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CONSTRUCTION OF TWO-LEVEL SATURATED SYMMETRICAL FACTORIAL DESIGNS OF RESOLUTION VI

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In this paper we have established the maximum number of constraints in a two-level orthogonal array of 192 runs of resolution VI.

1. Introduction. The problem of constructing fractional replicates of 2^m designs is not new in the literature. However, so far as the subject matter of this paper is concerned, the contributions made by Bose and Bush [3], Seiden and Zemach [11] and very recently by Draper and Mitchell [4], [5], [6] are of special interest. The basic concept pertains to the maximum number of factors that can be accommodated in a two-level symmetrical factorial design in which blocks are of size 2^m and no main effect or t-factor or lower order interaction is confounded. Since saturated designs of resolution VI with 2^{5+q} runs, $q = 1, 2, 3, 4, 5$, have been constructed by Draper and Mitchell, we have considered it worthwhile to undertake a similar investigation pertaining to the maximum number of variables in 192 runs resolution VI designs.

Let λ, $t\ (\geqq 2)$, and $k\ (\geqq t)$ be positive integers. Let A be a $k \times \lambda 2^t$ matrix of zeros and ones. Let B be a $t \times \lambda 2^t$ submatrix of A. Then A is called a two-level orthogonal array if, and only if, for each choice of t, each of 2^t possible column vectors occurs exactly λ times. An orthogonal array A of the weight λ, strength t and k constraints, may symbolically be denoted by $(\lambda 2^t, k, t)$.

If $k \times \lambda 2^t$ matrix A is of strength t, so is any subarray of k' rows if $t \leqq k' \leqq k$. Hence nonexistence of $(\lambda 2^t, k', t)$ implies the nonexistence of $(\lambda 2^t, k, t)$ if $k > k'$. Again, it may be observed that if a $k \times \lambda 2^t$ matrix A is of strength t, it is also of strength t' for all $t' \leqq t$. In a symmetrical factorial design with k factors each operating at two levels, $\lambda 2^t$ columns of an orthogonal array may be identified with treatment combinations or runs; k factors correspond to k rows of the array while an entry stands for the level of the factor against which it is shown. These $\lambda 2^t$ runs constitute a subset of 2^k possible treatment combinations needed in a complete factorial design.

2. Construction of orthogonal arrays. A necessary condition for an array to be orthogonal [3] is that for each nonnegative integer h not exceeding t, k must satisfy the following $t + 1$ linear inequalities.

(2.1.1) $$\textstyle\sum_{j=1}^{j=k} n_{ij} = \lambda 2^t - 1$$

(2.1.2) $$\textstyle\sum_{j=1}^{j=k} n_{ij} = C_h^{\,k}(\lambda 2^{t-h} - 1)\,, \qquad 1 \leqq h \leqq t$$

where $C_h^{\,j} = 0$ for $j < h$ and n_{ij} denotes the number of columns (other than ith)

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that have j coincidences with ith column of an orthogonal array, $i = 1, 2, \cdots, \lambda 2^t$; $j = 0, 1, \cdots, k$.

Recently, Blum, Schatz and Seiden [2] proved the following theorem.

THEOREM 2.1. *If λ is odd and $\lambda \leqq t - 1$, then the maximum number of variables in a resolution $(t + 1)$-design is exactly $t + 1$.*

Consider any $t + 1$ tuple and adjoin it to all the $2^t - 1$ $(t + 1)$-tuples that differ from it by an even number of elements. It is readily seen that for t odd, the columns of the array consist of either an even or an odd number of both zeros and ones. The array with the first column consisting of all zeros will be denoted by D whereas D^* will represent an array with an odd number of both zeros and ones. For $t = 5$, the two forms of the array (32, 6, 5) are given below:

D:

0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1
0 0 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 0 0 1
0 1 0 1 1 1 0 1 1 1 0 0 0 1 1 1 0 1 0 0 0 1 0 0 0 1 1 1 0 0 0 1
0 1 1 0 1 1 1 0 1 1 0 1 1 0 0 1 0 0 1 0 0 0 1 0 0 1 0 0 1 1 0 1
0 1 1 1 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 1 0 0 0 1 0 0 1 0 1 0 1 1
0 1 1 1 1 0 1 1 1 0 1 1 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 1 1 1

D^*:

0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0
1 0 1 1 1 1 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 1 1 1 1 0 1 0 0 0 0
1 1 0 1 1 1 0 1 1 1 0 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1 0 0 1 0 0 0
1 1 1 0 1 1 1 0 1 1 0 1 1 0 0 1 0 1 1 0 0 1 0 0 1 0 0 0 0 1 0 0
1 1 1 1 0 1 1 1 0 1 1 0 1 0 1 0 1 0 1 0 1 0 0 1 0 0 0 0 0 0 1 0
1 1 1 1 1 0 1 1 1 0 1 1 0 1 0 0 1 1 0 1 0 0 1 0 0 0 0 0 0 0 0 1

THEOREM 2.2. *For an orthogonal array* (96, k, 5), *the maximum value of k is six.*

This is a trivial consequence of Theorem 2.1. The array can be expressed as a juxtaposition of three arrays (32, 6, 5) each of unit weight.

THEOREM 2.3. *For an orthogonal array* (160, k, 5), *the maximum value of k is seven.*

Propositions 2.3 and 3.6 in [11], taken together, immediately imply this assertion.

THEOREM 2.4. *For an orthogonal array* (192, k, 5), *the maximum possible value of k is seven.*

We will enumerate first all the solutions of (2.1.1) and (2.1.2) for $k = 6$ and 7 and observe that each of these solutions represents an array.

Arrays admitting each of the solutions for $k = 6$ will not be exhibited separately since each of them is a subarray of an array for $k = 7$. Accordingly, we need to investigate which of the arrays with $k = 6$ can be extended to arrays with seven constraints.

TABLE 1

	n_{i0}	n_{i1}	n_{i2}	n_{i3}	n_{i4}	n_{i5}	n_{i6}	n_{i7}
6.1	1	30	15	100	15	30	0	
6.2	2	24	30	80	30	24	1	
6.3	3	18	45	60	45	18	2	
6.4	4	12	60	40	60	12	3	
6.5	5	6	75	20	75	6	4	
6.6	6	0	90	0	90	0	5	
7.1	0	19	12	75	40	33	12	0
7.2	1	13	27	55	55	27	13	0
7.3	2	7	42	35	70	21	14	0
7.4	3	1	57	15	85	15	15	0
7.5	0	20	6	90	20	48	6	1
7.6	1	14	21	70	35	42	7	1
7.7	2	8	36	50	50	36	8	1
7.8	3	2	51	30	65	30	9	1
7.9	0	21	0	105	0	63	0	2
7.10	1	15	15	85	15	57	1	2
7.11	2	9	30	65	30	51	2	2
7.12	3	3	45	45	45	45	3	2

We will assume throughout our discussion, unless stated otherwise, that the first column consists of all 0's and the array admits the solution under consideration in respect to the first column. The proof of the theorem and an effective construction of the arrays will require several steps.

(i) That the array 6.1 can be extended only to 7.1 and 7.2 follows from the fact that $n_{i0} = 1$ and $n_{i6} = 0$ in the solution 6.1; its extension must therefore have $n_{i0} \leqq 1$ and $n_{i7} = 0$. The solution 7.1 may be represented as under:

$$\begin{matrix} D & D^* & D^* & D^* & D^* & D^* \\ 1 & 0 & 0 & 0 & 1 & 1 \end{matrix} .$$

Zero or one below the array means that, to each column of the array the same element, either 0 or 1, is added. The structure of the array 7.2, as extended from 6.1, is complex. A completed array satisfying the solution 7.2 is given on the next page.

