A REPRESENTATION OF INDEPENDENT INCREMENT PROCESSES WITHOUT GAUSSIAN COMPONENTS

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1. Introduction and summary. It is the purpose of this paper to describe a simple way of representing processes with independent increments having no Gaussian components and no fixed points of discontinuity. As is well known, the only random part of such processes are the jump discontinuities occurring at random points with random heights. The representation appearing in this paper describes the joint distribution of the ordered heights of the jumps and of the points at which these jumps occur. In fact, such a process in represented as a countable sum of functions each with one random point of discontinuity at a random height (Formula (7)). There is an analogy to the way that Wiener [8] described the Brownian motion process $W_t$ on the interval $[0, \pi]$ as a countable sum,

$$W_t = iY_0 + 2\sum_{m=1}^{\infty} Y_m \frac{\sin mt}{m},$$

where $Y_0, Y_1, \ldots$ are independent normal random variables with zero means and unit variances. In the same way that certain almost sure properties of the sample paths of the Brownian motion process can be read from (1) (see, for example, Ito and McKean [4] page 21), so also may certain almost sure properties of the general process with independent increments be read from (7).

We use the notation $\mathcal{P}(\lambda)$ to represent the Poisson distribution with parameter $\lambda$, $\mathcal{G}(\alpha, \beta)$ to represent the gamma distribution with shape parameter $\alpha$ and scale parameter $\beta$, $\mathcal{U}(\alpha, \beta)$ to represent the uniform distribution on the interval $(\alpha, \beta)$, and $\mathcal{N}(\mu, \sigma)$ to represent the normal distribution with mean $\mu$ and variance $\sigma^2$. (See [2] Section 3.1 for this notation.) $I_S(x)$ denotes the indicator function of the set $S$: one if $x \in S$, and zero if $x \not\in S$. Expectations with subscripts always represent conditional expectations given the subscripted variables. $\mathbb{R}$ represents the real line and $\mathbb{R}^m$ Euclidean $m$-dimensional space.

Let $X_t$ denote a process with independent increments and no fixed points of discontinuity. For the purposes of this paper, we restrict the domain of $t$ to be the interval $[0, 1]$, and assume that $X_0 \equiv 0$. As is well known the increments of such processes have infinitely divisible distributions. Let $\psi_t(u)$ denote the logarithm of the characteristic function of $X_t$. The Lévy representation [5] of $\psi_t$ may be written as

$$\psi_t(u) = iu \int_0^t \lambda(t) dt + \int_0^t \left( e^{iuz} - 1 - \frac{iu z}{1 + z^2} \right) dM_t(z)$$

$$+ \int_0^t \left( e^{iuz} - 1 - \frac{iu z}{1 + z^2} \right) dN_t(z)$$

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where \( m(t) \) is continuous where \( \lambda(t) \) is non-decreasing and continuous, where \( M_t \) and \( N_t \) are measures on the Borel subsets of \((-\infty, 0)\) and \((0, \infty)\) respectively, such that \( M_t(A) \) and \( N_t(B) \) are non-decreasing and continuous in \( t \) for fixed Borel sets \( A \) and \( B \), and where

\[
\int_{-\infty}^{0} \frac{z^2}{1 + z^2} \, dM_t(z) < \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{z^2}{1 + z^2} \, dN_t(z) < \infty.
\]

The Gaussian component of this distribution is found in the term \(-\lambda(t)u^2\). We do not treat this component; we assume that \( \lambda(t) \equiv 0 \). The first term, \( iu m(t) \), is a component degenerate at \( m(t) \). This component is easily treated, so we assume that \( m(t) \equiv 0 \), also. Of the last two components, it is sufficient to consider just one, since the other may be treated by symmetry. Thus, we intend to represent \( X_t \) as a sum of a countable number of terms when \( X_t \) is the process with independent increments and log characteristic function

\[
\psi_t(u) = \int_{0}^{\infty} \left( e^{ixz} - 1 - \frac{izx}{1 + x^2} \right) \, dN_t(z).
\]

We relax one condition on \( N_t \) because it is not needed; namely continuity in \( t \). Thus we allow the process \( X_t \) to have some fixed points of discontinuity provided the lengths of the jumps at these points from the left and the right have infinitely divisible distributions. There is at least one application that requires such a generalization [3].

