CONVERGENCE IN DISTRIBUTION OF RANDOM MEASURES

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Let $(S, \mathcal{F})$ be a measurable space, $M$ the set of all finite measures on $\mathcal{F}$, $\mathcal{F}_M$ the $\sigma$-algebra generated by the family of all measurable cylindrical sets $\bigcap_{i=1}^n \{\mu \in M : \mu(A_i) \leq a_i\}$. With each probability measure $P$ on $\mathcal{F}_M$ the family $(P_{A_1}, \ldots, A_k)$ of all finite-dimensional probability measures of the cylindrical sets is associated. The following problem is considered: Given a sequence $P^{(n)}$ of probability measures on $\mathcal{F}_M$ such that each sequence $P_{A_1}, \ldots, A_k$ converges weakly to a $k$-dimensional probability measure $P_{A_1}, \ldots, A_k$, does the family $(P_{A_1}, \ldots, A_k)$ generate a probability measure $P$ on $\mathcal{F}_M$? It is proved that the answer is affirmative if $(S, \mathcal{F})$ is the Euclidean $n$-space with the $\sigma$-algebra of Borel sets.

1. Introduction. In the whole paper, $R = (-\infty, \infty)$, $\mathcal{B}^{(k)} = \sigma$-algebra of all Borel sets in $R^k$, $R = [0, \infty)$. Let $T$ be an arbitrary index set, $F$ a subset of $R^T$ = the set of all real functions on $T$, $\mathcal{F}_p$ the $\sigma$-algebra containing all sets $\{f \in F : f(t) \in E\}$, $t \in T$, $E \in \mathcal{B}^{(1)}$. If $P$ is a probability measure on $\mathcal{F}_p$, then the probability measures $P_{t_1, \ldots, t_k}$ on $\mathcal{B}^{(k)}$ $(k = 1, 2, \ldots, i \in T)$ defined by $P_{t_1, \ldots, t_k}(E_1 \times \cdots \times E_k) = P(\bigcap_{i=1}^k \{f \in F : f(t_i) \in E_i\})$, $E_i \in \mathcal{B}^{(1)}$, will be called the finite-dimensional probability distributions (f.d.p.d.’s) of $P$. We shall adopt the following two definitions:

DEFINITION 1. Let $P^{(n)}$, $P$ be probability measures on $\mathcal{F}_p$. We shall say that $P^{(n)}$ converge in distribution (or D-converge) to $P$, if each f.d.p.d. of $P^{(n)}$ converges weakly to the corresponding f.d.p.d. of $P$; $P$ will be called the D-limit of $P^{(n)}$.

DEFINITION 2. A sequence $P^{(n)}$ of probability measures on $\mathcal{F}_p$ will be called fundamental in distribution (or D-fundamental), if each f.d.p.d. of $P^{(n)}$ converges weakly to a finite-dimensional probability measure. It is clear that the D-limit $P$, if it exists, is unique. Using the well-known Kolmogorov theorem we can see easily that each D-fundamental sequence is D-convergent if $F = R^T$. This need not be true if $F$ is a proper subspace of $R^T$; e.g., take $T = [0, 1]$, $F = \{0, 1\}$, the set of all continuous functions on $T$, $P^{(n)}$ = the probability measure concentrated on the one-point set $\{f_n\}$, where $f_n(t) = t^n$. It is therefore rather a surprising fact that each D-fundamental sequence $P^{(n)}$ is D-convergent, if $F$ is the set of all finite measures on $(R^n, B^{(n)})$ or, more generally, on any measurable space $(S, \mathcal{F})$ satisfying the conditions C listed below. This is the main result of this paper—see Theorem 2.

2. Random measures. Let $S$ be an arbitrary set, $\mathcal{A}$ a $\sigma$-algebra of subsets of $S$, $M$ the set of all finite measures on $\mathcal{A}$. Since $M$ is a subset of $R^T$, $\mathcal{F}_M$ is the
least σ-algebra containing all sets \( \{ \mu \in M : \mu(A) \in E \} \), \( A \in \mathcal{A} \), \( E \in \mathcal{B}(1) \). If \( P \) is a probability measure on \( \mathcal{F}_M \), then the probability field \( \{ M, \mathcal{F}_M, P \} \) is one of the possible models for a random measure. It is easy to see that the corresponding f.d.p.d.'s \( P_{A_1, \ldots, A_k} \) (with \( A_i \in \mathcal{A} \)) satisfy the following conditions \( M \):

(M1) \[ P_{A_1, \ldots, A_k}(E_1 \times \cdots \times E_k) = P_{A_1, \ldots, A_k, A_{k+1}}(E_1 \times \cdots \times E_k \times R); \]

(M2) \[ P_{A_1, \ldots, A_k}(R^k) = 1; \]

(M3) If \( A_1 \cap A_2 = \emptyset \) and \( A_3 = A_1 \cup A_2 \), then \( P_{A_1, A_2, A_3} \) is concentrated on the plane \( x_3 = x_1 + x_2 \) in \( R^3 \).