(ii) The array 6.2 is extended to the solutions 7.2 and 7.5 only. Clearly, seven-rowed arrays with $n_{i0} = 3$ cannot be considered for possible extension of the six-rowed array with $n_{i0} = 2$. Further, since arrays 7.10 and 7.11 are obtainable from the solutions 7.4 and 7.8 respectively by a permutation of 0 and 1, it follows that the array 6.2 cannot be extended to 7.4 and 7.8. One can similarly obtain 7.6 from the solution 7.3. Thus, it remains to be shown that 6.2 cannot be extended to 7.1, 7.3, 7.7 and 7.9.

1	00000	11111	11111	11111	11111	11111	000001111111111	00000	00000	00000	00000
1	11111	00000	11111	11111	11111	11111	011110000111111	00000	00000	00000	00000
1	11111	11111	00000	11111	11111	11111	101110111000111	00000	11111	11111	11111
1	11111	11111	11111	00000	11111	11111	110111011011001	11111	00000	11111	11111
1	11111	11111	11111	11111	00000	11111	111011101101010	11111	11111	00000	11111
1	11111	11111	11111	11111	11111	00000	111101110110100	11111	11111	11111	00000
1	11100	11000	11000	11100	11000	11000	011111111000011	11000	11000	11000	11000

00000	00000	00000	00000	00000	00000	11111	11111	11111	11111	11111	11111	11111
11111	11111	11111	11111	11111	11111	00000	00000	00000	11111	11111	11111	11111
00000	00000	00000	11111	11111	11111	11111	11111	11111	11111	00000	00000	00000
00000	11111	11111	00000	11111	00000	00000	00000	11111	00000	11111	00000	00000
11111	00000	11111	11111	00000	00000	00000	11111	00000	00000	00000	11111	00000
11111	11111	00000	00000	00000	11111	11111	00000	00000	00000	00000	00000	11111
00000	00000	00000	00000	00000	11111	00000	11111	11111	11100	11100	11100	11100

11111	11111	11111	111110000000000	11111	00000	00000	00000	00000	00000
00000	00000	00000	100001111000000	00000	11111	00000	00000	00000	00000
00000	00000	00000	010001000111000	00000	00000	11111	00000	00000	00000
11111	00000	11111	001000100100110	00000	00000	00000	11111	00000	00000
00000	11111	11111	000100010010101	00000	00000	00000	00000	11111	00000
11111	11111	00000	000010001001011	00000	00000	00000	00000	00000	11111
11111	11111	11111	100000000001111	00011	00111	00111	00111	00111	00111

The first six rows of the following matrix are part of the array 6.2:

			(a)				(b)
11	0000	1111	1111	1111	1111	1111	0000000000
11	1111	0000	1111	1111	1111	1111	0011111111
11	1111	1111	0000	1111	1111	1111	1100111111
11	1111	1111	1111	0000	1111	1111	1111001111
11	1111	1111	1111	1111	0000	1111	1111110011
11	1111	1111	1111	1111	1111	0000	1111111100

The extended row would have two 0's added to the first two columns if 6.2 is extended to 7.1. This necessitates $7n_1$'s (going in n_2's) to be divided into six sets consisting of four columns having zeros in different rows. This, in turn, implies an existence of at least one set that has to have two zeros in the extended row. Further, since $n_{i2} = 12$ in 7.1, five columns of 6.2 that have two zeros in each column would have 1's added in the extended row. Without any loss of generality, we may assume that the last two columns of (a) and columns 1, 3, 5, 7, and 9 in (b) have 0's and 1's in the extended row while the remaining columns in (a) and (b) have 1's and 0's respectively. If the first two rows are dropped, there results a five tuple $(11111)'$ appearing seven times, a contradiction.

Suppose that the array 6.2 is extended to 7.9. As before, we would have two 0's in the first two columns in the extended row. Of $24n_1$'s in 6.2, 19 would have to have 1's added while the remaining five columns have 0's in the extended row. This gives $n_{i2} > 0$, but 7.9 has $n_{i2} = 0$.

