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ROBBINS-MONRO PROCEDURE WITH BOTH VARIABLES SUBJECT TO EXPERIMENTAL ERROR

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The limiting behavior of the one-dimensional RM procedure is investigated when the prescribed x-levels of sequential experiments can be realized with random errors only. The errors of constant variance and of variance decreasing to zero are studied in Sections 2 and 3, respectively.

0. Introduction. In the Robbins-Monro procedure for finding the root of the equation $M(x)=0$ it is assumed that for each x the value $M(x)$ is observable subject to an experimental error (with zero expectation). However, situations may occur where even the precise setting of the x-level of an experiment is impossible without error. These situations are studied in the present paper; Section 2 deals with errors unaffected by the experimenter, while in Section 3 it is assumed that the error in x-level can be made arbitrarily small for an inversely proportional price. In the latter case, it is intuitively clear that it is needless to pay for high precision at the starting steps of the procedure; the precision should be increased in the course of the approximation process.

1. Assumptions and notation. Let U_x and V_x be two families of random variables, the parameter space being the real line R. For each Borel set A, let the probability distributions $P^{U_x}(A)$, $P^{V_x}(A)$ be measurable functions of x. Further suppose $EU_x = EV_x = 0$ for all $x \in R$.

Let $M(x)$ be a measurable function, let θ be the unique root of $M(x) = 0$, the location of which is to be found.

Let us define the RM procedure with errors in setting the x-levels as follows: Let x_1 be a constant or a rv with $Ex_1^2 < +\infty$; for $n \geqq 1$ set recursively

$$x_{n+1} = x_n - a_n(M(x_n + u_n) + v_n)\,, \tag{1.1}$$

where a_n, $n \geqq 1$, is a sequence of positive numbers satisfying

$$\textstyle\sum_{n=1}^{\infty} a_n = +\infty\,, \qquad \sum_{n=1}^{\infty} a_n^2 < +\infty\,,$$

u_n, $n \geqq 1$, are rv's whose conditional distributions, given $x_1, u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}$, coincide with those of U_{x_n}, and v_n, $n \geqq 1$, are rv's whose conditional distributions, given $x_1, u_1, \cdots, u_n, v_1, \cdots, v_{n-1}$, coincide with those of $V_{x_n+u_n}$. (Note that x_n are uniquely determined by $x_1, u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}$.)

Throughout the paper, $C, C_1, C_2, \cdots$ will denote positive constants. The symbols O and O^{-1} will denote the upper and lower order estimates, i.e., for sequences of positive numbers, $f_n = O(g_n)$ will stand for $f_n \leqq Cg_n$, $n \geqq 1$, and $f_n = O^{-1}(g_n)$ for $f_n \geqq Cg_n$, $n \geqq 1$.

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2. Case of irreducible errors. The usual conditions ensuring the convergence of RM method are no more sufficient for the convergence of the sequence (1.1). (See Remark 2.1 below.) A set of strengthened conditions is given in the following

THEOREM 2.1. *Suppose the following conditions are satisfied:*

(2.1) M *is odd with respect to* θ, *i.e.*,

$$M(\theta + u) = -M(\theta - u) \qquad \textit{for all} \quad u \geqq 0 ;$$

(2.2) M *is strictly increasing;*

(2.3) $|M(x_2) - M(x_1)| \leqq C_1 + C_2|x_2 - x_1| \qquad$ *for all* $\quad x_1, x_2 \in R$;

(2.4) U_x *is a symmetric* rv *for each* $x \in R$, *i.e.*,

$$P(U_x \leqq u) = P(U_x \geqq -u) \qquad \textit{for all} \quad u \in R ;$$

(2.5) $\operatorname{Var} U_x \leqq C_3 \qquad$ *for each* $\quad x \in R$;

(2.6) $\operatorname{Var} V_x \leqq C_4 \qquad$ *for all* $\quad x \in R$.

