_1, \tau_2, \cdots$ satisfying* (2.5) *so that for D_n as in* (2.6)

$$P(d(D_n, W_0) \geqq 12(\log n)n^{-\frac{1}{4}}) = O(n^{-\frac{1}{2}}) . \tag{2.10}$$

PROOF. This has been indicated above, apart from (2.8) which is established by a reasonably straightforward truncation argument which we will not give here.

We require two additional preliminary results.

LEMMA 2.2. $P(\sup_{0 \leqq t \leqq 1} |D_n(t)| \geqq 12(\log n)^{\frac{1}{2}}) = O(n^{-\frac{1}{2}})$.

PROOF. From (2.4) it follows that,

$$
\begin{aligned}
(2.11)\qquad & P(\sup_{0\le t\le 1} |D_n(t)| \geqq 12(\log n)^{\frac{1}{2}}) \\
&\leqq P(\sup_{0\le t\le 1} |S_{n+1}(t) - tS_{n+1}(1)| \geqq 12(\log n)^{\frac{1}{2}} Y_{n+1}/(n+1)) \\
&\leqq P(\sup_{0\le t\le 1} |S_{n+1}(t) - tS_{n+1}(1)| \geqq 12(\log n)^{\frac{1}{2}}(1-\alpha_n)\} \\
&\quad + P\{|Y_{n+1} - (n+1)| \geqq (n+1)\alpha_n\} \\
&= P_n^{(1)} + P_n^{(2)}, \text{ say.}
\end{aligned}
$$

Choose $\alpha_n = 3(\log n)^{\frac{1}{2}}/(n+1)^{\frac{1}{2}}$. Then by Lemma 4(i) of [4] it follows immediately that

$$(2.12)\qquad P_n^{(2)} = O(n^{-\frac{1}{2}}).$$

Since for sufficiently large n, $(1-\alpha_n) \geqq \frac{2}{3}$ it follows that

$$
\begin{aligned}
(2.13)\qquad P_n^{(1)} &\leqq P\{\sup_{0\le t\le 1} |S_{n+1}(t) - tS_{n+1}(1)| \geqq 8(\log n)^{\frac{1}{2}}\} \\
&\leqq P\{\max_{1\le k\le 1} |\textstyle\sum_{j=1}^{k} (X_j - 1)/(n+1)^{\frac{1}{2}}| \geqq 4(\log n)^{\frac{1}{2}}\} \\
&\leqq 2P\{|Y_{n+1} - (n+1)| \geqq 4[(n+1)\log n]^{\frac{1}{2}} - (2(n+1))^{\frac{1}{2}}\} \\
&\leqq 2P\{|Y_{n+1} - (n+1)| \geqq (n+1)\alpha_n\} \qquad \text{for large enough } n, \\
&= O(n^{-\frac{1}{2}}), \qquad \text{by Lemma 4(i) of [4].}
\end{aligned}
$$

The result follows from (2.11), (2.12) and (2.13).

LEMMA 2.3. *For some* $0 < \lambda < 1$ *and* $0 < \alpha \leqq \frac{1}{2}$, *let*

$$(2.14)\qquad \alpha_n = 12(1-\lambda)^{-1}(\log n/n)^{\frac{1}{2}}$$

and

$$
(2.15)\qquad
\begin{aligned}
q_\alpha(t) &= t^{\frac{1}{2}-\alpha}, & 0 \leqq t \leqq \tfrac{1}{2} \\
&= q_\alpha(1-t), & \tfrac{1}{2} \leqq t \leqq 1.
\end{aligned}
$$

For any constant $C > 0$,

$$(2.16)\qquad P(\sup_{0\le t\le \alpha_n} (|W_0(t)|/q_\alpha(t) \geqq C(\log n)^{3/4+\alpha/2} n^{-\alpha/2}) = O(n^{-1}).$$

PROOF.

