RATES OF CONVERGENCE FOR WEIGHTED SUMS OF RANDOM VARIABLES

BY F. T. WRIGHT

The University of Iowa

For $N = 1, 2, \ldots$ let $S_N = \sum_{k=1}^{N} a_{N,k} X_k$ where $a_{N,k}$ is a real number for $N, k = 1, 2, \ldots$ and $\{X_k\}$ is a sequence of not necessarily independent random variables. For the case $0 < t < 1$, with assumptions closely related to $E |X_k|^t < \infty$ it is shown that the rate of convergence of $P(|S_N| > \varepsilon)$ to zero is related to $\sum_k |a_{N,k}|^t$. The theorems presented here extend some of the results in the literature to not necessarily independent sequences $\{X_k\}$.

1. Introduction and summary. Let $X_k$ for $k = 1, 2, \ldots$ be a sequence of random variables (not necessarily independent), let $a_{N,k}$ for $N, k = 1, 2, \ldots$ be real numbers, let $0 < t < 1$ and $\rho_N$ be a sequence of positive numbers such that $\sum_k |a_{N,k}|^t \leq \rho_N$, and let $S_{N,M} = \sum_{k=1}^{M} a_{N,k} X_k$ for $N, M = 1, 2, \ldots$. In Section 2 with assumptions closely related to $E |X_k|^t < \infty$, we show that for each $N$, $S_{N,M}$ has an almost sure limit $S_N$ as $M \to \infty$ and that the rate at which $P(|S_N| > \varepsilon)$ converges to zero is related to $\rho_N$. We conclude with some remarks about the case $t = 1$.

The results of this paper are similar to those of [2], [3], [4], and [7]. In the references cited above the random variables were assumed to be independent. However in [4] it was observed that Theorems 1a and 2a of that paper were valid if the assumption of independence was omitted. Since Theorems 1a and 2a of [4] were generalizations of Theorems 1 and 2 of [2], the question is raised as to whether Theorems 3 and 4 of [2] can be generalized to include dependent sequences $\{X_k\}$ for $0 < t < 1$. In [7] Theorem 4 of [2] was generalized to the case $0 < t < 1$ but the sequence $\{X_k\}$ was still assumed to be independent. The above question is answered in the affirmative by Theorems 3 and 4 of this paper.

2. Results. Using the notation of Section 1, define for $y \geq 0$,

$$F_k(y) = P(|X_k| \geq y) \quad \text{and} \quad F(y) = \sup_k F_k(y).$$

Throughout this paper $C$ will denote various positive constants whose exact values do not matter. Where appropriate, summations will be taken over those values of $k$ for which $a_{N,k} \neq 0$ and integrals will be Lebesgue-Stieltjes integrals.

We now prove the following

**Lemma.** If $y^t F(y) \leq B < \infty$ for all $y > 0$, then for each $N$ as $M \to \infty$ $S_{N,M}$ has an a.s. limit which we will denote by $S_N$.

**Proof.** We define $Y_{N,k} = X_k I_{|a_{N,k} X_k| < 1}$ and observe that for each $N$ and for

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each $\varepsilon > 0$

(1) \[ P(\sup_{j \geq 1} |S_{N,M+j} - S_{N,M}| > \varepsilon) \leq \sum_{k=M+1}^{N} P(a_{N,k} X_k \geq 1) + P(\sup_{j \geq 1} |\sum_{k=M+1}^{N+j} a_{N,k} Y_{N,k}| > \varepsilon). \]

Since $y^d F(y)$ is bounded for all $y > 0$, we see that the second expression in (1) is bounded by $C \sum_{k=M+1}^{N} |a_{N,k}|^t$. Using the inequality of Theorem 1 of [5] with $c_i \equiv 1$ and $r = 1$, the third expression in (1) is bounded by

\[ C \sum_{k=M+1}^{N} |a_{N,k}| \int_{0}^{1} |a_{N,k}|^{-1} x |dF_k(x)| \]

(where the last integral is taken with respect to the Lebesgue-Stieltjes measure corresponding to $-F_k$). Integrating by parts we see the last expression is bounded by

\[ C \sum_{k=M+1}^{N} |a_{N,k}| \int_{0}^{1} |a_{N,k}|^{-1} x^{-t} dt \]

So expression (1) tends to zero as $M \to \infty$ and hence $S_{N,M}$ has an almost sure limit (cf. page 115 of [6]).