Consider next a possibility of an extension of 6.2 to 7.3. The first two columns would have 1's in the extended row. Further, since 6.3 is a subarray of 7.3, we have to have a 1 and three 0's in columns four through six so that dropping of the first row results in a subarray with $n_{i0} = 3$. Of $17n_1$'s going in n_2's, selection of three 0's in columns four through six leaves $14n_1$'s to be distributed in the remaining five sets in (a). This demonstrates an existence of at least three sets that have to have zeros in the extended row. This shows that $n_{i7} \geqq 2$, contradicting that 7.3 has $n_{i7} = 0$.

An array satisfying 7.7 must satisfy 7.3 in respect to at least one of the columns having one coincidence with the ith column, since $n_{i1} = 8$ in 7.7. Since 6.2 cannot be extended to the array 7.3 and 7.7 must include a column in respect to which it satisfies 7.3, it is clear that 6.2 cannot be extended to 7.7.

The arrays 7.2 and 7.5, when extended from 6.2, are of the following nature:

$$7.2: \begin{matrix} D & D & D^* & D^* & D^* & D^* \\ 1 & 0 & 1 & 1 & 0 & 0 \end{matrix} \qquad 7.5: \begin{matrix} D & D & D^* & D^* & D^* & D^* \\ 0 & 0 & 1 & 1 & 1 & 0 \end{matrix}.$$

(iii) The array 6.3 can be extended to all the seven-rowed arrays 7.1 through 7.12. This can easily be verified by dropping an appropriate row from each array.

(iv) An array satisfying solution 6.4 can be extended to an array with seven constraints with $n_{i0} + n_{i1} \leqq 16$. Further, since 7.3 and 7.11 are obtainable from 7.6 and 7.8 by interchanging the two elements 0 and 1, it suffices to show that 6.4 cannot be extended to the arrays 7.6 and 7.8. Consider now an array 7.2. If the first row is dropped, a six-rowed array remains with $n_{i0} = 4$ and $n_{i1} = 16$, contradicting that $n_{i1} = 12$ in the solution 6.4. It may also be observed that the arrays 7.6 and 7.11 come only from the subarray 6.3 while an array 7.12 is extended only from the solutions 6.3 or 6.5. We now give below arrays satisfying solutions 7.4, 7.7 and 7.10 as extended from the array 6.4.

7.4						7.7						7.10					
D	D	D	D	D^*	D^*	D	D	D	D	D^*	D^*	D	D	D	D	D^*	D^*
1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	1	1	1

(v) The array 6.5 can be extended to 7.8 and 7.11.

Since an extension of 6.5 must have $n_{i0} + n_{i1} \leqq 11$, it is clear that 7.1, 7.2, 7.5, 7.6, 7.9 and 7.10 cannot be considered as possible extensions of 6.5. Further, arrays 7.3 and 7.4 can be obtained from 7.6 and 7.10 by interchanging two elements.

An array 6.5 has five columns of all 1's, one column of each possible type with a zero in a different row as in (a), and one column of every type with a 1 in a different row as in (c) respectively. Seventy-five columns of n_{i4} in 6.5 can be split into 15 sets so that each set of five identical columns has four coincidences with the column af all 0's. The first six rows of the following matrix are part of such an array and the seventh row is the extension, if possible:

(a)			(b)			(c)	
11111	011111		11111	00000		100000	00000
11111	101111		11111	00000		010000	00000
11111	110111		00000	00000		001000	00000
11111	110111	...	00000	00000	...	000100	00000
11111	111101		00000	11111		000010	00000
11111	111110		00000	11111		000001	00000
11000	111110		00011	00111		000001	11100

The elements in the extended row are determined by the solution 7.7. Thirty-five columns of n_{i4} that have to have zeros in the extended row could be obtained by adding two 0's in any of the ten sets and three 0's in the remaining five sets. We may assume that the first set in (b) has three 0's and two 1's in the extended row while the second set has two 0's and three 1's added. If the sixth row is dropped, there results a six-rowed subarray with $n_{i0} = 2$ which should, therefore, satisfy the solution 6.2. But we have noticed earlier that the array 6.2 cannot be extended to 7.7. In case any row, one through five, is deleted, a six-rowed array remains with $n_{i0} = 3$ and thus satisfies the solution 6.3. Since each column with five 0's in 6.3 is repeated three times, we have three 1's and two 0's in the extended row in the last five columns. If any one of the first two rows is deleted, a six tuple (100000)′ appears four times, a contradiction. The fact that 6.5 cannot be extended to 7.12 follows from a similar argument. It now remains to be shown that one can in fact construct arrays satisfying solutions 7.8 and 7.11.

