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THE STRUCTURE OF THE OPTIMAL STOPPING RULE IN THE S_n/n PROBLEM[1]

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The proof of the existence of the optimal stopping rule for S_n/n for the case where the i.i.d. random variables X_i have a moment of order greater than one has been obtained by B. Davis. In the present paper the asymptotic growth of the boundary of the optimal stopping region is studied. The method used generalizes one of Shepp (1969), and involves comparison with the corresponding problem for an infinitely divisible process, obtained as a limit of processes $(S_{[nt]}/a_n, t \geqq 0)$ for properly chosen norming constants a_n. When the X_i are in the domain of attraction of a random variable which is stable with exponent greater than one, an explicit asymptotic expression for the curve defining the boundary is obtained.

1. Introduction. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with mean 0 on a probability space $(\Omega, \mathscr{F}, P)$, and let $S_n = \sum_{i=1}^n X_i$. Suppose that we observe the process $(S_n, \mathscr{F}(n), n \geqq 1)$, where $\mathscr{F}(n)$ is the sigma-field generated by $X_1, \cdots, X_n$, and are allowed to stop at any stage n we please, basing our decision only on the past and present of the process at time n. If we stop at stage n, we receive the "payoff" S_n/n. The corresponding "optimal stopping problem" is to find if possible a stopping procedure which maximizes our expected payoff.

More formally, let $\mathscr{M}$ be the collection of finite valued stopping times τ relative to the family $(\mathscr{F}(n), n \geqq 1)$ for which $E(S_\tau/\tau)$ is defined (possibly infinite). The problem is to find if possible $\sigma \in \mathscr{M}$ such that

$$E(S_\sigma/\sigma) = \sup[E(S_\tau/\tau) : \tau \in \mathscr{M}]\,. \tag{1.1}$$

Burgess Davis [3] has shown that if $E(X_1 \log^+ X_1) = \infty$ (where $\log^+ a = \log a$ if $a \geqq 1$, and 0 if $a < 1$), then there is a $\sigma \in \mathscr{M}$ for which $E(S_\sigma/\sigma) = \infty$. This σ is clearly optimal in the sense of (1.1). On the other hand, it is well known that if $E(X_1 \log^+ X_1) < \infty$, then $E(\sup_{n \geqq 1}(S_n^+/n)) < \infty$. (This follows from inequality (3.7) on page 517 of [5] and the fact that the process $(\cdots, S_3^+/3, S_2^+/2, S_1^+)$ is a submartingale.) Therefore, if $E(X_1 \log^+ X_1) < \infty$, $\mathscr{M}$ is the class of all finite stopping times relative to $(\mathscr{F}(n), n \geqq 1)$; and general optimal stopping theory tells us a good deal more.

The process $X = ((S_n, n), n \geqq 1)$ may be regarded as a Markov process with state space $R \times [0, \infty)$, stationary transition probabilities, and initial distribution that of X_1 on the line $\{(x, 1) : x \in R\}$. (Here R denotes the real numbers.)

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[1] Several of the results in this paper were obtained independently in the doctoral dissertations of the two authors.

Suppose that $E(\sup_n (S_n^+/n)) < \infty$, and for each $(x, s) \in R \times (0, \infty)$ let

$$h(x, s) = (x/s)^+ \vee \sup [E((x + S_\tau)/(s + \tau))^+ : \tau \in \mathscr{M}] . \tag{1.2}$$

Let $D = \{(x, s) \in R \times (0, \infty) : h(x, s) = x^+/s\}$. It can be proved as in Theorem 8 of [2] that $(h(x + S_n, s + n), \mathscr{F}(n), n \geqq 1)$ is the minimal supermartingale above $((x + S_n)^+/(s + n), \mathscr{F}(n), n \geqq 1)$ for each $(x, s) \in R \times [0, \infty)$. If $\tau \in \mathscr{M}$ and $\bar{\tau} = \inf [n \geqq \tau : S_n > -x]$, then $\bar{\tau} \in \mathscr{M}$ ([7], page 380) and

$$E\left(\frac{x + S_\tau}{s + \tau}\right) \leqq E\left(\frac{x + S_{\bar{\tau}}}{s + \bar{\tau}}\right) = E\left(\left(\frac{x + S_{\bar{\tau}}}{s + \bar{\tau}}\right)^+\right). \tag{1.3}$$

This being true, $D \subset (0, \infty) \times (0, \infty)$, and it follows by the Corollary to Theorem 6 of [2] that if for $(x, s) \in R \times [0, \infty)$ the stopping time $\tau(x, s) = \inf [n \geqq 1 : (x + S_n, s + n) \in D]$ is finite, it maximizes $E((x + S_\tau)/(s + \tau))$ among $\tau \in \mathscr{M}$.