$$
\begin{aligned}
& P\{|W_0(t)| \geqq C(\log n)^{3/4+\alpha/2} n^{-\alpha/2} q_\alpha(t) \text{ for some } 0 \leqq t \leqq \alpha_n\} \\
&\leqq P\left\{|W(t)| \geqq \frac{C}{2}(\log n)^{3/4+\alpha/2} n^{-\alpha/2} q_\alpha(t) \text{ for some } 0 \leqq t \leqq \alpha_n\right\} \\
&\quad + P\left\{|W(1)| \geqq \frac{C}{2}(\log n)^{3/4+\alpha/2} n^{-\alpha/2} \alpha_n^{-\frac{1}{2}-\alpha}\right\} \\
&= P_n^{(1)} + P_n^{(2)}, \text{ say.}
\end{aligned}
$$

Let $C_n = \frac{1}{2}C(\log n)^{3/4+\alpha/2} n^{-\alpha/2}$. Then

$$(2.17)\qquad P_n^{(1)} \leqq 2P\{W(t) \geqq C_n t^{\frac{1}{2}-\alpha} \text{ for some } 0 < t \leqq \alpha_n\}.$$

Let r be a real number between 0 and 1 and let $k(\alpha_n)$ denote the largest integer k such that $r^k \geqq \alpha_n$. Then

$$
\begin{aligned}
P_n^{(1)} &\leqq 2 \sum_{k\geqq k(\alpha_n)} P\{\max_{r^{k+1}\leqq t\leqq r^k} W(t) \geqq C_n r^{(k+1)(\frac{1}{2}-\alpha)}\} \\
&\leqq 2 \sum_{k\geqq k(\alpha_n)} P\{\max_{0\leqq t\leqq r^k} W(t) \geqq C_n r^{(k+1)(\frac{1}{2}-\alpha)}\} \\
&\leqq 4 \sum_{k\geqq k(\alpha_n)} P\{W(1) \geqq C_n r^{(k+1)(\frac{1}{2}-\alpha)-k/2}\} \\
&\leqq [4(2\pi)^{-\frac{1}{2}}/C_n] \sum_{k\geqq k(\alpha_n)} r^{k\alpha-(\frac{1}{2}-\alpha)} \exp\{-\tfrac{1}{2}(C_n^2 r^{-2k\alpha+2(\frac{1}{2}-\alpha)})\}\,.
\end{aligned} \tag{2.18}
$$

The last sum is bounded by the following integral,

$$[4(2\pi)^{-\frac{1}{2}}/C_n] \int_{k(\alpha_n)-1}^{\infty} r^{x\alpha-(\frac{1}{2}-\alpha)} \exp\{-\tfrac{1}{2}(C_n^2 r^{-2x\alpha+2(\frac{1}{2}-\alpha)})\}\, dx\,. \tag{2.19}$$

Make the change of variable $y = f(x)$ where

$$f(x) = C_n r^{-x\alpha+(\frac{1}{2}-\alpha)}\,. \tag{2.20}$$

A routine calculation transforms (2.17) into

$$4(2\pi)^{-\frac{1}{2}} \int_{\beta_n}^{\infty} y^{-2} \exp(-y^2)\, dy, \tag{2.21}$$

where

$$\beta_n = f[k(\theta_n) - 1] \geqq (\log n)^{\frac{1}{2}} \qquad \text{for large enough } n. \tag{2.22}$$

It is clear then that (2.21), and hence $P_n^{(1)}$, is $O(n^{-1})$. $P_n^{(2)}$ is easily estimated to be $O(n^{-1})$.

3. Rates of convergence. For $0 < \lambda < 1$ define $V_n(\lambda)$ by

$$V_n(\lambda) = n^{\frac{1}{2}} \int_{\alpha_n}^{1-\alpha_n} \{h[F_n^{-1}(t)] - h(t)\}\, d\nu(t)\,, \tag{3.1}$$

with $\alpha_n = 12(1-\lambda)^{-1}(\log n/n)^{\frac{1}{2}}$. Let $G_n(\lambda; x)$ denote the distribution function of $V_n(\lambda)$. That is