The following theorems give rates of convergence for $P(|S_N| > \varepsilon)$.

**Theorem 1.** If $y^d F(y) \leq B < \infty$ for all $y > 0$, then for every $\varepsilon > 0$

\[ P(|S_N| > \varepsilon) = O(\rho_N). \]

**Theorem 2.** If $y^d F(y) \to 0$ as $y \to \infty$ and if $\max_k |a_{N,k}| \to 0$ as $N \to \infty$, then for every $\varepsilon > 0$

\[ P(|S_N| > \varepsilon) = o(\rho_N). \]

For Theorems 3 and 4 we assume that $\rho_N$ is of the form $CN^{-\rho}$ and hence there exists a constant $\beta$ such that

(2) \[ \max_k |a_{N,k}| \leq CN^{-\beta}. \]

For Theorem 3 let $s$ be a constant such that $0 < s < t$ and let $\alpha$ be a constant such that $\sum_k |a_{N,k}|^s \leq CN^s$. As in [4] it can be shown that we may assume $\beta \geq \rho/t$, $\beta \geq -\alpha/s$, and $\rho \geq \beta(t - s) - \alpha$.

**Theorem 3.** If $\beta > 0$ and if $F$ satisfies

(3) \[ \lim_{y \to \infty} F(y) = 0 \quad \text{and} \quad \int_{0}^{\infty} y^d |dF(y)| < \infty, \]

then for every $\varepsilon > 0$

\[ \sum_{N} N^{\beta(l-s)-\alpha-1} P(|S_N| > \varepsilon) < \infty. \]

**Theorem 4.** If $\beta > 0$ and if there exists a non-increasing real valued function $G(x)$ satisfying (3) and such that $G(x) \geq F(x)$ for all $x > 0$ and

(4) \[ \sup_{x \geq 1} |\sup_{y \geq x} y^d F(y)/(x^d G(x))| < \infty, \]
then for every $\varepsilon > 0$

$$\sum_N N^{\alpha - 1} P(|S_N| > \varepsilon) < \infty.$$  

Note. Theorem 1 was proved in [4] and has been included here for completeness. Theorem 2 was proved in [4] under the assumption that $\rho_N \to 0$ as $N \to \infty$; however, examining that proof we see that the weaker assumption $\max_k |a_{N,k}| \to 0$ as $N \to \infty$ would suffice. Rohatgi in [7] extended Theorem 4 of [2] to the case $0 < t < 1$, but did not give an extension of Theorem 3 of [2]. Theorem 4 of this paper extends the above work to dependent sequences $\{X_k\}$ and removes assumption (6) of [7]. Theorem 3 of this paper extends Theorem 3 of [2] to the case $0 < t < 1$ and to not necessarily independent sequences $\{X_k\}$.

We now prove Theorems 3 and 4.

Proofs. First we observe that

$$P(|S_N| > \varepsilon) \leq \sum_k F(|a_{N,k}|^{-1}) + P(|\sum_k a_{N,k} Y_{N,k}| > \varepsilon)$$

where $Y_{N,k}$ is defined as in the proof of the lemma. The proofs of Theorems 3 and 4 are completed by showing that the last two expressions in (5) behave as specified in the theorems.

To show that the second expression in (5) behaves as specified in Theorem 3 one only needs to mimic the proof given for Theorem 3 of [3] found on pages 446 and 447. For Theorem 4 the argument on pages 351 and 352 of [2] suffices. It should be noted that the two arguments cited do not require $\beta(t - s) - \alpha > 0$ or $\rho > 0$ but only that $\beta > 0$.