(M4) If \( A_j \supset A_{j+1}, \bigcap_{j=1}^{n} A_j = \emptyset \), then for each \( x > 0 \)
\[ P_{A_j}(x, \infty) \to_{j \to \infty} 0. \]

We shall say that the measurable space \( \{ S, \mathcal{A} \} \) satisfies conditions \( C \) if there exists a countable algebra \( \mathcal{A} \) and a class \( \mathcal{C} \) of subsets of \( S \) such that

(C1) \( \mathcal{A} \) is generated by \( \mathcal{C} \);

(C2) \( C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 \in \mathcal{C} \);

(C3) \( C_j \in \mathcal{C}, \quad C_j \supset C_{j+1}, \quad C_j \neq \emptyset, \quad k = 1, 2, \ldots \Rightarrow \bigcap_{j=1}^{n} C_j \neq \emptyset \);

(C4) To each \( A \in \mathcal{A} \) there exist sequences \( B_{j, A} \in \mathcal{A} \) and \( C_{j, A} \in \mathcal{C} \) such that \( B_{j, A} \subset C_{j, A} \subset A\) and \( \bigcap_{j=1}^{n} B_{j, A} = A \).

\( \{ R^m, \mathcal{B}^{(m)} \} \) and some other metric or topological spaces satisfy conditions \( C \).

The following theorem is a generalization of Theorem 3.1 Chapter III of [2]. The Appendix 1 to Chapter III of [2] contains the basic ideas of its proof and some details are implicitly contained in the proof of slightly different assertion in [3], Theorem 1.4. We shall therefore present the main steps of the proof only and omit the details.

**Theorem 1.** Let measurable space \( \{ S, \mathcal{A} \} \) satisfy the condition \( C \) and let a family \( \{ P_{A_1, \ldots, A_k} \} \) of probability measures on \( \mathcal{B}^{(k)} \) \( (k = 1, 2, \ldots, A_i \in \mathcal{A} \) satisfy the conditions \( M \). Then there exists exactly one probability measure \( P \) on \( \mathcal{F}_M \) such that \( P_{A_1, \ldots, A_k} \) are its f.d.p.d.'s.

**Proof.** The uniqueness follows from the fact that the family of all sets \( \bigcap_{i=1}^{n} \{ \mu \in M : \mu(A_i) \in E_i \} \) is a half-ring generating \( \mathcal{F}_M \). The existence follows from the following construction:

**Step 1.** Let \( F_0 = R^\mathcal{A} \) = the set of all set functions on the algebra \( \mathcal{A} \) of (C1). Using (M1) and the well-known Kolmogorov theorem we can construct a probability measure \( P^{(0)} \) on \( \mathcal{F}_{F_0} \) such that \( P_{A_1, \ldots, A_k} \) (with \( A_i \in \mathcal{A} \)) are its f.d.p.d.'s.

**Step 2.** Let \( F_1 \) be the set of all finite, nonnegative and two-additive set functions on \( \mathcal{A} \). Since \( \mathcal{A} \) is countable, \( F_1 \subset \mathcal{F}_{F_0} \) and (M2) and (M3) imply that \( P^{(0)}(F_1) = 1 \). Hence, the restriction \( P^{(1)} \) of \( P^{(0)} \) to \( \mathcal{F}_{F_1} = F_1 \cap \mathcal{F}_{F_0} \) is a probability measure on \( \mathcal{F}_{F_1} \) such that \( P_{A_1, \ldots, A_k} \) are its f.d.p.d.'s.
Step 3. Let $F_2$ be the set of all $\nu \in F_1$ such that for each $A \in \mathcal{A}$

\[(1) \quad \nu(A - B_{k, \nu}) \to_{k \to \infty} 0,\]

