A THRESHOLD FOR LOG-CONCAVITY FOR PROBABILITY GENERATING MOMENT INEQUALITIES

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Let \( \{p_n\}_{n=0}^{\infty} \) be a discrete distribution on \( 0 \leq n \leq N \) and let \( g(u) = \sum_{n=0}^{\infty} p_n u^n \) be its pgf. Then for \( 0 \leq t < \infty \) \( g(u) = g(u+t)/g(1+t) = \sum_{n=0}^{\infty} p_n(t) u^n \) is a family of pgfs indexed by \( t \). It is shown that there is a unique value \( t^* \) such that \( \{p_n(t)\}_{n=0}^{\infty} \) is log-concave \( (PF) \) for all \( t \geq t^* \) and is not log-concave for \( 0 < t < t^* \). As a consequence one finds the infinite set of moment inequalities \( \frac{\mu_{(r+1)}}{r!} \geq \frac{\mu_{(r+1)}(r+1)!}{(r+1)!} \) \( r = 1, 2, \ldots \) etc. where \( \mu_{(r)} \) is the \( r \)th factorial moment of \( \{p_n\}_{n=0}^{\infty} \) when the lattice distribution is log-concave. The known set of inequalities for the continuous analogue is shown to follow from the discrete inequalities.

0. Introduction and summary. A set of nonnegative masses \( \{p_n\}_{n=0}^{\infty} \) on the lattice of integers is \( PF \) [5], and “log-concave” or “strongly unimodal” [8] if \( p_n \geq p_{n+1}p_{n-1} \) and there are no gaps in the domain of positive support. Such strongly unimodal sequences play an important role in probability theory [8]. Let \( P(z) = \sum_{k=0}^{\infty} p_k z^k \) be the generating function for a set of nonnegative masses on the lattice interval \([0, N]\), with \( p_N > 0 \). Then \( P(z) = P(z + t) = \sum_{k=0}^{\infty} p_k(t) z^k \) is such a generating function for all \( t \geq 0 \). It will be seen that the semi-infinite interval \([0, \infty)\) has precisely one value \( t^* \), such that \( \{p_n(t)\}_{n=0}^{\infty} \) is log-concave for \( t \geq t^* \), and is not log-concave for \( t < t^* \). As one direct consequence, we provide new proofs of two important sets of inequalities, perhaps deserving of more attention than they have received. The first states that if \( \{p_n\}_{n=0}^{\infty} \) is a log-concave probability distribution, then

\[
\left\{ \frac{\mu_{(r+1)}}{(r+1)!} \right\}^{1/(r+1)} \leq \left\{ \frac{\mu_{(r)}}{r!} \right\}^{1/r}, \quad r = 1, 2, \ldots.
\]

Here \( \mu_{(r)} = \sum_{k=0}^{\infty} p_k(k-1) \cdots (k-r+1) \) is the factorial moment of order \( r \) [9]. In particular, one has that for all log-concave probability lattice distributions with nonnegative support,

\[
\mu_2 \leq 2\mu_1^2 + \mu_1
\]

where \( \mu_1 \) and \( \mu_2 \) are the ordinary first and second moments.

The inequality (0.2) has the continuous analogue

\[
\mu_2 \leq 2\mu_1^2
\]

where \( \mu_k = \int_{0}^{\infty} x^k f(x) \, dx \) and \( f(x) \) is any probability density function with purely positive support such that \( \log f(x) \) is a concave function on its interval of support. Such probability density functions are also “strongly unimodal” and of interest.

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to probability theory [4], [7]. Equation (0.3) is a special case of the analogue of (0.1) for such density functions, which takes the form

\[(0.4) \quad \left( \frac{\mu_{k+1}}{(k+1)!} \right)^{1/(k+1)} \leq \left( \frac{\mu_k}{k!} \right)^{1/k} ; \quad k = 1, 2, \ldots .\]

The inequalities (0.1) are implicit in Theorem 2 of [6] by Karlin, Proschan and Barlow. The inequalities (0.4) for continuous time have been given explicitly in [6], and in Barlow, Marshall and Proschan [2] under weaker conditions.