Thus, we assume that the Lévy function \( N_t \) is, for each \( t \in [0, 1] \), a measure on the Borel subsets of \((0, \infty)\) that satisfies the conditions

**Condition 1.** \( N_0 \equiv 0 \).

**Condition 2.** For every Borel set \( B \), \( N_{t_1}(B) \leq N_{t_2}(B) \) whenever \( 0 \leq t_1 < t_2 \leq 1 \), and

**Condition 3.** \( \int_{0}^{\infty} (z^2/(1 + z^2)) \, dN_t(z) < \infty \).

It is convenient to use the distribution function form of the measure \( N_t \). Condition 3 implies that \( N_t(z, \infty) < \infty \), for all \( z > 0 \), so we define

\[
N_t(z) = -N_t([-\infty, z])\).
\]

Then, \( N_t(z) \) is a non-decreasing function on \((0, \infty)\) such that \( \lim_{z \to -\infty} N_t(z) = 0 \). In this case, Condition 3 becomes simply

\[
\int_{0}^{\infty} z^2 \, dN_t(z) < \infty.
\]

The jumps of the independent increment process \( X_t \) with log characteristic (2) are all positive. We intend to describe \( X_t \) as the sum of a countable number of functions of the form, \( J_{f(t)}(T_j) - c_j(t) \), where \( J_j > 0 \) represents the height of a jump, \( T_j \) represents its position, and \( c_j(t) \) is a given function (nonrandom). We order the heights of the jumps \( J_1 \geq J_2 \geq \ldots \).

The main theorem states that the distribution of the ordered heights of the
jumps, $J_1, J_2, \ldots$ depends only on $N_1$ and not otherwise on $N_i$, as follows. The distribution of $J_1, J_2, \ldots$ is the same as the distribution of $N_1^{-1}(-S_1), N_1^{-1}(-S_2), \ldots$, where $S_1, S_2, \ldots$ is a Poisson point process at unit rate—that is, $S_1, S_2 - S_1, S_3 - S_2, \ldots$ are independent identically distributed with negative exponential distribution $\mathcal{E}(1, 1)$. The inverse function $N_1^{-1}(y) = \inf\{z : N_1(z) \geq y\}$ is well-defined except at an at most countable number of points $y < 0$ (the images of the intervals measure zero under $N_1$) so that the random variables $J_1, J_2, \ldots$ are well defined almost surely.

The actual distribution of the ordered jumps may easily be obtained from this, provided $N_i(z)$ is continuous in $z$ as follows. The largest jump, $J_1$, has distribution function, for $x > 0$,

$$P(J_1 \leq x) = P(N_1^{-1}(-S_1) \leq x) = P(S_1 \geq -N_1(x)) = e^{N_1(x)}.$$ 

If $\lim_{z \to -\infty} N_i(z) = N_i(0) > -\infty$, then the distribution of $J_1$ has mass exp $N_i(0)$ at the origin and is otherwise continuous. To find the conditional distribution of $J_2$ given $J_1 = x_1$, we compute, for $0 < x_2 < x_1$,

$$P(J_2 \leq x_2 \mid J_1 = x_1) = P(N_1^{-1}(-S_2) \leq x_2 \mid N_1^{-1}(-S_1) = x_1)
= P(S_2 \geq N_1(x_2) \mid S_1 = N_1(x_1))
= \exp\{N_1(x_2) - N_1(x_1)\}.$$

Thus, the conditional distribution of $J_2$, given $J_1 = x_1$, is the same as the distribution of $J_1$ truncated above at $x_1$. This procedure is easily continued. Thus, the distribution of $J_j$, given $J_{j-1} = x_{j-1}, \ldots, J_1 = x_1$, is the same as the distribution of $J_1$ truncated above at $x_{j-1}$.