Dvoretzky's proof in ([6] Section 3) of the following theorem, describing the set D, is valid; it does not use the general assumption of his paper that $EX_1^2 < \infty$.

THEOREM 1.1. *Let $EX_1 \log^+ X_1$ be finite. Then there is a strictly increasing positive function $f(s)$ on $[0, \infty)$ such that for $(x, s) \in R \times [0, \infty)$*

$$\begin{aligned} \frac{x}{s} &< \sup\left[E\left(\frac{x + S_\tau}{s + \tau}\right) : \tau \in \mathscr{M}\right] \quad \text{if } x < f(s) , \\ \frac{x}{s} &= \sup\left[E\left(\frac{x + S_\tau}{s + \tau}\right) : \tau \in \mathscr{M}\right] \quad \text{if } x = f(s) , \\ \frac{x}{s} &> \sup\left[E\left(\frac{x + S_\tau}{s + \tau}\right) : \tau \in \mathscr{M}\right] \quad \text{if } x > f(s) . \end{aligned} \tag{1.4}$$

Dvoretzky [6] and Teicher and Wolfowitz [14] showed that if $EX_1^2 < \infty$, then $\tau(0, 0)$ is finite, and therefore optimal. Burgess Davis [4] has succeeded in proving the conjecture of Dvoretzky that the same result holds in the case where $E|X_1|^q < \infty$ for some $q > 1$. In addition, it has been shown in [16] and in [4] that if $E|X_1|^q < \infty$ for some $q > 1$ then

$$f(n) \leqq K_q ||S_n||_q , \qquad n \geqq 1 , \tag{1.5}$$

for some $K_q > 0$ (not depending on the distribution of X_1). Here $||S_n||_q$ denotes $[E(|S_n|^q)]^{1/q}$.

When X_1 has finite variance, then $||S_n||_2$ is asymptotically equivalent to $n^{\frac{1}{2}}||X_1||_2$, and Shepp [12] and Walker [17] have shown that if $EX_1^2 = \sigma^2 < \infty$, then actually $f(n) \sim c\sigma n^{\frac{1}{2}}$ for a positive finite constant c, which they evaluated. Our purpose is to study the asymptotic behavior of $f(n)$ in the more general case where $E|X_1|^q < \infty$ for some $q > 1$. It turns out that the limiting behavior of $f(n)$ is very closely tied in with the limiting behavior of the distributions of the partial sums S_n. In the case where the random variable X_1 belongs to the domain of attraction of a stable random variable with exponent α greater than one we obtain the asymptotic expression

$$f(n) \sim c_p ||S_n||_p \tag{1.6}$$

as $n \to \infty$, where p is any number greater than one and less than α. The constant c_p is the same for all random variables X_1 in the domain of attraction of the same stable distribution. (It will be apparent from the proof that $||S_n||_p \sim Ka_n$ for some $K > 0$, where a_n are the norming constants for the attraction. See [7] page 302, for the definition of the a_n.)

2. Some inequalities. We shall make use of two important inequalities, as set forth in the following lemmas. Again, for any random variable Y and $b > 0$ the notation $||Y||_b$ means $[E|Y|^b]^{1/b}$.

LEMMA 2.1. *Suppose* $1 \leqq b \leqq 2$. *Then there is a positive finite constant* N_b *such that if* $X_1, X_2, \cdots$ *are any independent and identically distributed random variables with* $E|X_1|^b < \infty$, *then*

$$||S_n||_b \leqq N_b n^{1/b} ||X_1||_b , \qquad n \geqq 1 . \tag{2.1}$$

PROOF. This is an easy consequence of Theorem 5 of [10].

LEMMA 2.2. *If* $b > a > 1$, *there is a finite positive constant* L_{ab} *such that if* $X_1, X_2, \cdots$ *independent and identically distributed with* $EX_1 = 0$ *and* $E|X_1|^b < \infty$ *then*

$$P(S_n > L_{ab} ||S_n||_a) > L_{ab} \tag{2.2}$$

for infinitely many n.

PROOF. This is Corollary 2 of [4].

COROLLARY. *If K is any constant, and* $b > a > 1$, *there is a positive number* ρ_{ab} *depending on K such that under the conditions of Lemma* 2.2

$$P(S_n > K||S_n||_a) > \rho_{ab} \tag{2.3}$$

for infinitely many n.