The Pólya frequency sequences of order two and Pólya frequency functions of order two that have nonnegative support and total mass one are equivalent to the log-concave sequences and log-concave density functions above. They are described at length by Karlin [5] in his book on total positivity. The properties of $PF_2$ sequences needed for this paper are presented in a simple self-contained form in [8] oriented towards probability theory and unimodality.

1. **Real polynomials and log-concavity.** It is widely known that if $P(z) = \sum_{0}^{N} p_k z^k$ is a polynomial of degree $N$ with real coefficients having all real zeros, then (cf. [1] Section 2.22), with the convention $p_{-1} = p_{N+1} = 0$,

\[(1.1) \quad p_k^2 \geq p_{k+1} p_{k-1} , \quad 0 \leq k \leq N .\]

The inequalities of (1.1) permit one to speak of the coefficients $p_k$ as being "log-concave" when the coefficients are positive in that the sequence $\{ \log p_k \}^N_0$ has nonpositive second differences. The log-concavity is always latent for polynomials with real coefficients, whether or not the zeros are real, in the sense of the following definition and theorem.

**Definition.** A polynomial $P(z) = \sum_{0}^{N} p_k z^k$ for which $p_k \geq 0, p_N > 0$, will be said to be of type $\mathcal{D}_N$ if its coefficients $\{p_k\}^N_0$ are a log-concave sequence (cf. [8]), i.e. if the coefficients satisfy (1.1) and the set \{ $k : p_k > 0$ \} is connected. All such polynomials will be said to be of type $\mathcal{D}$.

**Theorem 1.** Let $P_0(z) = \sum_{0}^{N} p_{0k} z^k$ be a polynomial of degree $N$ with $p_{0N} > 0$ and all coefficients $p_{0k}$ real. Let the sequence $\{p_r(t)\}^N_0$ be defined for all $t \geq 0$ by

\[(1.2) \quad P_r(z) = P_0(z + t) = \sum_{0}^{N} p_k(t) z^k , \quad \text{so that} \]

\[p_r(t) = \sum_{k=0}^{N} (k!)^{t^k-r} p_{0k} .\]

Then the set of nonnegative numbers $t$ has some smallest finite value $t^*$ such that $P_0(z + t^*)$ is of type $\mathcal{D}_N$. Moreover, $P_0(z + t)$ is of type $\mathcal{D}_N$ for all $t \geq t^*$.

The proof of the theorem is based on the following lemma of some interest in its own right. (cf. Chapter 8, Section 7 of [5].)

**Lemma.** Let $P_0(z)$ be defined as in Theorem 1. Then there exists an $A \geq 0$ such that

\[(1.3) \quad P_0(z + t) = p_{0N} \prod_{i} \{ z + \alpha_i(t) \} \prod_{i} \{ z^2 + \beta_i(t) z + \gamma_i(t) \} \]

where for all $t \geq A$. 


\[ \alpha_i(t) \geq 0; \quad \beta_i(t) \geq 0; \quad \gamma_i(t) \geq 0; \quad \hat{\beta}_i(t) \geq \gamma_i(t). \]

As proof we note that we may write

\[ P_d(z) = p_{0N} \prod_i (z - r_i) \prod_j \{(z - w_j)(z - \tilde{w}_j)\} \]

where the \( r_i \) are real zeros of \( P_d(z) \) and the \( w_j \) and \( \tilde{w}_j \) are complex zeros taken in conjugate pairs. Hence

\[ P_d(z + t) = p_{0N} \prod_i \{z + (t - r_i)\} \prod_j \{(z + \xi_j)(z + \tilde{\xi}_j)\} \]

where for \( w_j = x_j + iy_j \), \( \xi_j = (t - x_j) + iy_j \). When \( A_i = \max_{i,j} \{r_i, x_j\} \) and \( t \geq A_i \), coefficients in the monomial and quadratic factors of \( P_d(z + t) \) will all be nonnegative, so that \( P_d(z + t) \) will be a polynomial in \( z \) with nonnegative coefficients. Consider the quadratic term

\[ Q_j(z) = z^2 + (\xi_j + \tilde{\xi}_j)z + |\xi_j|^2 \]

\[ = z^2 + 2(t - x_j)z + (t - x_j)^2 + y_j^2 \]

\[ = z^2 + c_{ij}z + c_{0j}. \]