Condition 2 implies that $N_1$ is absolutely continuous with respect to $N_2$ whenever $0 \leq t_1 \leq t_2 \leq 1$. Hence, the Radon-Nikodym derivative of $N_1$ with respect to $N_1$, call it $n_1(z)$,

$$n_1(z) = \frac{dN_1}{dN_1}(z),$$

exists and is determined up to equivalence $dN_1$. It is shown in Lemma 3 that there is a determination of $n_1(z)$ such that for all $z \in (0, \infty)$, $n_1(z)$ is a non-decreasing function of $t$ on $[0, 1]$ with $n_1(0) \equiv 0$ and $n_1(1) \equiv 1$. One is tempted to describe such a function of $t$ as a distribution function on $[0, 1]$; however, the specific values assumed by $n_1(z)$ at points of discontinuity in $t$ are important—they play a role in determining the distribution of the left and right hand jumps of $X_t$ at the fixed points of discontinuity.

The conditional distribution of the points $T_1, T_2, \ldots$ at which the respective jumps $J_1, J_2, \ldots$ occur, given $J_1, J_2, \ldots$, is, essentially, as independent random variables with respective distribution functions, $n_1(J_1), n_1(J_2), \ldots$. This description is valid if all the $n_i$ are continuous in $t$ (more generally, right-continuous in $t$). However, a more precise description is needed at points of discontinuity of $n_1$ because these are the fixed points of discontinuity of the process and part
of the jump at such points may be due to a discontinuity on the left and the rest of the jump due to a discontinuity on the right. This more precise description, as found in the main theorem, is as follows. Let \( U_1, U_2, \ldots \) be independent identically distributed \( \mathcal{U}(0, 1) \) random variables, independent of \( J_1, J_2, \ldots \). The \( j \)th jump, \( J_j \), occurs at the point \( t \) at which the jump in

\[
I_{(0,n_t(J_j))}(U_j)
\]

occurs. If this occurs at a point \( t_0 \) of discontinuity of \( n_t(J_j) \), then \( J_j \) is part of the left discontinuity at \( t_0 \) if \( U_j < n_{t_0}(J_j) \), and part of the right discontinuity if \( U_j \geq n_{t_0}(J_j) \).

If \( \int_0^1 z \, dN_i(z) < \infty \), then the sum of the jumps \( \sum_{j=1}^\infty J_j \) is finite with probability one. In such a case,

\[
X_t = \sum_{j=1}^{\infty} J_j I_{(0,n_t(J_j))}(U_j)
\]

already represents a process with independent increments having log characteristic function

\[
\phi_t(u) = \int_0^\infty (e^{itu} - 1) \, dN_i(z).
\]

However, when \( \int_0^1 z \, dN_i(z) = \infty \), the sum of the jumps \( \sum_{j=1}^\infty J_j \) is infinite so that the series (4) is infinite also (at least at \( t = 1 \)). As is well known, it is sometimes possible to center such divergent series (each term centered at its mean, say), and obtain series convergent with probability one, even though the series is not absolutely convergent. This is possible under Condition 3. One possible centering constant for the \( j \)th term is

\[
c_j(t) = \int_{1-}^1 \frac{N_i^{-1}(-v)}{1 + N_i^{-1}(v)^2} n_i(N_i^{-1}(-v)) \, dv.
\]

With these centering constants, the main theorem states that

\[
X_t = \sum_{j=1}^{\infty} (J_j I_{(0,n_t(J_j))}(U_j) - c_j(t))
\]

converges with probability one for each \( t \in [0, 1] \), and represents a process with independent increments and log characteristic function (2).

This paper is incomplete in one aspect. We would like to be able to state that the set of measure zero outside of which the series (7) converges may be taken to be independent of \( t \). Under the condition

\[
\int_0^1 z \, dN_i(z) < \infty,
\]

this is the case because (4) itself converges for \( t = 1 \) with probability 1, and is obviously non-decreasing. (In addition, \( \sum_{j=1}^{\infty} c_j(t) < \infty \) under (8).) Thus, when (8) is satisfied, the following well-known property of the sample paths of the process (7) is visible: almost every sample path of \( X_t - \sum_{j=1}^{\infty} c_j(t) \) is a pure jump function (see Breiman [1] page 314). However, when condition (8) is not satisfied, we do not know whether or not the series (7) converges for all \( t \) almost surely, although we suspect it does.