7.8						7.11					
D	D	D	D	D	D^*	D	D	D	D	D	D^*
1	1	1	0	0	0	0	0	0	1	1	1

(vi) The array 6.6 can be extended to 7.12.

Any extension of 6.6 would have to have $n_{i0} + n_{i0} \leqq 6$. This leaves 7.4 and 7.8 for further consideration, but these are obtained from 7.10 and 7.11 by permuting 0 and 1. We demonstrate below the array 7.12 as extended from 6.6.

D	D	D	D	D	D
0	0	0	1	1	1

REMARK. Before completing the proof of this theorem, we wish to note that one can always construct an array (64, 7, 6) and repeat it three times. That the array so obtained cannot be extended follows from Theorem 2.1. The main point of our proposition is to show that *none* of the seven rowed arrays can be extended further. We will, however, eliminate arrays 7.6, 7.10 and 7.11 from further considerations since these are equivalent to 7.3, 7.4 and 7.8 respectively in the sense that one can be obtained from the other by permuting the elements 0 and 1.

TABLE 2

	n_{i0}	n_{i1}	n_{i2}	n_{i3}	n_{i4}	n_{i5}	n_{i6}	n_{i7}	n_{i8}
8.1	0	8	25	18	95	4	39	2	0
8.2	0	9	19	33	75	19	33	3	0
8.3	1	3	34	13	90	13	34	3	0
8.4	1	4	28	28	70	28	28	4	0
8.5	0	11	7	63	35	49	21	5	0
8.6	1	5	22	43	50	43	22	5	0
8.7	0	12	1	78	15	64	15	6	0
8.8	1	6	16	58	30	58	16	6	0
8.9	2	0	31	38	45	52	17	6	0
8.10	1	7	10	73	10	73	10	7	0
8.11	2	1	25	53	25	67	11	7	0
8.12	2	2	19	68	5	82	5	8	0
8.13	0	12	2	72	30	44	30	0	1
8.14	1	6	17	52	45	38	31	0	1
8.15	2	0	32	32	60	32	32	0	1
8.16	1	7	11	67	25	53	25	1	1
8.17	2	1	26	47	40	47	26	1	1
8.18	1	8	5	82	5	68	19	2	1
8.19	2	2	20	62	20	62	20	2	1
8.20	2	3	14	77	0	77	14	3	1

In order to investigate whether any of the seven-rowed arrays can be extended further, we enumerate in Table 2 solutions of (2.1.1) and (2.1.2) for $k = 8$.

LEMMA 2.4.1. *The solution* 8.4 *does not represent an orthogonal array.*

PROOF. An array satisfying the solution 8.4 has one column of ones, four columns with one zero in a different row and twenty-eight columns of each type having two coincidences with the column of all zeros. The following thirty-three columns constitute a part of the solution:

$$\begin{array}{ccccccccccccccccccccccccccccccccc}
1&0&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&0&1&1&1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&0&1&1&1&1&1&0&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&1&1&0&1&1&1&1&1&0&1&1&1&1&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&0&1&1&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&1&0&0&0&1&1&1\\
1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&0&1&1&0&0&1\\
1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&0&1&0&1&0\\
1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&0&1&1&1&1&0&1&1&1&0&1&1&0&1&0&0
\end{array}$$

If any one of the rows 1 through 4 is deleted, there results a seven-rowed sub-array with $n_{i0} = 2$ and $n_{i1} = 10$. In case any row five through eight is dropped, the resulting array would have $n_{i0} = 1$ and $n_{i1} = 11$. Since none of these parameters satisfy a seven-rowed orthogonal array, it follows that 8.4 cannot possibly be considered as an extension of any seven-rowed array.