We note that

\[ c_{ij}/c_{0j} = 4[1 + y_j^2(t - x_j)^2]^{-1} \geq 1 \]

when \( y_j^2(t - x_j)^2 \leq 3 \). It follows that each quadratic factor will be a polynomial of type \( \mathcal{P} \) whenever

\[ \max_j \{y_j^2(t - x_j)^2\} \leq 3 \]

i.e., whenever

\[ t \geq A_2 = \max_j \{x_j + |y_j|/(3)^{1/2}\}. \]

The validity of the lemma then follows for \( A = \max (A_1, A_2) \).

It is known [5], [8] that the product of polynomials of type \( \mathcal{P} \) is itself a polynomial of type \( \mathcal{P} \). It then follows from Theorem 2 that \( P_d(z + t) \) is such a polynomial in \( z \) for all \( t \geq A \). A polynomial of type \( \mathcal{P} \) need not have factors of type \( \mathcal{P} \). (To verify this the reader need only examine the case \( P(z) = (z^2 + \alpha z + 1)^2 \) to find that \( P(z) \) is in the set \( \mathcal{P}_4 \) when \( \alpha^2 \geq \frac{2}{\phi} \). The factor \( z^2 + \alpha z + 1 \) is not of type \( \mathcal{P} \) unless \( \alpha^2 \geq 1 \).) To prove Theorem 1, we must show that if \( P_d(z) \) is of type \( \mathcal{P}_N \) then \( P_d(z + t) \) will also be for all \( t \geq 0 \).

**Proof of Theorem 1.** Let \( P_d(z) = \sum_{k=0}^{N} p_{ok} z^k \) be of type \( \mathcal{P}_N \) with \( p_{0N} > 0 \); \( \Delta_{ok} = p_k^2 - p_{k+1} p_{k-1} \geq 0, 0 \leq k \leq N \). We note that, for \( P_d(t) \) defined by (1.2), we have for all \( t \geq 0 \) \( p_{0}(t) = p_{0N} \), and \( \Delta_{0}(t) = p_{0N}^2 \). Further,

\[ P_d(z + t) = P_d(z + t) = \sum_{k=0}^{N} p_{ok}(z + t)^k = \sum_{k=0}^{N} \sum_{0}^{N} p_{ok}(t)z^k \]

\[ = \sum_{k=0}^{N} p_r(t)z^k. \]

Hence,

\[ p_r(t) = \sum_{k=0}^{N} (t)z^{k-r} p_{ok} = p_{0r} + (r + 1)p_{0r+1} t + \cdots. \]
and
\[
(1.12) \quad p_r'(t) = \sum_{k=0}^r (k - r)_p t^{k-r-1} p_{r+k}.
\]
We see that for \(0 \leq r \leq N - 1\), \(p_r'(t)\) is positive for \(t \geq 0\), so that \(p_r(t)\) is positive and monotonic increasing for \(t > 0\). We note in particular that \(p_r'(0) = (r + 1)p_{r+1} r\). Clearly \(P_t(z + \varepsilon) = P_t(z + t + \varepsilon)\). It follows that
\[
(1.13) \quad p_k'(t) = (k + 1)p_{k+1}(t), \quad 0 \leq k \leq N.
\]
From simple algebra we then have
\[
(1.14) \quad \frac{d}{dt} \Delta_k(t) = \frac{d}{dt} \left[ p_k^2(t) - p_{k+1}(t)p_{k-1}(t) \right] = (k + 2)[p_k(t)p_{k+1}(t) - p_{k+1}(t)p_{k-1}(t)],
\]
and this may be rewritten as
\[
(1.15) \quad p_k(t)\Delta_k'(t) = (k + 2)[p_{k+1}(t)\Delta_k(t) + p_{k-1}(t)\Delta_{k+1}(t)].
\]
Equation (1.15) may be written in vector matrix form as
\[
(1.16) \quad \frac{d}{dt} \Delta(t) = \Delta(t)A(t).
\]
When \(p_k(0) > 0\), for \(0 \leq k \leq N\), then \(p_k(t) > 0\) for all \(t \geq 0\) and the matrix \(A(t)\) has finite nonnegative components for all \(t \geq 0\). If further \(\Delta(0)\) has nonnegative components (one always has \(\Delta_N(0) = p_N^2 > 0\)), (1.16) will have a unique solution \(\Delta(t)\) and all of its components will be nonnegative as required for the log-concavity stipulated. We note from (1.11) that the vector \(p(t)\) is a continuous function of the vector \(p(0)\), and hence that \(\Delta(t)\) is a continuous function of the vector \(p(0)\). If some components of \(p(0)\) are zero, we may consider a sequence of vectors \(p^{(n)}(0)\) with all positive components converging to \(p(0)\). By the above reasoning \(\Delta^{(n)}(t)\) will be nonnegative and its limit \(\Delta(t)\) will be nonnegative.