The process with independent increments and log characteristic function (2)
is said to be homogeneous if its Lévy function is linear in \(t\), \(N_t(z) = tN_1(z)\). There is a simplification in the representation (7) when the process is homogeneous or, more generally, when

\[ N_t(z) = G(t)N_1(z) \]

for some non-decreasing function \(G(t)\) on \([0, 1]\) such that \(G(0) = 0\) and \(G(1) = 1\). In this case, \(n_t(z) = dN_t(z)/dN_1(z) = G(t)\) independent of \(z\). Hence the points at which the jumps \(J_1, J_2, \ldots\) occur are independent identically distributed random variables, independent of \(J_1, J_2, \ldots\), having distribution function \(G(t)\) (with the difficulties previously noted at points of discontinuity of \(G(t)\)). In addition, \(c_j(t) = G(t)c_j\) where

\[ c_j = \int_{1}^{N_1^{-1}(-v)} \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2} dv. \]

Hence, the representation (7) becomes

\[ X = \sum_{j=1}^{\lambda} (J_{\theta(j)}(U_j) - G(t)c_j). \]

2. The proof. We precede the proof of the main theorem by four lemmas. We have defined a Poisson point process at unit rate as a sequence of random variables, \(S_1, S_2, S_3, \ldots\) such that \(S_1, S_2 - S_1, S_3 - S_2, \ldots\) are independent identically distributed with the negative exponential distribution, \(\mathcal{E}(1, 1)\). An alternative definition of the Poisson point process at unit rate is as the times of the jumps in a Poisson process at unit intensity. The following well-known lemma is based in part upon this fact, and its proof is omitted (see Parzen [6] Section 4–4).

**Lemma 1.** Let \(S_1, S_2, \ldots\) be a Poisson point process at unit rate.

(a) The conditional distribution of \(S_1, \ldots, S_k\) given \(S_{k+1}\), is as the order statistics of a sample of size \(k\) from \(\mathcal{E}(0, S_{k+1})\).

(b) Let \(K\) be the largest integer \(k\) such that \(S_k < \lambda\), where \(\lambda > 0\). Then \(K \sim \mathcal{E}(\lambda)\), and the conditional distribution of \(S_1, \ldots, S_k\) given \(K = k\) is as the order statistics of a sample of size \(k\) from \(\mathcal{E}(0, \lambda)\).

The next lemma appears to be new and interesting in its own right. It is used to show that the process \(X_t\) defined in (7) has independent increments.

Let \(\Theta_m = \{\theta \in \mathbb{R}^n : \theta_j \geq 0, \sum_j^n \theta_j = 1\}\). A random \(m\)-dimensional vector \(M\) is said to be multinomial with probability vector \(\theta \in \Theta_m\) if \(P_n(M = e_j) = \theta_j\), where \(e_j\) is the unit vector with \(j\)th coordinate one and the remaining coordinates zero (or, equivalently, if \(E \exp iu'M = \sum_j^n \theta_j e^{iu_j}\)).

**Lemma 2.** Let \(K, Y_1, Y_2, \ldots\) be random variables, \(K \sim \mathcal{E}(\lambda)\), and given \(K = k\) let \(Y_1, Y_2, \ldots, Y_k\) be independent, identically distributed with common distribution function \(F(y)\). Let \(\theta(y)\) be a measurable map of \(\mathbb{R}\) into \(\Theta_m\). Let \(M_1, M_2, \ldots\) be a sequence of \(m\)-dimensional random vectors whose conditional distribution given \(K, Y_1, Y_2, \ldots\) is as independent multinomials with respective probability vectors \(\theta(Y_1), \theta(Y_2), \ldots\). Let \(Z = \sum_{j=1}^{K} Y_jM_j\). Then \(Z_1, \ldots, Z_m\) are stochastically independent.
PROOF. We show that the characteristic function of $Z$ factors:

$$
\varphi_Z(u) = E \exp[iu'Z] = E \prod_{j=1}^{K} E_{K_j} \exp[iY_j u_j] \\
= E \prod_{j=1}^{K} E_{K_j} (\sum_{i=1}^{n} \theta_i(Y_j) \exp[iY_j u_j]) \\
= E(\sum_{i=1}^{n} \theta_i(y)e^{iuv} dF(y))^K \\
= \exp[\lambda \sum_{i=1}^{n} \theta_i(y)e^{iuv} dF(y) - \lambda] \\
= \prod_{i=1}^{n} \exp[\lambda \theta_i(y)(e^{iuv} - 1) dF(y)],
$$

completing the proof.

Condition 2 on the Lévy function $N_t$ implies that $N_t \ll N_1$, so that by the Radon-Nikodym theorem there is a measurable function $n_t = dN_t/dN_1$ determined up to an equivalence $dN_1$ such that

$$
\int_A n_t(z) \, dN_1(z) = N_1(A)
$$

for all Borel sets $A \subset (0, \infty)$. For $t_1 < t_2$, then,

$$
\int_A n_{t_1}(z) \, dN_1(z) \leq \int_A n_{t_2}(z) \, dN_1(z)
$$

for all Borel sets $A \subset (0, \infty)$, so that $n_{t_1}(z) \leq n_{t_2}(z)$ for almost all $z(dN_1)$. It is important to establish that $n_t(z)$ can be chosen so that it is non-decreasing in $t$ for $t \in [0, 1]$ and for all $z \in (0, \infty)$ i.e., that the null set on which $n_{t_1}(z) > n_{t_2}(z)$ can be assumed independent of $t_1$ and $t_2$. That this is the case is the content of the following lemma.

LEMMA 3. There exists a determination of the Radon-Nikodym derivative $n_t = dN_t/dN_1$ such that for all $x \in [0, 1]$, $n_t(x)$ is non-decreasing in $t$, $n_t(x) \equiv 0$, and $n_t(x) \equiv 1$.

PROOF. Let $D$ be a denumerable dense set in $[0, 1]$. Include in $D$ the points 0 and 1, and all fixed discontinuity points of the process (i.e. points $t_0$ for which there exist a Borel set $A$ such that

$$
\lim_{t \uparrow t_0} N_t(A) \neq \lim_{t \downarrow t_0} N_t(A).
$$

There are only a countable number of such points. To see this, let $T_m$ be the set of all $t$ for which there exists a Borel set $A_t \subset (1/m, \infty)$ such that

$$
\lim_{t \downarrow A_t} N_t(A_t) - \lim_{t \uparrow A_t} N_t(A_t) > \frac{1}{m}.
$$

Then $T_m$ is a finite set. (If not, then $N_t(1/m, \infty)$ being at least the sum of these jumps would be equal to infinity, contradicting Condition 3.) The sets $T_m$ are non-decreasing in $m$, and the limit as $m \to \infty$ is exactly the set of all fixed points of discontinuity, which must therefore be countable.

Find, for each $t \in D$, $n_t(x)$ such that $\int_A n_t \, dN_1 = N_1(A)$ for all Borel sets. $A$. Then, for $t_1 < t_2$, $n_{t_1}(x) \leq n_{t_2}(x)$ a.e. $(dN_1)$. Redefine $n_t(x)$ for $t \in D$ if necessary so that $n_{t_1}(x) \leq n_{t_2}(x)$ for all $x \in [0, \infty)$ and all $t_1 \in D$, $t_2 \in D$, $t_1 < t_2$, and so that $n_0(x) \equiv 0$ and $n_1(x) \equiv 1$. Define for $t \notin D$

$$
n_t(x) = \lim_{t', t' \in D} n_{t'}(x).
$$
Then, for all \( x \in [0, \infty) \), \( n_t(x) \) is non-decreasing in \( t \). Furthermore, for \( t \not\in D \)
\[ \int_{A} n_t dN_t = \lim_{t^+, t^- \in D} \int_{A} n_t dN_t = \lim_{t^+, t^- \in D} N_t(A) = N_t(A) \]
for all Borel sets \( A \), showing that \( n_t(x) \) is a Radon-Nikodym derivative \( dN_t/dN_t \), and completing the proof.