PROOF. Using (2.1) we obtain

$$P(S_{mn} > K||S_{mn}||_a) \geqq P(S_{mn} > KN_a m^{1/a} ||S_n||_a) \tag{2.4}$$

for all m and n. Hence

$$\begin{aligned} P(S_{mn} > K||S_{mn}||_a) &\geqq P(S_{mn} > (KL_{ab}^{-1} N_a m^{-1+1/a}) m L_{ab} ||S_n||_a) \\ &> L_{ab}^m > 0 \end{aligned} \tag{2.5}$$

for all n satisfying (2.2) and a fixed m sufficiently large.

COROLLARY. *Under the conditions of Lemma* 2.2

$$P(\limsup_{n\to\infty} S_n/||S_n||_a = \infty) = 1 . \tag{2.6}$$

3. Limiting distributions for $S_n/||S_n||_p$. In this section, let us assume that the X_i of Section 1 are nondegenerate, and that $E|X_1|^q < \infty$ for some $q > p > 1$. For each n let $F_n(x)$, $-\infty < x < \infty$, be the distribution function of the random variable $S_n/||S_n||_p$. It is easy to show using a diagonal argument that given any subsequence of the functions F_n we must have some further subsequence tending to a right-continuous limit function F at all continuity points of F.

Now clearly

$$P(|S_n| > x\|S_n\|_p) < x^{-p} \tag{3.1}$$

for all x, and hence any such limit distribution F cannot be improper. Moreover, if $F_n \to F$ as $n \to \infty$ through Q, a subsequence of the natural numbers, then (3.1) and the dominated convergence theorem imply that

$$\lim_Q E(S_n/\|S_n\|_p) = \int_{-\infty}^{\infty} x\,dF(x)\,,$$

where $\lim_Q$ denotes the limit as $n \to \infty$ through Q. Thus if $F = \lim_Q F_n$ is degenerate, it must assign mass 1 to the point 0. But therefore if Q is a subset of the set of those n for which (2.3) holds, with q and p instead of b and a, the limiting distribution F must be nondegenerate. The following theorem is now evident.

THEOREM 3.1. *If $X_1, X_2, \cdots$ are independent and identically distributed random variables with $EX_1 = 0$ and $0 < E|X_1|^q < \infty$ for some $q > p > 1$, then some subsequence of the random variables $S_n/\|S_n\|_p$ converges in distribution to a nondegenerate random variable.*

The limiting distribution function F is infinitely divisible, and it follows from Theorem 3.1 that any random variable X_1 satisfying its hypotheses is in the "domain of partial attraction" of some infinitely divisible distribution. (See [7] page 555.)

We cannot say in general that the sequence $\{F_n\}$ is stochastically compact, or in other words that for any subsequence of the F_n there is a further subsequence which converges to a nondegenerate limit. In fact, as Feller has shown in [8], a necessary condition for stochastic compactness of $\{F_n\}$ is that the truncated variance

$$U(x) = \int_{(-x)-}^{x} y^2\,dP(X_1 \leqq y) \tag{3.2}$$

be of "dominated variation," i.e., that it satisfy

$$\frac{U(tx)}{U(t)} < Cx^{2-\nu}\,, \qquad x > 1\,, \quad t > T \tag{3.3}$$

for some constants $\nu > 0$, C and $T > 0$. It is not difficult to show (cf. the Corollary on page 274 of [7]) that this implies that for any $\delta > 0$ the function $U(x)$ has the form

$$U(x) = d(x) \exp \int_1^x \frac{\varepsilon(w)}{w}\,dw\,, \qquad x > T \tag{3.4}$$

where $d(x)$ is Borel measurable, positive and bounded away from 0 and ∞, and $\varepsilon(x)$ is Borel measurable, positive, and bounded above by $2 - \nu + \delta$ for $x > T$. Conversely, if U is of dominated variation with $\nu > 1$, then $E|X_1|^p < \infty$ for $1 < p < \nu$. Feller proves in [8] that there is a sequence $\{a_n\}$ of positive constants for which the sequence $\{S_n/a_n\}$ is stochastically compact, and it is shown in [16] that for this sequence there is a constant C_0 independent of n and x such that

$$P(|S_n| \geqq xa_n) < C_0 x^{-\nu} . \tag{3.5}$$

It follows that the sequence $\{||S_n||_p/a_n\}$ is bounded away from 0 and ∞, and therefore that the sequence $\{S_n/||S_n||_p\}$ is also stochastically compact.