$$G_n(\lambda; x) = P(S_n(\lambda) \leqq x)\,.$$

Denote by Φ the standard normal distribution function. Let

$$q_\alpha'(t) = \{t(1-t)\}q_\alpha(t)\,.$$

Our first result is

THEOREM 3.1. *Suppose $h'(t)$ exists on $(0, 1)$ and that for some $0 < \alpha \leqq \frac{1}{2}$ and $0 < \lambda < 1$ the following conditions are satisfied.*

$$\int_0^1 |h'(t)| q_\alpha(t)\, d|\nu| < \infty\,. \tag{3.2}$$

$$|h'(t_1) - h'(t_2)| \leqq K_\lambda(t)|t_1 - t_2| \quad \textit{for } t_1, t_2 \textit{ in } \quad [\lambda t, 1-\lambda(1-t)]\,, \quad 0 < t < 1\,, \tag{3.3}$$

where K is a nonnegative function, non-increasing on $[0, \frac{1}{2}]$, with $K(t) = K(1-t)$ and

$$\int_0^1 K_\lambda(t) q_\alpha'(t)\, d|\nu| < \infty\,, \tag{3.4}$$

where

$$
\begin{aligned}
K_\lambda(t) &= K(\lambda t)\,, && 0 < t \leqq \tfrac{1}{2} \\
&= K[\lambda(1-t)]\,, && \tfrac{1}{2} \leqq t < 1\,.
\end{aligned} \tag{3.5}
$$

Then

$$\max_{-\infty<x<\infty} |G_n(\lambda; x) - \Phi(x/\sigma)| = O\{(\log n)^{3/4+\alpha/2} n^{-\alpha/2}\} \tag{3.6}$$

where σ *is defined by*

$$\sigma^2 = \int_0^1 \int_0^t h'(t)h'(s)s(1-t)\, d\nu(s)\, d\nu(t)\,. \tag{3.7}$$

PROOF. Let $b_n = C(\log n)^{3/4+\alpha/2} n^{-\alpha/2}$ where

$$C = 36 \max \left(\int_0^1 |h'(t)| q_\alpha(t)\, d|\nu|,\ \int_0^1 K_\lambda(t) q_\alpha'(t)\, d|\nu|\right)\,.$$

The first step in the proof is to show that

$$P(|V_n(\lambda) - \int_0^1 h'(t) W_0(t)\, d\nu| \geqq b_n) = O(n^{-\frac{1}{2}})\,. \tag{3.8}$$

The left-hand side of (3.8) is bounded above by $P_n^{(1)} + P_n^{(2)} + P_n^{(3)}$ where

$$\begin{aligned} P_n^{(1)} &= P\{d(D_n, W_0)(\alpha_n)^{-\frac{3}{2}+\alpha} \int_{\alpha_n}^{1-\alpha_n} |A_n - h'(t)| q_\alpha'(t)\, d|\nu| \geqq b_n/3\}\,, \\ P_n^{(2)} &= P\{|\int_{\alpha_n}^{1-\alpha} (\Delta_n - h') W_0\, d\nu| \geqq b_n/3\} \\ P_n^{(3)} &= P\{d_{q_\alpha}(I_n D_n, W_0) \int_0^1 |h'(t)| q_\alpha(t)\, d|\nu| \geqq b_n/3\}\,, \end{aligned} \tag{3.9}$$

where I_n is the indicator function of the interval $[\alpha_n, 1-\alpha_n]$, and

$$d_{q_\alpha}(x, y) = \sup(|x(t) - y(t)|/q_\alpha(t);\ 0 \leqq t \leqq 1)\,. \tag{3.10}$$

Let B_n be the event that $\sup|F_n^{-1}(t) - t| < 12(\log n/n)^{\frac{1}{2}}$. Then on B_n for all t in $[\alpha_n, 1-\alpha_n]$,

$$|A_n(t) - h'(t)| \leqq K_\lambda(t)|F_n^{-1}(t) - t| \leqq K_\lambda(t) 12(\log n/n)^{\frac{1}{2}} \tag{3.11}$$

so that, by Lemma 2.2,

$$\begin{aligned} P_n^{(1)} &\leqq P\{d(D_n, W_0) \geqq 12(\log n) n^{-\frac{1}{4}}\} + O(n^{-\frac{1}{2}}) \\ &= O(n^{-\frac{1}{2}}) \qquad \text{by Lemma 2.1.} \end{aligned} \tag{3.12}$$