**Lemma 4.** Let \( g \) be a non-decreasing square-integrable real-valued function defined on \([0, \infty)\). Let \( X_1, X_2, \ldots \) be a sequence of independent random variables with \( \text{EX}_i = 0 \) and \( \text{Var} X_i = \sigma_i^2 < \infty \). Let \( S_n = \sum_1^n X_i \), let \( a_n = \sum_1^n \sigma_i^2 \), and assume \( a_n \to \infty \) as \( n \to \infty \). Then
\[ \int_{a_n + S_n}^{a_n + S_n + 1} g(x) \, dx \to a_n, 0 \quad \text{as} \quad n \to \infty. \]

**Proof.** Let \( \varepsilon > 0 \). Let \( t_n = \sup \{ t : \int_{a_n + t}^{a_n + t + 1} g(x) \, dx < \varepsilon \} \), and let \( \hat{t}_n = \sup \{ t : \int_{a_n + t}^{a_n + t + 1} g(x) \, dx < \varepsilon \} \). We are to show \( P[S_n \geq t_n \text{ i.o.}] = 0 \) and \( P[S_n \geq -\hat{t}_n \text{ i.o.}] = 0 \).

Since \( g \) is non-increasing, we have \( \hat{t}_n \leq t_n \), so by the symmetry of the problem with respect to \( S_n \), it is sufficient to show, say, \( P[S_n \geq \hat{t}_n \text{ i.o.}] = 0 \). Let \( t_n' = \min(\hat{t}_n, a_n/2) \). It is sufficient to show \( P[S_n \geq t_n' \text{ i.o.}] = 0 \). Let \( b(n) = \inf \{ b : a_b \geq 2^n \} \). Then, for all \( k \)
\[ P[S_n \geq t_n' \text{ i.o.}] \leq P[\bigcup_{j \geq b(n)} \{ S_j \geq t_j' \}] \]
\[ \leq \sum_{n \geq k} P\left[ \bigcup_{j \geq b(n) + 1} \{ S_j \geq t_j' \} \right] \]
\[ \leq \sum_{n \geq k} P\left[ \bigcup_{j \geq b(n) + 1} \{ S_j \geq t_j' \} \right] \]
\[ \leq \sum_{n \geq k} P\left[ \max_{j < b(n) + 1} S_j \geq t_j' \right] \]
\[ \leq \sum_{n \geq k} a_{b(n)} \left( \frac{t_j'}{a_{b(n)}} \right)^{-2} \quad \text{(Kolmogorov’s inequality)} \]
\[ \leq \sum_{n \geq k} 2^{n+1} \left( t_j'/a_{b(n)} \right)^{-2} \]
\[ = \sum_{n \geq k, n \in \mathbb{E}} 2^{n+1} \left( t_j'/a_{b(n)} \right)^{-2} + \sum_{n \geq k, n \in \mathbb{E}} 2^{n+1} \left( t_j'/a_{b(n)} \right)^{-2} \]
where \( E = \{ n : \hat{t}_b(n) \leq a_{b(n)}/2 \} \). If \( n \in E^c \), \( t_j'/a_{b(n)} \geq 2^{n-1} \), so that
\[ \sum_{n \geq k, n \in \mathbb{E}} 2^{n+1} \left( t_j'/a_{b(n)} \right)^{-2} \leq \sum_{n \geq k} 2^{n+1} (2^{n-1})^{-2} < \infty. \]