If the sequence $\{F_n\}$ itself has a limiting distribution F, the random variable X_1 is said to belong to the domain of attraction of F. Then F is a stable distribution function with exponent α, where $2 \geqq \alpha \geqq q$ if $E|X_1|^q < \infty$, and the truncated variance $U(x)$ has a representation of the form of (3.4), where this time $d(x)$ approaches a positive finite limit as $x \to \infty$, and $\varepsilon(x) \to 2 - \alpha$ as $x \to \infty$. (See [7] page 302 ff.) Conversely, if U has this form for some $\alpha > 1$, the distribution of $S_n/||S_n||_p$ approaches a stable random variable with exponent α for any p, $1 < p < \alpha$.

4. Lipschitz continuity of the boundary of the optimal stopping set. Consider the nonnegative, increasing function $f(s)$ on $(0, \infty)$ defined by (1.4). We have placed an upper bound on its growth in (1.5), and now we shall determine its local behavior, assuming that $E|X_1|^q < \infty$ for some $q > p > 1$.

LEMMA 4.1. *For any* $s \geqq 1$ *and* $h > 0$,

$$|f(s+h) - f(s)| \leqq 4hK_p N_p ||X_1||_p s^{-1+1/p} . \tag{4.1}$$

where K_p *is defined in* (1.5) *and* N_p *in Lemma* 2.1.

PROOF. For each $\tau \in \mathscr{M}$ which has the property that $P(S_\tau > 0) = 1$, define $f^\tau(s)$ to be the solution for x of

$$\frac{x}{s} = E\left(\frac{x + S_\tau}{s + \tau}\right). \tag{4.2}$$

That $f^\tau(s)$ for $s > 0$ is well-defined, finite valued and non-decreasing follows from Lemmas 5 and 8 of [6]. It is not difficult to prove that the derivative $df^\tau(s)/ds$ exists, and in fact the computation

$$\frac{1}{s}\frac{df^\tau(s)}{ds} - \frac{f^\tau(s)}{s^2} = E\left(\frac{1}{s+\tau}\right)\frac{df^\tau(s)}{ds} - E\left(\frac{f^\tau(s) + S_\tau}{(s+\tau)^2}\right) \tag{4.3}$$

shows that

$$\begin{aligned} \frac{df^\tau(s)}{ds} &\leqq f^\tau(s) E\left[\frac{1}{s^2} - \frac{1}{(s+\tau)^2}\right] \Big/ E\left[\frac{1}{s} - \frac{1}{s+\tau}\right] \\ &\leqq \frac{2f^\tau(s)}{s} . \end{aligned} \tag{4.4}$$

The result follows easily from (1.5), (2.1) and (4.4) once we note that $f(s) = \sup[f^\tau(s) : \tau \in \mathscr{M}, P(S_\tau > 0) = 1]$.

5. The analogous continuous parameter optimal stopping problem. Suppose that Y is an infinitely divisible random variable in one dimension with a finite qth moment for some $q > 1$, and mean 0. Let $Y^* = (y(t), t \geqq 0)$ with $y(0) \equiv 0$ be a right-continuous realization with left limits of the homogeneous process with independent increments on the real line R, induced by the random variable Y. (See [1] page 18.) Suppose that the probability space underlying Y^* is complete, and let $(\mathscr{F}^*(t), t \geqq 0)$ be the increasing right-continuous family of sigma-fields, with $\mathscr{F}^*(0)$ containing all sets of zero measure, generated by Y^*. Let $\mathscr{N}$ be the class of finite valued stopping times with respect to $(\mathscr{F}^*(t), t \geqq 0)$.

Consider the problem of maximizing for each (x, s) in $R \times (0, \infty)$ the quantity $E[(x + y(\tau))/(s + \tau)]$ for $\tau \in \mathscr{N}$. Any τ_0 which maximizes this quantity will be called "optimal" for the starting point (x, s). If we define

$$h^*(x, s) = \sup\left[E\left(\frac{x + y(\tau)}{s + \tau}\right) : \tau \in \mathscr{N}\right] \tag{5.1}$$

it is easy to show that $h^*(x, s)$ is a continuous, nonnegative function of (x, s) in $R \times (0, \infty)$. Furthermore, if

$$D^* = \{(x, s) \in R \times (0, \infty) : h^*(x, s) = x/s\}, \tag{5.2}$$

then D^* is of the form $\{(x, s) \in R \times (0, \infty) : x \geqq f^*(s)\}$, where $f^*(s)$ is a positive and non-decreasing function of $s \in (0, \infty)$. It follows from general optimal stopping theory for Markov processes (see [15], Theorem 10.1) that if $h^*(x, s)$ is finite and if an optimal stopping time for (x, s) exists, then there is a minimal one, given by