Then x_n *converges to* θ *with probability* 1 *as well as in mean-square.*

PROOF. Introducing the function

$$M^*(x) = EM(x + U_x) , \qquad x \in R , \tag{2.7}$$

we may rewrite (1.1) as

$$x_{n+1} = x_n - a_n M^*(x_n) - a_n y_n , \tag{1.1$'$}$$

where y_n stands for $M(x_n + u_n) - M^*(x_n) + v_n$.
We have $E(y_n \mid x_n) = 0$; to see that, take first the conditional expectation of y_n given x_n, u_n, make use of $E(v_n \mid x_n, u_n) = 0$ and then take conditional expectation given x_n only. Thus (1.1)$'$ may be viewed as the usual RM procedure for finding the root of $M^*(x) = 0$. We first observe that $M^*(x)$ is defined for all $x \in R$ (this follows from (2.3) and (2.5)), and that θ, the root of M, is a root of M^*, too, (this is a consequence of (2.1) and (2.4)). Further, $M^*(x)$ is a measurable function, as follows from the measurability of P^{U_x}.

To prove the convergence $x_n \rightarrow \theta$ (w.p. 1 and in mean-square), we shall use Dvoretzky Theorem, specialized for the RM method in Result 1 of [3], page 50. That means to prove

$$E(y_n^2) \leqq C_5 , \qquad n \geqq 1 ; \tag{2.8}$$

$$|M^*(x)| \leqq C_6|x| + C_7 , \qquad x \in R ; \tag{2.9}$$

$$\inf_{x-\theta>\eta} M^*(x) > 0 , \qquad \sup_{x-\theta<-\eta} M^*(x) < 0 \qquad \text{for every} \quad \eta > 0 . \tag{2.10}$$

We first show that $\operatorname{Var} M(x + U_x)$ is bounded. By (2.3) we have

$$|M(x + U_x) - M(x)| \leqq C_1 + C_2|U_x| ,$$

hence

$$\operatorname{Var} M(x+U_x) \leqq E(M(x+U_x)-M(x))^2 \leqq 2C_1^2 + 2C_2^2 E U_x^2 \leqq 2C_1^2 + 2C_2^2 C_3 .$$

Now, by (2.6)

$$E(y_n^2 \mid x_n, u_n) \leqq (M(x_n+u_n) - M^*(x_n))^2 + C_4 ;$$

hence (2.8) follows:

$$E(y_n^2) \leqq 2C_1^2 + 2C_2^2 C_3 + C_4 (= C_5, \text{ say}) .$$

To prove (2.9), we first use (2.4) and (2.1) to rewrite $M^*(x)$ as

$$\begin{aligned}(2.11) \qquad M^*(x) &= \textstyle\int M(x+u) P^{U_x}(du) = M(x)P(U_x = 0) \\ &\quad + \textstyle\int_{(0,+\infty)} \{M(x+u) + M(x-u)\} P^{U_x}(du) = M(x)P(U_x=0) \\ &\quad + \textstyle\int_{(\theta,+\infty)} \{M(w+(x-\theta)) - M(w-(x-\theta))\} P^{W_x}(dw) ,\end{aligned}$$

with $W_x = U_x + \theta$.
Hence, by (2.3), $|M^*(x)| \leqq C_1 + 2C_2|x-\theta|$ follows.

To prove (2.10), let $\eta > 0$ be given; then for each $x > \theta + \eta$, the relation (2.11) together with the monotoneity of M imply

$$\begin{aligned}(2.12) \qquad M^*(x) &\geqq M(\theta+\eta)P(U_x = 0) \\ &\quad + \textstyle\int_{(\theta,+\infty)} \{M(w+\eta) - M(w-\eta)\} P^{W_x}(dw) .\end{aligned}$$

By Čebyšev inequality, $P(|U_x| \leqq C_8) \geqq \frac{1}{2}$ for a C_8, and the integrand in (2.12) is bounded from below by a $C(\eta) > 0$ on $(\theta, \theta + C_8]$, as follows from the strict monotoneity of M. Owing to the symmetry of U_x, we thus obtain

$$M^*(x) \geqq \tfrac{1}{4} \min (M(\theta+\eta), C(\eta)) .$$

Hence $\inf_{x-\theta>\eta} M^*(x) > 0$. The inequality $\sup_{x-\theta<-\eta} M^*(x) < 0$ follows in the same way.

REMARK 2.1. The assumptions of the oddness of M and of the symmetry of U_x are essential; if either is violated, then the approximations x_n may converge to a value different from θ, since $M^*(\theta)$ is, in general, no longer zero.