We have established that the set \(\mathcal{T}\) of nonnegative values of \(P_t(z + t)\) is of type \(\mathcal{P}_N\) is connected and unbounded. To complete the proof of the theorem we must show that the set \(\mathcal{T}\) is closed on the left. The case \(t^* = 0\) is trivial.

Let \(t_i\) be positive and let \(P_t(z + t_i + \varepsilon)\) of type \(\mathcal{P}_N\) for all \(\varepsilon > 0\). Consider a sequence of positive \(\varepsilon_j\) converging to zero. It is known [5, 8] that convergent log-concave sequences have log-concave limits. This implies that \(P_t(z + t_i)\) is of type \(\mathcal{P}_N\) and \(\mathcal{T}\) is closed on the left.

2. A set of moment inequalities for log-concave lattice distributions. Many of the basic lattice distributions of importance to statistics and probability are log-concave sequences with connected support and \(\Delta_n = p_n^2 - p_{n+1}p_{n-1} \geq 0\) for all \(n\). A discussion of this prevalence and its underlying origins may be found in [8].

\[1\] This may also be seen from \(P_t(z) = P_0(z + t)\) so that \(\partial/\partial t P_t(z) = \partial/\partial z P_t(z)\). Hence \(\sum p_k'(t)z^k = \sum kp_k(t)z^{k-1} = \sum (k + 1)p_{k+1}(t)z^k\).
The important moment inequalities of Karlin, Proschan and Barlow (Theorem 2 of [6]) may be obtained from Theorem 1.

**Theorem 2.** Let \( \{p_n\}_0^\infty \) be a log-concave probability distribution on the lattice of nonnegative integers, and let \( \mu_r = \sum_n^\infty p_n [n(n-1) \cdots (n-r+1)] \) be its \( r \)th factorial moment [9]. Then the sequence \( \{\frac{\mu_r}{r!}\}_1^\infty \) is also log-concave. Moreover,

\[
\frac{\mu_1}{1!} \geq \left(\frac{\mu_2}{2!}\right)^{\frac{1}{2}} \cdots .
\]

Equality in (2.1) holds for all \( r \) when \( \{p_n\}_0^\infty \) is geometric, i.e., when \( p_n = (1 - \theta)\theta^n \), \( 0 \leq \theta < 1 \).

**Proof.** It is known [5], [8] that all moments of integral order \( \mu_r = \sum_n^\infty n^r p_n \) are finite for such lattice distributions. Consequently all factorial moments will be finite as well. Suppose now that \( \{p_n\}_0^\infty \) is log-concave, i.e., is such a distribution with all support on \( 0 \leq n \leq N \). Then \( P(z) = \sum_0^N p_n z^n \), the probability generating function for \( \{p_n\}_0^\infty \) is a polynomial of the type described in Theorem 1 for which the value \( r^* = 0 \). Hence by Theorem 1, \( P(z+1) = P_t(z) \) also has log-concave coefficients. But it is known that \( P(z+1) \) is the generating function for the factorial moments, i.e. one has [9]

\[
P(z+1) = \sum_0^\infty \{\frac{\mu_r}{r!}\} z^r .
\]