$$\tau_{00} = \inf[t \geqq 0 : (x + y(t), t + s) \in D^*]. \tag{5.3}$$

Moreover ([15], Theorem 7.3 and Section 10), if we can show that $\sup[(x + y(t))/(s + t) : t \geqq 0]$ is integrable and that the stopping time τ_{00} is a.s. finite, then τ_{00} will be optimal for (x, s). Now

$$\begin{aligned} E\left[\sup_{t \leqq 0}\left|\frac{y(t)}{t + s}\right|^q\right] &\leqq E\left[\sup_{0 \leqq t \leqq s}\left|\frac{y(t)}{t + s}\right|^q\right] + E\left[\sup_{t > s}\left|\frac{y(t)}{t + s}\right|^q\right] \\ &\leqq E\left[\sup_{0 \leqq t \leqq s}\left|\frac{y(t)}{t + s}\right|^q\right] + E\left[\sup_{t > s}\left|\frac{y(t)}{t}\right|^q\right]. \end{aligned} \tag{5.4}$$

But the processes $(|y(t)|^q, t \geqq 0)$ and $(|y(-t)/(-t)|^q, 0 < t)$ are both submartingales, and hence for some constant C

$$E\left[\sup_{t \geqq 0}\left|\frac{y(t)}{t + s}\right|^q\right] \leqq \frac{CE|y(s)|^q}{s^q}. \tag{5.5}$$

Applying Jensen's inequality we conclude that

$$E\left[\sup_{t \geqq 0}\left|\frac{y(t)}{t + s}\right|\right] \leqq \frac{C'\|y(s)\|_q}{s} \tag{5.6}$$

for some constant C'. It follows from (5.6) that the process

$$\left(\frac{x + y(t)}{s + t}, \mathscr{F}^*(t), t \geqq 0\right) \tag{5.7}$$

is bounded both above and below by integrable random variables.

Because the process (5.7) is bounded below in this way, we can approximate the continuous parameter problem with discrete parameter problems in the following way ([15], Section 10). Let $\mathscr{N}_N$ be the collection of finite valued stopping times with respect to $(\mathscr{F}^*(t), t \geqq 0)$ taking values of the form $k2^{-N}$ where k is an integer with probability one. Let

$$h_N^*(x, s) = \sup\left[E\frac{x + y(\tau)}{s + \tau} : \tau \in \mathscr{N}_N\right] \tag{5.8}$$

and

$$D_N^* = \{(x, s) \in R \times (0, \infty) : h_N^*(x, s) = x/s\}, \tag{5.9}$$

and define $f_N(s)$ by

$$D_N^* = \{(x, s) \in R \times (0, \infty) : x \geqq f_N(s)\}. \tag{5.10}$$

Then $\lim_{N\to\infty} h_N^*(x, s) = h^*(x, s)$ for all (x, s) and $f_N(s) \nearrow f^*(s)$ as $N \to \infty$.

Now suppose $1 < p < q$. It follows easily from (1.5) that $f_N(s) < K_p\|y(s + 2^{-N})\|_p$ for each N and hence that

$$f^*(s) < K_p\|y(s)\|_p. \tag{5.11}$$

From Lemma 2.2 we see that for any (x, s) τ_{00} is a.s. finite.

We have now shown that τ_{00} is an optimal stopping time, and that it is minimal. We may ask next whether the optimal stopping time is unique. In order to apply the uniqueness argument used by Shepp ([12], page 1005) for the case where Y^* is the Wiener process, it is enough to show that $f^*(s)$ is Lipschitz continuous locally and that

$$P(\limsup_{t\to 0} y(t)/t > K) = 1 \tag{5.12}$$

for any $K > 0$.

LEMMA 5.1. *The function $f^*(s)$ is Lipschitz continuous for $s \in [a, \infty)$ for any $a > 0$.*

PROOF. We argue as in Lemma 4.1 to prove that

$$|f_N(s + h) - f_N(s)| \leqq 2hK_p\|y(s + 1)\|_p s^{-1} \tag{5.13}$$

for all N, $s > 0$ and $h > 0$, from which follows the result.

Equation (5.12) does not hold in general. B. A. Rogozin in [11] has given a necessary and sufficient condition for (5.12) to hold in terms of the Lévy measure of the random variable Y.