Similarly, by intersecting the event of $P_n^{(2)}$ with the event B_n one obtains

$$\begin{aligned} P_n^{(2)} &\leqq P\{12(\log n/n)^{\frac{1}{2}}(\alpha_n)^{-1+\alpha}|\int_0^1 K_\lambda(t) W_0(t)^{1-\alpha}\, d\nu| \geqq b_n/3\} + O(n^{-\frac{1}{2}}) \\ &\leqq P\{|\int_0^1 K_\lambda(t) W_0(t) t^{1-\alpha}\, d\nu| \geqq (\log n)^{\frac{3}{4}}\} + O(n^{-\frac{1}{2}}) \\ &= O(n^{-\frac{1}{2}}) \end{aligned} \tag{3.13}$$

since by condition (3.4) $\int_0^1 K_\lambda(t) W_0(t) t^{1-\alpha}\, d\nu$ is a well-defined normal random variable with zero mean. Now

$$\begin{aligned} P_n^{(3)} &\leqq P\{d(D_n, W_0) \geqq 12(\log n)^{3/4+\alpha/2} n^{-\alpha/2} q_\alpha(\alpha_n)\} \\ &\quad + P\{|W_0(t)| \geqq C q_\alpha(t) b_n \quad \text{for some} \quad 0 < t \leqq \alpha_n\} \\ &= P\{d(D_n, W_0) \geqq 12(\log n) n^{-\frac{1}{4}}\} + O(n^{-1})\,, \qquad \text{by Lemma 2.3} \\ &= O(n^{-\frac{1}{2}})\,, \qquad \text{by Lemma 2.1.} \end{aligned} \tag{3.14}$$

Hence (3.8) holds. The limiting random variable $\int_0^1 h'(t) W_0(t)\, d\nu(t)$ is clearly normal with mean 0 and variance σ^2 given by (3.7). Now

$$|\Phi(x_1/\sigma) - \Phi(x_2/\sigma)| \leqq (2\pi\sigma^2)^{-\frac{1}{2}}|x_1 - x_2| \tag{3.15}$$

and this together with (3.8) yields

$$\Phi(x/\sigma) - r(n) \leqq G_n(\lambda; x) \leqq \Phi(x/\sigma) + r(n) \tag{3.16}$$

where

$$\begin{aligned} r(n) &= O(n^{-\frac{1}{2}}) + (2\pi\sigma^2)^{-\frac{1}{2}} b_n \\ &= O[(\log n)^{3/4+\alpha/2} n^{-\alpha/2}] . \end{aligned} \tag{3.17}$$

The authors are indebted to one of the referees for pointing out that the "right" exponent of t in the function $q_\alpha'(t)$ is $\frac{3}{2} - \alpha$ rather than the authors' original $1 - \alpha$. Also the idea of the "shifted" function K_λ is due to Shorack in [5].

In some applications the weights are given by $C_{jn} = J(j/n+1)$, $1 \leqq j \leqq n$, $n > 1$, for some Borel measurable function J on $(0, 1)$. Let $G_n(x)$ be the distribution function of the normalized statistic $n^{\frac{1}{2}}(\dot{T}_n - \mu_n)$ where for some $0 < \lambda < 1$,

$$\mu_n = \textstyle\int_{\alpha_n}^{1-\alpha_n} h(t)J(t)\, dt . \tag{3.18}$$

THEOREM 3.2. *Suppose that for some* $0 < \alpha \leqq \frac{1}{2}$ *and* $0 < \lambda < 1$ *the conditions of Theorem* 3.1 *hold with* ν *defined by*