In considering the last expression in (5), we define $\delta_{N,M} = \text{card.} \{k : M^{-1} \leq |a_{N,k}|\}$ for $N, M = 1, 2, \ldots$. Using the Markov Inequality we see that

$$\sum_N N^{\beta(t-s)-\alpha-1} P(|\sum_k a_{N,k} Y_{N,k}| > \varepsilon)$$

$$\leq C \sum_N N^{\beta(t-s)-\alpha-1} E \sum_k a_{N,k} Y_{N,k}$$

$$\leq C \sum_N N^{\beta(t-s)-\alpha-1} \sum_k |a_{N,k}| \sum_{j \geq 1} 1_{\{a_{N,k} \leq \varepsilon^{-1} j\}} |x| dF(x)$$

$$\leq C \sum_N N^{\beta(t-s)-\alpha-1} \sum_{j \geq 1} 1_{\{a_{N,k} \leq \varepsilon^{-1} j\}} F(x) dx$$

$$\leq C \sum_N N^{\beta(t-s)-\alpha-1} \sum_{j \geq 1} 1_{\{a_{N,k} \leq \varepsilon^{-1} j\}} F(x) dx$$

$$+ C \sum_N N^{\beta(t-s)-\alpha-1} \sum_{j \geq 1} (\delta_{N,M} - \delta_{N,M-1})(M - 1)^{1-\beta} \sum_{j \geq 1} F(x) dx$$

where the prime on the summation in expression (7) indicates it is to be taken over those values of $k$ for which $|a_{N,k}| \geq 1$. Since $\beta > 0$ expression (7) is finite. Expression (8) is bounded by

$$C \sum_N N^{\beta(t-s)-\alpha-1} \sum_{j \geq 1} (\delta_{N,M} - \delta_{N,M-1})(M - 1)^{1-\beta} \sum_{j \geq 1} F(j - 1)$$

$$\leq C \sum_{j \geq 1} F(j - 1) \sum_{M = j}^{\infty} M^{-2} \sum_{N = 1}^{\infty} N^{\beta(t-s)-\alpha-1} \delta_{N,M}.$$  

We now obtain estimates for $\delta_{N,M}$. Since $\sum_k |a_{N,k}|^\alpha \leq CN^\alpha$ and $\max_k |a_{N,k}| \leq CN^{-\beta}$, we see that $\delta_{N,M} = 0$ unless $N \leq CM^\beta$ and $\delta_{N,M} \leq CN^\alpha M^\beta$. Therefore
(9) is bounded by
\[
C \sum_{j=1}^{\infty} F(j-1) \sum_{M=j}^{\infty} M^{-(2-\kappa)} \sum_{N=1}^{[CN^{1/\theta}]} N^{\theta(t-\kappa)-1} \\
\leq C \sum_{j=1}^{\infty} j^{-1} F(j-1) \leq C + C \sum_{j=1}^{\infty} j^{-1} F(j) \\
\leq C + C \int_0^\infty x^t |dF(x)| < \infty.
\]