6. The asymptotic behavior of $f(n)$. Suppose that the hypotheses of Theorem 3.1 are satisfied, and that some subsequence of the $S_n/\|S_n\|_p$ converges in distribution to an infinitely divisible random variable Y. Let $Y^* = (y(t), t \geqq 0)$ be the induced homogeneous process with independent increments as defined in the beginning of Section 5. Then it is a well-known fact (see [9] page 479) that for any $s > 0$ the random variables $S_{[ns]}/\|S_n\|_p$ for the same values of n converge in distribution to $y(s)$.

In what follows let $f^*(s)$ and the stopping times τ_{00} be defined as in Section 5, and $f(s)$ be defined as in (1.4). We shall prove two theorems, generalizing a result of Shepp [12] and Walker [17], who both considered the case where the X_i were in the domain of attraction of a normal random variable.

THEOREM 6.1. *Let Q be a subset of the natural numbers such that the random variables $S_n/||S_n||_p$ converge in distribution to Y as $n \to \infty$ through Q. Then*

$$\liminf_Q f(n)/||S_n||_p \geqq f^*(1) . \tag{6.1}$$

PROOF. To prove this, we suppose that $\liminf_Q f(n)/||S_n||_p = \gamma' < f^*(1)$, and choose γ such that $\gamma' < \gamma < f^*(1)$. Consider τ_{00} defined for the point $(\gamma, 1)$, i.e., let

$$\tau_{00} = \inf[t \geqq 0 : \gamma + y(t) \geqq f^*(1 + t)] ; \tag{6.2}$$

and let

$$\tau(n) = \inf[m \geqq 1 : \gamma||S_n||_p + S_m \geqq ||S_n||_p f^*(1 + m/n)] \tag{6.3}$$

for each $n \in Q$. Then because $\gamma < f^*(1)$, τ_{00} being minimal as an optimal stopping time implies that

$$E\left(\frac{\gamma + y(\tau_{00})}{1 + \tau_{00}}\right) > \frac{\gamma}{1} . \tag{6.4}$$

But

$$E\left(\frac{\gamma||S_n||_p + S_{\tau(n)}}{n + \tau(n)}\right) \leqq \frac{\gamma||S_n||_p}{n} \tag{6.5}$$

for infinitely many $n \in Q$ by our original supposition. Thus if we could show that

$$\lim_Q \frac{n}{||S_n||_p} E\left(\frac{\gamma||S_n||_p + S_{\tau(n)}}{n + \tau(n)}\right) = E\left(\frac{\gamma + y(\tau_{00})}{1 + \tau_{00}}\right) \tag{6.6}$$

we would have a contradiction. The proof of (6.6) goes through in the same way as Shepp's proof of his Equation (8.1), using the invariance principle of Skorokhod [13] where Shepp uses the invariance principle of Donsker.

THEOREM 6.2. *Suppose that the conditions of Theorem* 6.1 *hold and in addition that for every finite B*

$$\lim_{\varepsilon \to 0} P(y(\varepsilon) \leqq B\varepsilon) = 0 . \tag{6.7}$$

[This condition is stronger than (5.12).] *Then*

$$\limsup_Q f(n)/||S_n||_p \leqq f^*(1) . \tag{6.8}$$

As a first step in the proof, we let $f_n(s) = f(ns)/||S_n||_p$ for $n \in Q$, and establish the following lemma.

LEMMA 6.1. *The functions $f_n(s)$ for $n \in Q$ are equicontinuous.*

PROOF. Using (4.4), we see easily that

$$|f_n(s + h) - f_n(s)| \leqq 4hK_p s^{-1}||S_{[ns]}||_p/||S_n||_p .$$

Using Lemma 2.1 we obtain

$$|f_n(s+h) - f_n(s)| \leqq 4hK_pN_p s^{-1+1/p},$$

and hence the result.

Continuing with the Proof of Theorem 6.2, we suppose that as $n \to \infty$ through some subset Q' of Q, $\lim f(n)/||S_n||_p = \gamma_1 > f^*(1)$. By Lemma 6.1 and the Arzelà–Ascoli theorem there is a subsequence Q'' of Q' and an $f_0(s)$ continuous and increasing such that $\lim_{Q''} f_n(s) = f_0(s)$ for all $s > 0$. Suppose $f^*(1) < \gamma < \gamma_1$. For each $n \in Q''$, let