$$\nu((a, b]) = \textstyle\int_a^b J(t)\, dt . \tag{3.19}$$

If in addition there exist nonnegative functions $\bar{h}$ *and* $\bar{J}$, *increasing on* $[\frac{1}{2}, 1]$ *and symmetric about* $\frac{1}{2}$ *such that* $|h(t)| \leqq \bar{h}(t)$ a.s. $|\nu|$ *and*

$$|J(t_1) - J(t_2)| \leqq \bar{J}_\lambda(t)|t_1 - t_2| \quad \textit{for } t_1, t_2 \textit{ in } \quad [\lambda t, 1 - \lambda(1-t)]$$

and

$$\begin{aligned} &\textstyle\int_0^1 \bar{h}_\lambda(t)\bar{J}_\lambda(t)[t(1-t)]^{\frac{1}{2}}\, dt < \infty \\ &\textstyle\int_0^1 \bar{h}_\lambda(t)J(t)[t(1-t)]^{\frac{1}{2}}\, dt < \infty \end{aligned} \tag{3.20}$$

then

$$\max_{-\infty<x<\infty} |G_n(x) - \Phi(x/\sigma)| = O(n^{-\alpha/2}(\log n)^{3/4+\alpha/2}) . \tag{3.21}$$

PROOF.

$$n^{\frac{1}{2}}(T_n - \mu_n) = S_n(\lambda) + R_n \tag{3.22}$$

where

$$R_n = n^{\frac{1}{2}} \textstyle\int_{\alpha_n}^{1-\alpha_n} h[F_n^{-1}(t)][(n+1)/nJ_n(t) - J(t)]\, dt . \tag{3.23}$$

Let b_n be as in the proof of Theorem 2. Then

$$\begin{aligned} &P\{|n^{\frac{1}{2}}(T_n - \mu_n) - \textstyle\int_0^1 h'(t)W_0(t)J(t)\, dt| \geqq 2b_n\} \\ &\quad\leqq P\{|S_n(\lambda) - \textstyle\int_0^1 h'(t)W_0(t)J(t)\, dt| \geqq b_n\} + P\{|R_n| \geqq b_n\} \\ &\quad= O[n^{-\alpha/2}(\log n)^{3/4+\alpha/2}] + P\{|R_n| \geqq b_n\} , \qquad \text{by Theorem 3.1.} \end{aligned} \tag{3.24}$$

It is now a fairly straightforward matter to see that the additional conditions in the hypothesis of the theorem imply

$$P(|R_n| \geqq b_n) = O(n^{-\frac{1}{2}}) . \tag{3.25}$$

The theorem follows from (3.24) and (3.25).

If $h = [t(1-t)]^{-a}$ and $d\nu = J = [t(1-t)]^{-\frac{1}{2}+a+2\alpha}$ then with $\bar{h} = h$ and $\bar{J} = J'$, $K = h''$ the conditions of Theorem (3.2) hold.

4. The "right" rate. In general the rate of convergence will depend on the weight function. However, one may conjecture that at least in some situations the "right" rate is $n^{-\frac{1}{2}}$. Our methods yield a rate $n^{-\frac{1}{4}}$ when the conditions of Theorems 3.1 and 3.2 are satisfied for $\alpha = \frac{1}{2}$. This will be the case, for example, when the extreme order statistics are given zero weight; that is, for some $0 < \delta < \frac{1}{2}$, $\nu((\delta, 1-\delta)') = 0$. Moreover, $n^{-\frac{1}{4}}$ is the largest possible rate that can be obtained by these methods. This is so because the rate is bounded by the sequence of constants ε_n for which $P(d(D_n, W_0) \geqq \varepsilon_n) \to 0$ for any constant $\varepsilon > 0$

$$\liminf_{n\to\infty} P(d(D_n, W_0) \geqq \varepsilon n^{-\frac{1}{4}}) > 0 . \tag{4.1}$$