For Theorem 4 an argument similar to the one beginning at (6) shows that it is sufficient to consider
\[
(10) \quad \sum_N N^{-\theta} \sum_k |a_{N,k}| \int_0^{\infty} x^{t-1} F(x) \, dx.
\]
From (2) we see that there exists a positive constant \(A\) such that \(|a_{N,k}|^{-1} \geq AN^\theta\) for \(N, k = 1, 2, \ldots\). Expression (10) is equal to
\[
(11) \quad \sum_N N^{-\theta} \sum_k |a_{N,k}| \int_0^{\infty} F(x) \, dx + \sum_N N^{\theta-1} \sum_k |a_{N,k}| \int_0^{\infty} x^{t-1} F(x) \, dx.
\]
The first expression in (11) is bounded by
\[
(12) \quad C \sum_N N^{-1-\theta(1-t)} \sum_k \int_0^{\infty} F(A(k-1)^\theta) [k^\theta - (k - 1)^\theta] \, dx.
\]
Applying the Mean Value Theorem to the function \(1 - (1 - x)^\theta\), one can show that there exists a constant \(C\) depending only on \(\theta\) such that \(k^\theta - (k - 1)^\theta \leq CK^{\theta-1}\). Hence expression (12) is bounded by
\[
C \sum_{k=1}^{\infty} k^{\theta-1} F(A(k-1)^\theta) \sum_{N=1}^{\infty} N^{-1-\theta(1-t)} \sum_k \int_0^{\infty} F(A(k+1)^\theta) - F(A(k-1)^\theta) \, dx \leq C + C \int_0^{\infty} x^t |dF(x)| < \infty.
\]
Choose \(N_0\) so that \(AN^\theta \leq 1\). Using (4) we see that
\[
\sum_{N=N_0}^{\infty} \sum_k |a_{N,k}| \int_0^{\infty} x^{t-1} F(x) \, dx \leq C \sum_{N=N_0}^{\infty} N^{\theta+\theta-1} G(N^\theta) \sum_k |a_{N,k}| \int_0^{\infty} x^{t-1} \, dx \leq C \sum_{N=1}^{\infty} N^{\theta-1} G(N^\theta) \leq C \int_0^{\infty} x^t |dG(x)| < \infty.
\]
We have shown that the second expression in (11) is finite and the proofs are completed.

In [3] (see Theorem 6), it was shown that \(\int_0^{\infty} x^t \log x \, |dF(x)|\) finite implies the existence of the hypothesized \(G\) of Theorem 4. In [3] and [4] the sharpness of these theorems has been investigated for sequences of independent random variables.

For the case \(t = 1\) it was shown in [4] that Theorems 1 and 2 are not valid for independent random variables even if it is assumed that \(\rho_N \to 0\) as \(N \to \infty\). However for \(t = 1\) it was shown that with the additional hypotheses that \(\rho_N \to 0\) as \(N \to \infty\) and \(\lim \sup_{r \to \infty} \sup_k \left\{ \int_{[-r,r]} x \, dP(X_k \leq x) \right\} < \infty\) the conclusions of Theorems 1 and 2 hold for independent variables. The following example shows that this is not the case for dependent variables.

**Example.** Let \(Z, Y_1, Y_2, \ldots\) be independent random variables such that \(P(Z = -1) = P(Z = 1) = \frac{1}{2}\), \(Y_1, Y_2, \ldots\) are identically distributed, \(P(Y_1 \geq 0) = 1\), \(yP(Y_1 \geq y) \to 0\) as \(y \to \infty\), and \(EY_1 = \infty\). Set \(X_k = ZY_k\) for \(k = 1, 2, \ldots\). Clearly \(yP(y) \to 0\) as \(y \to \infty\) and \(\int_{[-r,r]} x \, dP(X_k \leq x) = 0\) for all \(T\) and \(k = \ldots\)
1, 2, \ldots. Now $N^{-1} \sum_{k=1}^{N} Y_k \to_{a.s.} \infty$ since $EY_1 = \infty$ and so there exists a sequence of positive numbers $\delta_N \to \infty$ such that
\[ P(|N^{-1} \sum_{k=1}^{N} Y_k| > \delta_N) \geq \frac{1}{2} \quad \text{for } N = 1, 2, \ldots. \]

Let $a_{N,k}$ be $(N\delta_N)^{-1}$ for $1 \leq k \leq N$ and zero for $k > N$. For this example $\rho_N = \delta_N^{-1}$ and $\rho_N^{-1}P(|S_N| > 1) \to \infty$.

In [1] an example of a stationary ergodic sequence $X_k$ was given for which $EX_1 = 0$, $|X_k| = 1$, and
\[ \sum_{N} N^{-1}P(|N^{-1} \sum_{k=1}^{N} X_k| > \varepsilon) = \infty. \]

Hence Theorems 3 and 4 do not hold for $t = 1$ if $\rho = 0$. It would be of interest to know if they hold in the case $t = 1$ for $\beta(t - s) - \alpha > 0$ or $\rho > 0$, respectively.

REFERENCES