LEMMA 2.4.2. *Arrays* (192, 7, 5) *cannot be extended to eight constraints.*

PROOF. First, we will consider the seven-rowed arrays with $n_{i7} = 2$. An array with solution 7.9 can be extended to an array of eight constraints with $n_{i0} = 0$ and $n_{i7} + n_{i8} = 2$. Accordingly, 8.1 is the only solution which needs to be considered. Further, an extension of 7.9 would have to have $n_{i1} + n_{i2} \leqq 21$, but solution 8.1 has $n_{i1} + n_{i2} = 33$, a contradiction. The solution 7.12 is obtained precisely by repeating three times an array (64, 7, 6). That the maximum number of constraints in this array is seven is a consequence of previously mentioned Theorem 2.1.

Next, we will consider 7.1, 7.4, 7.5 and 7.8. Since an extension of 7.1 must have $n_{i0} = n_{i8} = 0$ and $n_{i1} + n_{i2} \leqq 31$, we would consider solutions 8.2, 8.5 and 8.7 only for possible extension. In all these solutions, there are at least two identical columns having one coincidence with the column of all zeros. If the row in which this coincidence occurs is deleted, there results a seven-rowed array with $n_{i0} > 0$, contradicting that 7.1 has $n_{i0} = 0$. The array 7.4 has three columns of all ones followed by a column with a single zero. The solutions 8.9, 8.11 and 8.12 need only be considered as possible extensions of 7.4, since an extension of 7.4 must have $n_{i8} = 0$ and $n_{i0} + n_{i1} \leqq 4$. If 8.9 is to be considered as a possible extension, then the elements in the extended row are 1, 1, 0, 0. This yields $n_{i1} = 1$, but 8.9 has $n_{i1} = 0$. The array 7.4 cannot be extended to 8.11

or 8.12 since $\sum_{j=1}^{j=3} n_{ij}$ equals 81 in solution 8.11 and 91 in solution 8.12 respectively, whereas any extension of 7.4 would have to have $\sum_{j=1}^{j=3} n_{ij} \leqq 76$. Again, an extension of 7.5 would have $n_{i0} = 0$ and $n_{i1} + n_{i2} \leqq 26$. The solutions 8.5, 8.7 and 8.13, which satisfy this property, have at least two identical columns of type n_{i1} and deleting the row in which this coincidence occurs yields a seven-rowed subarray with $n_{i0} \geqq 1$, but $n_{i0} = 0$ in 7.5. An array satisfying solution 7.8 includes three identical columns and must, therefore, satisfy one or more solutions 7.9 and/or 7.12. Since none of the arrays with $n_{i7} = 2$ can be extended, we conclude that the array 7.8 cannot be extended further.

Consider next an array 7.3. It is obvious that any extension of 7.3 would have to have $n_{i8} = 0$ and $n_{i0} + n_{i1} \leqq 9$. Accordingly, we need to consider 8.1, 8.2, 8.3, 8.6, 8.8, 8.9, 8.10, 8.11 and 8.12. An array of type 8.1 has 8 columns of each possible type with one zero in a different row; dropping any one of the rows results in a seven-rowed subarray with $n_{i0} = 1$ but 7.3 has $n_{i0} = 2$. If 7.3 could be extended to the solution 8.3, then all the columns of type n_{i3} would have to go to n_{i4} of 8.3. This would leave fifty-five columns of type n_{i4} in 7.3 that would have one added in each of these columns in the extended row so that $n_{i4} = 90$ in the solution 8.3. The remaining fifteen columns (of type n_{i4}) in 7.3 that would have zero added in the extended row yields $n_{i5} = 15$ but 8.3 has $n_{i5} = 13$. That 7.3 cannot be extended to 8.10 and 8.12 follows easily, since $\sum_{j=0}^{j=3} n_{ij}$ equals 86 in 7.3 while $\sum_{j=0}^{j=3} n_{ij} = 91$ both in 8.10 and 8.12. The array 7.3 as extended from its subarray 6.3 must have the first fifty-one columns as follows:

11	0111111	000000000000111111111111111111111111111111
11	1011111	001111111111000000000011111111111111111111
11	1101111	110011111111001111111100000000111111111111
11	1110111	111100111111110011111100111111000000111111
11	1111011	111111001111111100111111001111001111000011
11	1111101	111111110011111111001111110011110011001100
11	1111110	111111111100111111110011111100111100111100

The first two columns have 0's while the next seven columns have 1's added in the extended row if 7.3 is to be extended to 8.2. Of 42 columns of type n_{i2} in 7.3, 23 have 0's while the remaining 19 columns have to have 1's in the extended row. This may, without loss of generality, be accomplished by having in the extended row 0's in columns 46—51 and 1's in alternate columns beginning with a one in column 10. If three rows, five through seven, are dropped, there results a five-rowed subarray with a five-tuple (11110)′ appearing eight times. In case 7.3 is extended to 8.6, then the elements in the first nine columns are 1, 0, 1, 1, 1, 1, 0, 0, 0 so that if any one of the rows, one through four, is deleted, we have a seven-rowed subarray with $n_{i0} = 2$. The elements in the first nine columns in the extended row are 1, 1, 1, 0, 0, 0, 0, 0, 0, if 8.11 is considered as a possible extension of 7.3. We may have, as before, 0's in columns 46—51 and

1's in the alternate columns beginning with a 1 in column 10, so that 23 columns have 0's and 19 columns have 1's in the extended row, if 7.3 is extended to 8.6 or 8.11. Again, a five-tuple $(11110)'$ appears more than six times in a five rowed subarray resulting from the deletion of the three rows five through seven. The fact that 7.3 cannot be extended to 8.8 or 8.9 follows from a similar argument.

We have remarked earlier that the array satisfying solution 7.7 satisfies solution 7.3 in respect to at least one of the columns having one coincidence with the ith column, since $n_{i1} = 8$. Since 7.7 includes a column in respect to which it satisfies 7.3, it follows that 7.7 cannot be extended.

An array satisfying solution 7.2 includes two identical columns and must, therefore, satisfy one or more solutions 7.5 through 7.8. Since we have established above that arrays with $n_{i7} = 1$ cannot be extended, it follows that 7.2 cannot be extended to eight constraints.

The proof of the theorem is now complete.

3. Projective geometry and saturated designs. In this section, we sketch briefly a relation between the Galois spaces, orthogonal arrays and saturated symmetrical factorial designs. The investigation pertaining to the maximum number of constraints in an orthogonal array $(\lambda 2^t, k, t)$ for $\lambda = 2^q$ becomes purely a geometric problem. Bose [1] established that the maximum number of factors in a symmetrical factorial design, in which each factor operates at two levels, blocks are of size 2^{n+1} and no main effect or a t-factor or lower order interaction is confounded, is given by the maximum number of distinct points in an n-dimensional projective space $PG(n, 2)$ based on 0 and 1 such that no t points among them are linearly dependent. This number is usually denoted by $m_t(n+1, 2)$. If $k = m_t(n+1, 2)$, then a premultiplication of a $k \times (n+1)$ matrix by another matrix consisting of all possible $2^{n+1}(n+1)$-tuples yields an orthogonal array of strength t, and hence a saturated design of resolution $t+1$.

The results in Theorems 3.1 and 3.2, when translated into the language of saturated designs, agree with those of Draper and Mitchell [4], [5], but we have considered it worthwhile to give independent proofs of their main results because their extensive use of 'number of words' in a 'word length' and 'dead-end' designs is rather involved and complicated, and partly to indicate that any results of this kind can scarcely be expected to be the best obtainable in as much as their proof is in a sense independent of algebraic and geometric considerations. The geometrical methods are sometimes direct and simpler as compared to those considered by these authors.