If \( n \in E \), \( t_j'/a_{b(n)} \leq a_{b(n)}/2 \), and
\[ \epsilon = \int_{a_n - \hat{t}_n}^{a_n} g(x) \, dx \leq g(a_n - \hat{t}_n). \]
Thus, \( t_j'/a_{b(n)} \leq \epsilon/g(a_{b(n)}) \leq \epsilon/g(a_{b(n)}/2) \geq \epsilon/g(2^{n-1}) \), so that
\[ \sum_{n \geq k, n \in \mathbb{E}} 2^{n+1} \left( t_j'/a_{b(n)} \right)^{-2} \leq \sum_{n \geq k} 2^{n+1} g(2^{n-1})^2. \]

But since \( g \) is non-increasing,
\[ \int_{0}^{\infty} g(x)^2 \, dx < \infty \Rightarrow \sum_{n} g(n)^2 < \infty \Rightarrow \sum_{n} 2^n g(2^n)^2 < \infty. \]

Thus both summations in (9) are finite and hence converge to zero as \( k \to \infty \).

**Theorem.** Let \( N_t \) satisfy Conditions 1, 2, and 3. Let \( J_n = N_1^{-1}(-S_j) \) \( j = 1, 2, \ldots \) where \( S_1, S_2, \ldots \) is a Poisson process at unit rate. Let \( U_1, U_2, \ldots \) be independent
identically distributed \( \sim (0, 1) \), independent of \( S_1, S_2 \ldots \). Let

\[
c_j(t) = \sum_{j=1}^{\infty} \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2} n_j(N_1^{-1}(-v)) \, dv
\]

where \( n_j \) is as in Lemma 3. Then for each \( t \in [0, 1] \), the series

\[
X_t = \sum_{j=1}^{\infty} (J_j I_{(0, n_j(j))}(U_j) - c_j(t))
\]

converges with probability 1, and \( X_t \) is a process with independent increments and log characteristic function

\[
\phi_t(u) = \int_0^\infty \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) dN_t(x).
\]

**Proof.** Let \( a_0 = 0 \) and \( a_n = \sum_{j=1}^{n} 1/j \). Let \( K_n \) be the largest integer \( k \) for which \( S_k \leq a_n \). With probability one \( K_1 \leq K_2 \leq \ldots \) and \( K_n \to \infty \). Since \( K_n - K_{n-1} \in : \mathcal{O}(1/n) \) \( P[K_n - K_{n-1} \geq 2] = O(1/n^2) \) so that \( \sum_{n=1}^{\infty} P[K_n - K_{n-1} \geq 2] < \infty \). The Borel-Cantelli Lemma implies that with probability one \( K_n - K_{n-1} \geq 2 \) only finitely often. In other words, with probability one the sequence \( K_1, K_2, \ldots \) contains all the integers from some integer on.

Let

\[
X_t^{(n)} = \sum_{j=1}^{K_n} (J_j I_{(0, n_j(j))}(U_j) - c_j(t)).
\]

We will show that \( X_t^{(n)} \) converges almost surely for each \( t \in [0, 1] \). Then by the above paragraph the series (10) converges almost surely as well. Let

\[
V_t^{(n)} = \sum_{j=1}^{K_n} J_j I_{(0, n_j(j))}(U_j) - \sum_{j=1}^{K_n} g_t(v) \, dv
\]

where for \( 0 < v < \infty \)

\[
g_t(v) = \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2} n_j(N_1^{-1}(-v)).
\]

We shall complete the proof by showing,

(i) \( X_t^{(n)} - V_t^{(n)} \to_{\mathbb{P}, n} 0 \) as \( n \to \infty \) for each \( t \in [0, 1] \)

(ii) for each \( n \), \( V_t^{(n)} \) is a process with independent increments, and

(iii) \( V_t^{(n)} \) converges with probability one for each \( t \in [0, 1] \) to a random variable with characteristic function (11).

\[
X_t^{(n)} - V_t^{(n)} = |\sum_{j=1}^{K_n} g_t(v) \, dv| \leq |\sum_{j=1}^{K_n} g_t(v) \, dv|.
\]