REMARK 2.2. If in addition to (2.4) and (2.5), the distribution of U_x is independent of x, then (2.2) can be weakened to

(2.2)′ M is non-decreasing for all $x \in R$ and $P^U(u : M$ is [at least one-sidedly] increasing at $\theta + u) > 0$.

The proof of the Remarks is immediate. As an illustration to Remark 2.2 consider the following

EXAMPLE. Let $M(x) = \operatorname{sign} x$, $V_x = 0$, $P(U_x = 1) = P(U_x = -1) = \frac{1}{2}$, for all $x \in R$. Then x_n does not converge (as $M^*(x) = 0$ for all $-1 < x < 1$). If however in the same example, $U_x = 1, 0, 1$ with respective probabilities $\frac{1}{2}(1-p)$, p, $\frac{1}{2}(1-p) (p > 0)$, then $x_n \to 0$ w.p. 1 and in mean-square, as now condition (2.2)′ is fulfilled.

3. Case of reducible errors. We shall retain the overall assumptions of Section 1, but we shall allow the rv's U_x to depend on n as well, denoting them by $U_{n,x}$. We shall understand that the conditional distribution of u_n is that of $U_{n,x}$.

THEOREM 3.1. *Suppose the following conditions are satisfied:*

$$|M(x)| \leqq C_9|x| + C_{10} \quad \textit{for all} \quad x \in R\,; \tag{3.1}$$

$$\inf_{x-\theta>\eta} M(x) > 0\,, \quad \sup_{x-\theta<-\eta} M(x) < 0\,, \qquad \textit{for every} \quad \eta > 0\,; \tag{3.2}$$

$$|U_{n,x}| \leqq K_n^{-1} \quad \textit{for some} \quad 0 < K_n \nearrow +\infty \quad \textit{and for all} \quad x \in R\,; \tag{3.3}$$

$$\operatorname{Var} V_x \leqq C_{11} \qquad \textit{for all} \quad x \in R\,. \tag{3.4}$$

Then $x_n \to \theta$ with probability 1 *and in mean-square.*

PROOF. This time we shall refer to the Generalization 5 of Dvoretzky Theorem ([3] page 49). Defining *random* transformations

$$T_n(x) = x - a_n M(x + U_{n,x})\,, \tag{3.5}$$

we may rewrite (1.1) as

$$x_{n+1} = T_n(x_n) - a_n v_n\,.$$

The conditions laid by the cited theorem upon v_n are satisfied: Denoting by t_n the value assumed by $T_n(x_n)$, we have $E(v_n \mid x_n, t_n) = 0$, as $E(v_n \mid x_n, t_n, u_n) = 0$; further, $E(v_n^2) \leqq C_{11}$ according to (3.4).

It remains to investigate the transformations T_n. Let ρ_n and η_n, $n \geqq 1$, be two null sequences of positive numbers such that

$$\textstyle\sum_{n=1}^{\infty} a_n \rho_n = +\infty \qquad \text{and} \qquad \inf_{|x-\theta|>\eta_n} |M(x)| > \rho_n\,.$$

If $|x - \theta| \leqq \eta_n + K_n^{-1}$, then (according to (3.1) and (3.3))

$$|T_n(x) - \theta| \leqq \eta_n + K_n^{-1} + C_{12} a_n = \alpha_n(\text{say})\,; \tag{3.6}$$

if $|x - \theta| > \eta_n + K_n^{-1}$, then

$$|T_n(x) - \theta| \leqq |x - \theta| - a_n \rho_n\,, \tag{3.7}$$

owing to (3.2) and the definition of ρ_n. Thus we have (combining (3.6) and (3.7))

$$|T_n(x) - \theta| \leqq \max\,(\alpha_n, |x - \theta| - a_n \rho_n)\,, \qquad n \geqq 1\,,$$

with $\alpha_n \to 0$ and $\sum a_n \rho_n = +\infty$ and for all $x \in R$ and all realizations of $T_n(x)$. But this is the condition required by the cited theorem. □

In the sequel, we shall replace the condition of uniform boundedness of $U_{n,x}$ by a null sequence, by an analogous condition on variances:

$$\operatorname{Var} U_{n,x} \leqq K_n^{-1} \quad \text{for some} \quad 0 < K_n \nearrow +\infty \quad \text{and for all} \quad x \in R\,. \tag{3.8}$$

An analog to Theorem 3.1 can be proved then for a slightly less general class of functions M; but instead of that, we shall investigate a special case in more detail and by different tools.