This relationship may also be seen directly from (1.10) for \( t = 1 \). Consequently

\[
\frac{\{\frac{\mu_r}{r!}\}^2}{r!} \geq \frac{\mu_{r+1}}{(r+1)!} \frac{\mu_{r-1}}{(r-1)!} ; \quad 1 \leq r .
\]

This together with \( p_0(1) = 1 \) implies (2.1) in the classical way, for lattice distributions on \( 0 \leq n \leq N \). For a log-concave distribution \( \{p_n\}_0^\infty \) one considers the sequence of distributions \( \{p_n^*\}_0^\infty \) where \( p_n^* = p_n/(p_0 + p_1 + \cdots + p_N) \) which converges to the given distribution. It is known [6] that log-concavity is unaffected by truncation and convergence so that (2.3) and (2.1) continue to hold.

That the inequality bounds are tight may be seen for the geometric case. Here \( P(z) = (1 - \theta)/(1 - \theta z) \) and

\[
P(z+1) = \frac{1 - \theta}{1 - \theta - \theta z} = \sum_0^\infty \left(\frac{\theta}{1 - \theta}\right)^k z^k
\]

so that

\[
\frac{\mu_r}{r!} = \left(\frac{\theta}{1 - \theta}\right)^r ; \quad 1 \leq r
\]

and the inequalities in (2.1) become equalities. This completes the proof of Theorem 2. \( \Box \)

The most important inequality for statistics is the case \( r = 2 \). One has for the ordinary moments \( \mu_r = \sum n^r p_n \) the following.
Corollary. If \( \{ p_n \}_0^\infty \) is a log-concave distribution, then

\[
\mu_2 \leq 2\mu_1^2 + \mu_1.
\]

Remark. The set of inequalities (2.1) does not characterize log-concavity. One may have the full set of inequalities (2.1), but \( \{ p_n \}_0^\infty \) need not be log-concave. The simplest example is provided by \( p_0 = (1 + \lambda)^{-1}, p_1 = 0, p_2 = \lambda(1 + \lambda)^{-1} \) for which \( P(z) = (1 + \lambda z^2)/(1 + \lambda) \). Then \( P(z + 1) = ((1 + \lambda) + 2\lambda z + \lambda z^2)/(1 + \lambda) \) and \( P(z) \in \mathcal{P}_3 \) if \( \lambda \geq \frac{1}{3} \).

3. Continuous distributions. The inequalities of (0.4) may be obtained from those of (0.1) with the aid of a simple lemma, and associated limiting argument.

Lemma. Let \( f(x) \) be log-concave with purely positive support and let \( F(x) = \int_0^x f(y) \, dy \) be its cumulative distribution function. Then the lattice distribution \( \{ g_n(a) \}_0^\infty \) where

\[
g_n(a) = F(na) - F(na - a)
\]

is log-concave for every \( a > 0 \).

Proof. Since \( f(x) \) is a log-concave function [4], and since the convolution of two such functions is also such a function [4], then

\[
g(a, x) = \int_0^x f(x - u) \, du
\]

is such a function. Moreover \( g(a, x) \) is continuous in \( x \) and its interval of positivity is connected. Consequently \( g(a, x + a)/g(a, x) \) is monotonic decreasing in the interior of the support interval \( I \) as \( x \) increases and \( \{ g(a, na) \}_0^1 \supseteq g(a, na + a)g(a, na - a) \). Hence \( \{ g(a, na) \}_0^\infty \) is log-concave. The lemma follows from the identification \( g_n(a) = g(a, na) \).

If we define \( \mu_{(K)}(a) \) to be the factorial moment of the lattice distribution with masses \( g_n(a) \), it is easy to establish that \( a^K \mu_{(K)}(a) \rightarrow \mu_K = \int x f(x) \, dx \), as \( a \rightarrow 0 + \), and to infer (0.4) from (0.1) thereby. The weak convergence of \( F_n(x) \) to \( F(x) \) and the standard lemma (Feller, II, page 245) provide the desired result.

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