where $B_{k, \nu}$ are the sets mentioned in (C4). Again $F_2 \in \mathcal{F}_F$ and $P^{(1)}(F_2) = 1$ because of (M4) and (C4). Hence, the restriction $P^{(2)}$ of $P^{(1)}$ to $\mathcal{F}_F = F_2 \cap \mathcal{F}_F$ is a probability measure on $\mathcal{F}_F$ such that $P_{A_1, \ldots, A_k}$ are its f.d.p.d.'s. Each $\nu \in F_2$ is finite, nonnegative and finitely additive set function on $\mathcal{A}$ satisfying (1) with $B_{k, \nu}$ satisfying (C4). Using this and the properties (C2) and (C3) we can prove that $\nu$ is countably additive on $\mathcal{A}$ and can be therefore extended uniquely to a finite measure $\bar{\nu}$ on $\mathcal{A}$. For each fixed $A \in \mathcal{A}$, $\bar{\nu}(A)$ is (as a function of $\nu \in F_2$) $\mathcal{F}_F$-measurable. This is trivial for $A \in \mathcal{A}$ and it can be proved for all $A \in \mathcal{A}$ by means of the monotone-class theorem. ([1], Chapter I, Section 6, Theorem B).

Step 4. Let $h$ be the transformation (of $F_2$ into $M$) assigning to each $\nu \in F_2$ its extension $\bar{\nu}$. The transformation $h$ is $\mathcal{F}_F - \mathcal{F}_M$-measurable and the set function $P$ on $\mathcal{F}_M$ defined by $P(D) = P(h^{-1}(D))$ is a probability measure on $\mathcal{F}_M$. It remains to show that $P_{A_1, \ldots, A_k}$ are its f.d.p.d.'s. This is trivial for $A \in \mathcal{A}$. To prove it generally, the monotone-class theorem is to be applied successively to each of the indices $A_1, \ldots, A_k$. The following four relations are essential in that part of the proof.

If $A_{k, i} \subset A_{k, i+1}$, $A_k = \bigcup_{j=1}^k A_{k, j}$, then

\[(2) \quad P(\{\mu \in M : \mu(A_i) \in E_i, i = 1, 2, \ldots, k-1, \mu(A_k) \leq x\})
= \lim_{j \to \infty} P(\{\mu \in M : \mu(A_i) \in E_i, i = 1, 2, \ldots, k-1, \mu(A_{k, j}) \leq x\})
\]

and

\[(3) \quad P_{A_1, \ldots, A_{k-1}, k}(E_1 \times \ldots \times E_{k-1} \times [0, x])
= \lim_{j \to \infty} P_{A_1, \ldots, A_{k-1}, k, j}(E_1 \times \ldots \times E_{k-1} \times [0, x]).\]

If $A_{k, i} \supset A_{k, i+1}$, $A_k = \bigcap_{j=1}^k A_{k, j}$, then

\[(4) \quad P(\{\mu \in M : \mu(A_i) \in E_i, i = 1, 2, \ldots, k-1, \mu(A_k) < x\})
= \lim_{j \to \infty} P(\{\mu \in M : \mu(A_i) \in E_i, i = 1, 2, \ldots, k-1, \mu(A_{k, j}) < x\})
\]

and

\[(5) \quad P_{A_1, \ldots, A_{k-1}, k}(E_1 \times \ldots \times E_{k-1} \times [0, x])
= \lim_{j \to \infty} P_{A_1, \ldots, A_{k-1}, k, j}(E_1 \times \ldots \times E_{k-1} \times [0, x]).\]

The relations (2) and (4) are easy; (3) and (5) must be derived directly from the conditions M. Let us do that for (3). To simplify writing, we shall omit the indices $A_1, \ldots, A_{k-1}$ (as if $E_1 = \ldots = E_{k-1} = R$) and we shall write $B_j, B$ instead of $A_{k, j}, A_k$. Using the 3-dimensional distribution $P_{B_{j, B-B_j, B}}$ and (M1)—(M3), we see that for any $x > 0$ and $\varepsilon > 0$

\[P_{B_j}([0, x]) \geq P_{B_{j+1}}([0, x]) \geq P_B([0, x])\]

and

\[P_{B_j}([0, x]) \leq P_B([0, x + \varepsilon]) + P_{B-B_j}((\varepsilon, \infty)).\]
Hence, by (M4),
\[ P_{\delta}(0, x] \leq \lim P_{B_\delta}[0, x] \leq P_{\delta}(0, x + \varepsilon) . \]
Since \( \varepsilon > 0 \) was arbitrary, \( P_{\delta}(0, x] = \lim P_{B_\delta}(0, x] \), which proves (3). The proof of (5) is similar.