We shall denote

$$N = \sum_{m=1}^{n} K_m , \qquad n \geqq 1 ,$$

(the total cost of first n experiments) and we shall study the asymptotic behavior of the mean-square error

$$B_N = E(x_n - \theta)^2$$

as a function of N, for N running to infinity through partial sums of the K_m's.

THEOREM 3.2. *Suppose conditions* (3.4) *and* (3.8) *are satisfied. Further suppose*

(3.9) $C_{13} \leqq (M(x_2) - M(x_1))/(x_2 - x_1) \leqq C_{14} , \qquad x_1, x_2 \in R ;$

(3.10) $a_n = an^{-1} , \qquad n \geqq 1 \quad$ *with* $\quad a > (2C_{13})^{-1} ;$

(3.11) $K_n = C_{15} n^{\alpha} , \qquad n \geqq 1 \quad$ *with* $\quad \alpha > 0 .$

Then

$$(3.12) \qquad B_N = O(N^{-\alpha/(1+\alpha)}) \qquad 0 < \alpha \leqq 1 ,$$
$$= O(N^{-1/(1+\alpha)}) \qquad \alpha \geqq 1 ,$$

and the choice $\alpha = 1$ (*leading to* $B_N = O(N^{-\frac{1}{2}})$) *is optimal in the following sense: If* $\alpha \neq 1$, *then there exist* U_x, V_x, $M(x)$, *satisfying all the conditions of the theorem and such that*

$$(3.13) \qquad B_N = O^{-1}(N^{-\frac{1}{2}+\varepsilon}) \qquad \textit{for some } \varepsilon > 0 .$$

PROOF. From (1.1) we get by subtracting θ, squaring, taking conditional expectations and using the basic assumptions

$$(3.14) \qquad E((x_{n+1} - \theta)^2 \mid x_n) = (x_n - \theta)^2 - 2a_n(x_n - \theta)E(M(x_n + u_n) \mid x_n) + a_n^2 E(M^2(x_n + u_n) \mid x_n) + a_n^2 E(v_n^2 \mid x_n) .$$

Denoting by Q_{x_1,x_2} the middle term of (3.9), we can write

$$M(x + U_{n,x}) = Q_{\theta,x}(x - \theta) + Q_{x,x+U_{n,x}} U_{n,x} ,$$

hence (using (3.8), (3.9) and Schwarz inequality)

$$(x - \theta)E(M(x + U_{n,x})) \geqq C_{13}(x - \theta)^2 - C_{14} K_n^{-\frac{1}{2}}|x - \theta|$$

and also

$$E(M^2(x + U_{n,x})) \leqq 2C_{14}\{(x - \theta)^2 + K_n^{-1}\} .$$

Inserting these inequalities into (3.14) and applying (3.4) to its last term, we get

$$(3.15) \qquad E((x_{n+1} - \theta)^2 \mid x_n) \leqq (1 - 2C_{13} a_n + 2C_{14} a_n^2)(x_n - \theta)^2 + 2C_{14} K_n^{-\frac{1}{2}} a_n |x_n - \theta| + a_n^2(2C_{14} K_n^{-1} + C_{11}) .$$

Now, take expectations on both sides of (3.15), then apply the inequality

$$E(|X|) \leqq \eta + \eta^{-1} E(X^2) , \qquad \eta > 0 ,$$

to rv's $x_n - \theta$ with

$$\eta_n = 2C_{14}/(\gamma C_{13} K_n^{\frac{1}{2}}) , \qquad \gamma > 0 .$$

Finally, insert for a_n and K_n from (3.10), (3.11) and choose γ so that $(2-\gamma)C_{13}a > 1$ holds and so that the inequality resulting (after the indicated operations) from (3.15) remains true (at least for lange n) even if the term of order $a_n^2 E(x_n - \theta)^2$ is deleted. Denoting $E(x_n - \theta)^2$ as b_n when considered as a function of n, we thus get

$$b_{n+1} \leqq (1 - (2 - \gamma)C_{13}a/n)b_n + C_{16}a/n^{1+\alpha} + C_{17}a^2/n^2 , \qquad n \geqq n_0 . \tag{3.16}$$

Hence by a lemma due to Chung ([1] Lemma 1) we get

$$\begin{aligned} b_n &= O(1/n^\alpha) , & 0 < \alpha \leqq 1 , \\ &= O(1/n) , & \alpha \geqq 1 ; \end{aligned}$$

since $N = \sum_{m=1}^n K_m \cong C_{18}n^{1+\alpha}$ (i.e., $n \cong C_{18}^{-1}N^{1/(1+\alpha)}$), the assertion (3.12) follows.