$$\tau^*(n) = \inf[m \geqq 1 : ||S_n||_p\gamma + S_m \geqq f(n+m)],$$

and

$$\eta(n) = (||S_n||_p\gamma + S_{\tau^*(n)})/(n + \tau^*(n)).$$

Let

$$\tau^*(0) = \inf[t \geqq 0 : \gamma + y(t) \geqq f_0(1+t)],$$

and

$$\eta(0) = (\gamma + y(\tau^*(0)))/(1 + \tau^*(0)).$$

Then $||S_n||_p\gamma/n \leqq E\eta(n)$ for all but a finite number of $n \in Q''$, and because the optimal stopping times for the continuous parameter problem are unique, $E(\eta(0)) < \gamma$. If we can show that

$$\lim_{Q''} (n/||S_n||_p)E(\eta(n)) = E(\eta(0)), \tag{6.9}$$

we shall have a contradiction.

Let us select an ε, $0 < \varepsilon < \gamma_1 - \gamma$, and assume that Q'' has been chosen so that $f_n(1) > \gamma$ for any $n \in Q''$. Choose N_0 so large that for all s, $1 \leqq s \leqq T$, where T is some positive number, we have $|f_n(s) - f_0(s)| < \varepsilon$ for all $n \geqq N_0$, $n \in Q''$. Define for $n \in Q''$

$$\begin{aligned} \tau(n, \varepsilon) &= \inf[m \geqq 1 : \gamma||S_n||_p + S_m \geqq ||S_n||_p[f_0(1 + m/n) + \varepsilon]], \\ \tau(n, -\varepsilon) &= \inf[m \geqq 1 : \gamma||S_n||_p + S_m \geqq ||S_n||_p[f_0(1 + m/n) - \varepsilon]], \\ \eta(n, \varepsilon) &= (||S_n||_p\gamma + S_{\tau(n,\varepsilon)})/(n + \tau(n, \varepsilon)), \\ \eta(n, -\varepsilon) &= (||S_n||_p\gamma + S_{\tau(n,-\varepsilon)})/(n + \tau(n, -\varepsilon)). \end{aligned} \tag{6.10}$$

Define also

$$\begin{aligned} \tau(0, \varepsilon) &= \inf[t \geqq 0 : \gamma + y(t) \geqq f_0(1+t) + \varepsilon], \\ \tau(0, -\varepsilon) &= \inf[t \geqq 0 : \gamma + y(t) \geqq f_0(1+t) - \varepsilon], \\ \eta(0, \varepsilon) &= (\gamma + y(\tau(0, \varepsilon)))/(1 + \tau(0, \varepsilon)), \\ \eta(0, -\varepsilon) &= (\gamma + y(\tau(0, -\varepsilon)))/(1 + \tau(0, -\varepsilon)). \end{aligned} \tag{6.11}$$

LEMMA 6.2. *For any y which is a continuity point of the right-hand side,*

$$\lim_{Q''} P(\tau^*(n)/n \leqq y) = P(\tau^*(0) \leqq y). \tag{6.12}$$

PROOF. For $n \geqq N_0$, $n \in Q''$,

$$P(\tau(n, \varepsilon) \wedge nT \leqq ny) \leqq P(\tau^*(n) \wedge nT \leqq ny) \leqq P(\tau(n, -\varepsilon) \wedge nT \leqq ny). \tag{6.13}$$

Applying Skorokhod's invariance principle and letting $T \to \infty$, we obtain

$$(6.14) \qquad P(\tau(0, \varepsilon) \leqq y) \leqq \liminf_{Q''} P(\tau^*(n) \leqq ny) \leqq \limsup_{Q''} P(\tau^*(n) \leqq ny) \leqq P(\tau(0, -\varepsilon) \leqq y)$$

for every y which is a continuity point of both extreme terms.

If we take a sequence ε_m decreasing to 0, we find, using the Lipschitz continuity of f_0, the property (5.12), and the quasi-left-continuity of Y^* (see [1] page 45), that $\lim_{m\to\infty} \tau(0, \varepsilon_m) - \tau^*(0) = 0$ a.s. and $\lim_{m\to\infty} \tau^*(0) - \tau(0, -\varepsilon_m) = 0$ a.s. Thus (6.12) holds at any continuity point of its right-hand side.

LEMMA 6.3. *If $\varepsilon > 0$, then for all y outside some countable set $S(\varepsilon)$,*

$$(6.15) \qquad \lim_{Q''} P(\eta(n, \varepsilon) > ||S_n||_p y/n) = P(\eta(0, \varepsilon) > y)$$

and

$$(6.16) \qquad \lim_{Q''} P(\eta(n, -\varepsilon) > ||S_n||_p y/n) = P(\eta(0, -\varepsilon) > y) .$$

PROOF. The proof is analogous to the proof of Shepp's ([12] Equation (8.1)), with the use of Skorokhod's invariance principle where Shepp uses Donsker's.