THEOREM 3.1. $$m_5(8, 2) = 12\,.$$

PROOF. Let $k = m_t(n+1, 2)$ have the usual meaning. Rao [9] has shown that $m_4(7, 2) = 11$. We have established elsewhere [7] that an increase in the value of n and t by one results in the corresponding increase in k by at most one. Thus, it follows that $m_5(8, 2)$ cannot exceed 12. We now demonstrate an existence of 12 points in $PG(7, 2)$ in the columns of the matrix A, no five of which are linearly dependent.

$$(3.1.1) \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad \square$$

It is recently established [8] that

$$(3.1.2) \qquad m_t(t, 2) = t + 1 \qquad \text{for} \quad t \geqq 2$$

$$(3.1.3) \qquad m_t(t + 1, 2) = t + 2 \qquad \text{for} \quad t \geqq 4$$

$$(3.1.4) \qquad m_t(t + 2, 2) = t + 4 \qquad \text{for} \quad t = 4, 5$$

$$(3.1.5) \qquad m_t(t + 2, 2) = t + 3 \qquad \text{for} \quad t \geqq 6\,.$$

THEOREM 3.2. [10] $m_5(n + 1, 2) \leqq 3(2^{n-6} - 1) + 9$ *for* $n \geqq 8$.

PROOF. Through a six-dimensional subspace, exactly $(2^{n-6} - 1)$ seven-dimensional spaces will pass. That $m_5(7, 2) = 9$ is a consequence of (3.1.4) above while the preceding Theorem 3.1 assures the existence of 12 points in $PG(7, 2)$, no five linearly dependent. It follows, therefore, that each of the seven-flats can have at most three points not lying on a six-flat. Hence, we have the upper bound as asserted. $\square$

We now give below an example of 18 points in $PG(8, 2)$ in the columns of the following matrix B satisfying the condition that no five of these points are linearly dependent.

$$(3.1.6) \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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REFERENCES

[1] BOSE, R. C. (1947). Mathematical theory of symmetrical factorial designs. *Sankhyā* **8** 107–166.

[2] BLUM, J. R., SCHATZ, J. A., and SEIDEN, E. (1970). On two levels of orthogonal arrays of odd index. *J. Combinatorial Theory* **9** 239-243.

[3] BUSH, K. A. and BOSE, R. C. (1952). Orthogonal arrays of strength two and three. *Ann. Math. Statist.* **23** 508-524.

[4] DRAPER, N. R. and MITCHELL, T. J. (1967). The construction of saturated 2_R^{k-p} designs. *Ann. Math. Statist.* **38** 1110-1126.

[5] DRAPER, N. R. and MITCHELL, T. J. (1968). Construction of the set of 256-run design of resolution $\geqq 5$ and the set of even 512-run designs of resolution $\geqq 6$ with reference to the unique saturated designs. *Ann. Math. Statist.* **39** 246-255.

[6] DRAPER, N. R. and MITCHELL, T. J. (1970). Construction of a set of 512-run designs of resolutions $\geqq 5$ and a set of even 1024-run designs of resolutions $\geqq 6$. *Ann. Math. Statist.* **41** 876-887.

[7] GULATI, B. R. (1969). On the packing problem and its applications. Ph. D. thesis. The Univ. of Connecticut.

[8] GULATI, B. R. and KOUNIAS E. G. (1972). On two level symmetrical factorial designs. To appear in *Ann. Inst. Statist. Math.* **24**(2).

[9] RAO, C. R. (1947). Factorial experiments derivable from combinatorial arrangements of arrays. *J. Roy. Statist. Soc., Suppl.* **9** 128-139.

[10] SEIDEN, E. (1964). On the maximum number of points no four on one plane in projective space PG(r-1, 2). Technical Report RM-117, Michigan State Univ.

[11] SEIDEN, E. and ZEMACH, R. (1966). On orthogonal arrays. *Ann. Math. Statist.* **37** 1355-1370.

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