The optimality proof is similar to that of Theorem 3 in [2]: If $\alpha > 1$, choose $U_x = 0$, $x \in R$, V_x satisfying (3.4) and Var $V_x \leqq C_{19}$, $x \in R$, and M satisfying (3.9). Then starting from (3.14) we obtain

$$b_{n+1} \geqq (1 - 2C_{14}a/n)b_n + C_{19}a^2/n^2 .$$

Hence, by [1], Lemma 2, $b_n \geqq C_{20}/n$, $n \geqq 1$, i.e., $B_N \geqq C_{21}/N^{1/(1+\alpha)}$, which gives (3.13), as $1/(1 + \alpha) < \frac{1}{2}$.

If $\alpha < 1$, choose $P(U_{n,x} = n^{-\alpha/2}) = P(U_{n,x} = -n^{-\alpha/2}) = \frac{1}{2}$; $V_x = 0$, $x \in R$; $M(x) = 2x$ for $x \leqq 0$, $M(x) = x$ for $x > 0$. (Thus $\theta = 0$, $C_{13} = 1$, $C_{14} = 2$, etc.) After easy calculations we get

$$E(M(x + U_{n,x})) \leqq 3x/2 - 1/(2n^{\alpha/2}) , \tag{3.17}$$

$$xE(M(x + U_{n,x})) \leqq 2x^2 - x/(2n^{\alpha/2}) , \tag{3.18}$$

for all $x \in R$. Combining (3.17) with the obvious relation $E(x_{n+1}) = E(x_n) - an^{-1}E(M(x_n + u_n))$, we have

$$E(x_{n+1}) \geqq (1 - 3a/2n)E(x_n) + a/(2n^{1+\alpha/2}) ,$$

hence $E(x_n) \geqq 1/(3n^{\alpha/2})$, $n \geqq n_1$, according to [1], Lemma 2, taking into account that $0 < \alpha < 1$, $a > \frac{1}{2}$. Inserting the last inequality into (3.18), we get $E(x_n M(x_n + u_n)) \leqq 2E(x_n^2) - 1/(6n^\alpha)$, and using this in (3.14),

$$b_{n+1} \geqq (1 - 4a/n)b_n + a/(3n^{1+\alpha}) , \qquad n \geqq n_1 .$$

Hence, by the same lemma, $b_n \geqq C_{22}/n^\alpha$, i.e., $B_N \geqq C_{23}/N^{\alpha/(1+\alpha)}$ which again gives (3.13), as now $\alpha/(1 + \alpha) < \frac{1}{2}$. □

THEOREM 3.3. *If the conditions of Theorem* 3.2 *are satisfied and if further the second derivative of M exists and is bounded for all $x \in R$, then*

$$\begin{aligned} B_N &= O(N^{-2\alpha/(1+\alpha)}) & 0 < \alpha \leqq \tfrac{1}{2} , \\ &= O(N^{-1/(1+\alpha)}) & \tfrac{1}{2} \leqq \alpha . \end{aligned}$$

The choice $\alpha = \frac{1}{2}$ *(leading to* $B_N = O(N^{-\frac{2}{3}})$*) is optimal; i.e., if* $\alpha \neq \frac{1}{2}$, *then there exist* U_x, V_x, $M(x)$, *satisfying all the conditions of the theorem and such that*

$$B_N = O^{-1}(N^{-\frac{2}{3}+\varepsilon}) \qquad \textit{for some} \quad \varepsilon > 0 .$$

PROOF. The proof is similar to that of Theorem 4 in [2] and will be omitted.

4. Generalization. The problem studied in the present paper can be formulated also for the Kiefer-Wolfowitz procedure for finding the maximum of a function. However, the results are of a very limited importance, since the presence of errors in setting the x-levels makes KW procedure practically inapplicable. On the other hand, a generalization to multidimensional RM procedure might be of interest.

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