To show that (4.1) holds it is sufficient to establish the corresponding result for $\bar{S}_n$ and W. Using the notation of Rosenkrantz [3] let

$$Z_{nk} = \sum_{j=1}^{k} (T_j - 1)/n$$

and B_n denote the event:

$$B_n = \left\{\max_{1\leqq k\leqq n}\left|W\left(\frac{k}{n} + Z_{nk}\right) - W\left(\frac{k}{n}\right)\right| \geqq \varepsilon n^{-\frac{1}{4}}\right\}$$

and note $B_n = B_{n1} \cup B_{n2}$ where

$$B_{n1} = B_n \cap \{\max_{1\leqq k\leqq n} |Z_{nk}| \leqq \delta_n\}$$

and

$$B_{n2} = B_n \cap \{\max_{1\leqq k\leqq n} |Z_{nk}| > \delta_n\}$$

where $\delta_n > 0$ is a sequence whose dependence on n will be specified later. Now

$$\begin{aligned} P(d(\bar{S}_n, W) \geqq \varepsilon n^{-\frac{1}{4}}) &= P(B_n) \\ &\geqq P(B_{n2}) \\ &\geqq P(\bar{S}_n(1) - W(1) \geqq \varepsilon n^{-\frac{1}{4}}, Z_{nn} < -\delta_n) \\ &= \int_{-\infty}^{-\delta_n} P(W(1+t) - W(1) \geqq \varepsilon n^{-\frac{1}{4}} / Z_{nn} = t) F_n(dt) , \end{aligned} \tag{4.2}$$

where $F_n(t) = P(Z_{nn} \leqq t)$. Use the strong Markov property for Brownian motion to write the right-hand side of (4.2) as

$$\begin{aligned} \int_{-\infty}^{-\delta_n} & P(W(1+t) - W(1) \geqq \varepsilon n^{-\frac{1}{4}}) F_n(dt) \\ &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{-\delta_n} \int_{\varepsilon n^{-\frac{1}{4}}(-t)^{-\frac{1}{2}}}^{\infty} e^{-s^2/2}\, ds F_n(dt) \\ &\geqq (2\pi)^{-\frac{1}{2}} \int_0^C e^{-s^2/2} P\left(Z_{nn} \leqq -\frac{\varepsilon^2 n^{-\frac{1}{2}}}{s^2}\right) ds \end{aligned} \tag{4.3}$$

for δ_n chosen so that $\varepsilon^2 n^{-\frac{1}{2}}/\delta_n \to \infty$ and n large enough that $\varepsilon^2 n^{-\frac{1}{2}}/\delta_n > C$, where C is an arbitrary constant. By the central limit theorem,

$$P\left(Z_{nn} \leqq -\frac{\varepsilon^2 n^{-\frac{1}{2}}}{s^2}\right) \to \Phi(-\varepsilon^2/\gamma s^2)\,, \tag{4.4}$$

where γ^2 is the variance of τ_1. By the dominated convergence theorem the right-hand side of (4.3) tends as $n \to \infty$ to

$$(2\pi)^{-\frac{1}{2}} \int_0^C e^{-s^2/2}\Phi(-\varepsilon^2/\gamma s^2)\, ds > 0\,. \tag{4.4}$$

It follows from (4.2), (4.3) and (4.4) that

$$\liminf_{n\to\infty} P(d(\bar{S}_n, W) \geqq \varepsilon n^{-\frac{1}{4}}) \geqq (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-s^2/2}\Phi(-\varepsilon^2/\gamma S^2)\, ds \tag{4.5}$$

where the right side of (4.5) is strictly greater than zero.

REFERENCES

[1] BREIMAN, L. (1968). *Probability*. Addison Wesley, Reading.

[2] PYKE, R. (1967). Applications of almost surely convergent constructions of weakly convergent processes. *Boeing Sci. Res. Lab. Doc.* D1-82-0747.

[3] ROSENKRANTZ, W. A. (1967). On rates of convergence for the invariance principle. *Trans. Amer. Math. Soc.* **129** 542-552.

[4] ROSENKRANTZ, W. A. (1969). A rate of convergence for the von Mises Statistic. *Trans. Amer. Math. Soc.* **139** 329-337.

[5] SHORACK, G. (1969). Asymptotic normality of linear combinations of order statistics. *Ann. Math. Statist.* **40** 2041-2050.