To complete the proof of Theorem 6.2, let $K_0 = 4K_pN_p$, with K_p from (1.5) and N_p from (2.1). Let $A_0 = \inf[f_n(1) : n \in Q'', n \geqq N_0] > 0$. Then for $n \geqq N_0$, $n \in Q''$

$$(6.17) \qquad P(n\eta(n, -\varepsilon)/||S_n||_p > y + \varepsilon) - P(n\eta(n)/||S_n||_p > y) \leqq P(\tau(n, -\varepsilon) > nT) + P(\tau^*(n) > nT) + p(n)$$

where $p(n) = P(\sup_{0 \leqq r \leqq \varepsilon A_1} S_{[nr]}/||S_n||_p \leqq 2\varepsilon + A_2\varepsilon)$, $A_1 = A_0/2y^2$ and $A_2 = K_0A_1$; and

$$(6.18) \qquad P(n\eta(n)/||S_n||_p > y) - P(n\eta(n, \varepsilon)/||S_n||_p > y - \varepsilon) \leqq P(\tau(n) > nT) + P(\tau(n, \varepsilon) > nT) + p(n) .$$

From (6.18) we obtain, again using Skorokhod's invariance principle,

$$(6.19) \qquad \limsup_{Q''} P(n\eta(n)/||S_n||_p > y) - P(\eta(0, \varepsilon) > y - \varepsilon) \leqq P(\tau^*(0) > T) + P(\tau(0, \varepsilon) > T) + p(0) ,$$

where $p(0) = P(\sup_{0 \leqq t \leqq \varepsilon A_1} y(t) \leqq 2\varepsilon + A_2\varepsilon)$, if T is outside some countable set S_0 and $y - \varepsilon \notin S(\varepsilon)$. Letting $T \to \infty$ through points of $R - S_0$ gives

$$(6.20) \qquad \limsup_{Q''} P(n\eta(n)/||S_n||_p > y) \leqq P(\eta(0, \varepsilon) > y - \varepsilon) + p(0) .$$

Now it is obvious if (6.7) is satisfied that $p(0)$ approaches 0 as $\varepsilon \to 0$. Therefore, if ε_m is a sequence of points decreasing to zero,

$$(6.21) \qquad \limsup_{Q''} P(n\eta(n)/||S_n||_p > y) \leqq \liminf_{m\to\infty} P(\eta(0, \varepsilon_m) > y - \varepsilon_m)$$

for any y such that $y - \varepsilon_m \notin S(\varepsilon_m)$ for all m. If in addition $y + \varepsilon_m \notin S(\varepsilon_m)$ for all m, we also have

$$(6.22) \qquad \liminf_{Q''} P(n\eta(n)/||S_n||_p > y) \geqq \limsup_{m\to\infty} P(\eta(0, -\varepsilon_m) > y + \varepsilon_m) .$$

For any m', the right-hand side of (6.21) is less than or equal to

$$\liminf_{m\to\infty} P(\eta(0, \varepsilon_m) > y - \varepsilon_{m'}) \qquad \text{or} \qquad P(\eta(0) > y - \varepsilon_{m'}) ,$$

if $y - \varepsilon_{m'}$ is a continuity point of the distribution of $\eta(0)$. The right-hand side of (6.22) is no less than $P(\eta(0) > y + \varepsilon_{m'})$ by quasi-left-continuity, under analogous conditions on y. Thus, if y is a continuity point of the distribution of $\eta(0)$, then

$$\lim_{Q''} P(n\eta(n)/||S_n||_p > y) = P(\eta(0) > y) . \tag{6.23}$$

Equation (6.9) follows, by a standard argument using the dominated convergence theorem. This completes the proof of Theorem 6.1.

7. The case of attraction to a stable law. Suppose that the X_i belong to the domain of attraction of some stable random variable with exponent $\alpha > 1$. Then the sequence Q of the previous section may be taken to be the set of all the natural numbers. Since the process Y^* is now a stable process with exponent α, it is easy to show, as Shepp has done in [12] for $\alpha = 2$, that $f^*(s) = C_1 s^{1/\alpha}$ for some positive constant C_1. Moreover, (6.7) is satisfied for stable processes with exponent greater than one, and therefore for this case Theorems 6.1 and 6.2 combine to produce the result (1.6), with $c_p = C_1$.

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