3. D-convergence of random measures.

**Theorem 2.** Let the measurable space \( \{S, \mathcal{S}\} \) satisfy the condition C. Then each D-fundamental sequence of probability measures \( P^{(n)} \) on \( \{M, \mathcal{M}\} \) is D-convergent.

**Proof.** Let us denote the f.d.p.d.'s of \( P^{(n)} \) by \( P^{(n)}_{A_1, \ldots, A_k} \) \( (n \geq 1) \). By assumption, to each sequence \( A_1, \ldots, A_k \) \( (A_i \in \mathcal{S}) \) there exists a probability measure \( P^{(0)}_{A_1, \ldots, A_k} \) on \( \mathcal{B}^{(k)} \) such that
\[ P^{(n)}_{A_1, \ldots, A_k} \to_{n \to \infty} P^{(0)}_{A_1, \ldots, A_k} \quad \text{(weakly)}. \]
By Theorem 1, it is sufficient to show that the family \( P^{(0)}_{A_1, \ldots, A_k} \) satisfies the conditions M. Since, for each \( n \geq 1 \), the family \( P^{(n)}_{A_1, \ldots, A_k} \) satisfies M, it is sufficient to show that the conditions M are preserved under (6). This is trivial for (M1), (M2) and (M3) and only (M4) remains to be proved. To an arbitrary \( \varepsilon > 0 \) there exists \( a > 0 \) such that
\[ P_{\delta}((a, \infty)) < \frac{1}{2} \varepsilon \quad \text{and} \quad P_{\delta}(\{a\}) = 0 . \]
Put
\[ f(x) = 0 \quad \text{if} \quad x \leq 0 \]
\[ = x \quad \text{if} \quad 0 \leq x \leq a \]
\[ = a \quad \text{if} \quad x \geq a \]
and
\[ M_\mu(A) = \int_{R \times (-\infty, a]} f(x) P^{(n)}_{A, \delta}(d(x, y)) \]
for all \( A \in \mathcal{S} \) and \( n \geq 0 \). The function \( f \), as a function of \((x, y)\), is bounded and continuous on the domain \( R \times (-\infty, a) \) and the boundary of this domain has \( P^{(0)}_{A, \delta} \)-measure zero by (M1) and (7). Hence the weak convergence (6) implies
\[ M_\mu(A) \to M_\delta(A) \quad \text{for each} \quad A \in \mathcal{S} . \]
Since \( \mu(A) \leq \mu(S) \) for all \( \mu \in M \),
\[ P^{(n)}_{A, \delta}((a, \infty) \times (-\infty, a)) = 0 \]
for all \( n \geq 1 \). Hence
\[ M_\mu(A) = \int_{R \times (-\infty, a]} x P^{(n)}_{A, \delta}(d(x, y)) = \int_{\{\mu \in M; \mu(S) < a\}} \mu(A) P^{(n)}(d\mu) \]
for all \( n \geq 1 \). The second integral shows that \( M_\mu \) is, for each \( n \geq 1 \), a measure on \( \mathcal{S} \) and that the sequence \( M_\mu \) is uniformly bounded (by \( a \)). Hence, by (8) and a well-known theorem, \( M_\delta \) is also a finite measure on \( \mathcal{S} \). (9) is clearly preserved under (6) and, therefore
\[ M_\delta(A) = \int_{R \times (-\infty, a]} x P^{(0)}_{A, \delta}(d(x, y)) \geq x P^{(0)}_{A, \delta}((x, \infty) \times (-\infty, a)) \]
for all \( A \in \mathcal{S} \) and an arbitrary \( x > 0 \).
Consider now a sequence \( A_j \in \mathcal{S} \), \( A_j \supset A_{j+1} \), \( \bigcap_{j=1}^{\infty} A_j = \emptyset \). Since \( M_0 \) is a finite measure, \( M_0(A_j) \to j \to \infty 0 \) and therefore, by (10), there exists \( j_0 \) such that
\[
P_{A_j, \delta}((x, \infty)) \times (-\infty, a)) \leq \frac{1}{2} \varepsilon \quad \text{for all } j \geq j_0.
\]
Finally, by (7), (11) and (M1)
\[
P_{A_j}((x, \infty)) = P_{A_j, \delta}((x, \infty) \times (-\infty, a)) + P_{A_j, \delta}((x, \infty) \times [a, \infty)) < \varepsilon
\]
for all \( j \geq j_0 \), which proves that (M4) holds for \( P_{A_1, \ldots, A_k} \).

REFERENCES


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