The function \( g_t(v) \) is square-integrable since

\[
\int_0^\infty g_t(v)^2 \, dv = \int_0^\infty \left( \frac{N_1^{-1}(-v)}{1 + N_1^{-1}(-v)^2} \right)^2 \, dv
\]

\[
= \int_0^\infty \left( \frac{z}{1 + z^2} \right)^2 \, dN_t(z)
\]

\[
\leq 2 \int_0^\infty \frac{z^2}{1 + z^2} \, dN_t(z) < \infty.
\]
Also \( g_i(v) \) is eventually non-increasing and so is bounded by a square integrable non-increasing function \( g \) say \( g_i \leq g \). Hence by Lemma 4

\[
|X^{(n)}_t - V^{(n)}_t| \leq \left| \int_0^t g(v) \, dv \right| \to 0, \quad 0 \text{ as } n \to \infty.
\]

(ii) Let \( t(0) = 0 < t(1) < t(2) < \cdots < t(m) = 1 \). Then \( V^{[n]}_{t(i)} \), \( V^{[n]}_{t(i+1)} - V^{[n]}_{t(i)} \), \( V^{[n]}_{t(i+1)} - V^{[n]}_{t(m)} \) are independent if \( Z_1, Z_2, \ldots, Z_m \) are independent where

\[
Z_n = \sum_{j=1}^m \int_{[a_{j-1}, a_j)} J_{\alpha}^{[n]}(t_j, s_{\alpha}) \, dU_j \quad \alpha = 1, \ldots, m.
\]

From Lemma 1, \( K_n \in \mathcal{G}(a_n) \) and the conditional distribution of \( S_1, \ldots, S_k \) given \( K_n = k \) is as the order statistics of a sample of size \( k \) from \( \mathcal{D}(0, \alpha_n) \). Thus, Lemma 2 applies and \( Z_1, Z_2, \ldots, Z_m \) are independent.

(iii) Let

\[
V_{n}(t) = \sum_{j=1}^m J_{\alpha}[0, a_{j-1}, a_j)(t)](U_j) - \int_{a_{j-1}}^{a_j} g_i(v) \, dv.
\]

Then, for each \( t \in [0, 1] \), \( V_{n}(t), V_{n}(t), \ldots \) are independent since \( V_{n}(t) \) is determined by those \( S_j \) that fall in the interval \([a_{j-1}, a_j)\). Therefore, \( V_{n}^{[n]} = \sum_{j=1}^m V_{n}(t) \) converges almost surely if it converges in law. (See Neveu [5] page 155.) The characteristic function of the first term of \( V_{n}^{[n]} \) is

\[
E \exp\left\{ iu \sum_{j=1}^m J_{\alpha}[0, a_{j-1}, a_j)(t)](U_j) \right\}
\]

\[
= E_{K_n} \prod_{j=1}^m \left[ \exp\left\{ \sum_{j=1}^m iu J_{\alpha}[0, a_{j-1}, a_j)(U_j) \right\} \right]
\]

\[
= E_{K_n} \prod_{j=1}^m \left[ \exp\left\{ iuN_i^{-1}(-v) - 1 \right\} n_i(N_i^{-1}(-v)) \right] \frac{1}{a_n} \, dv
\]

\[
= \left[ \exp\left\{ iuN_i^{-1}(-v) - 1 \right\} n_i(N_i^{-1}(-v)) \right] a_n
\]

The third equality follows since the distribution of the \( J_j \) given \( K_n \) is as the order statistics of a sample of size \( K_n \) from \( \mathcal{D}(0, \alpha_n) \), and the product involves the \( J_j \) symmetrically, so that the expectation given \( K_n \) may be computed as if \( J_j \) were independent \( \mathcal{D}(0, \alpha_n) \). The characteristic function of \( V_{n}^{[n]} \) is therefore

\[
E \exp\left\{ iu V_{n}^{[n]} \right\}
\]

\[
= \exp\left\{ \int_0^\infty \left( \exp\left\{ iuN_i^{-1}(-v) \right\} - 1 - \frac{iuN_i^{-1}(-v)}{1 + N_i^{-1}(-v)^2} n_i(N_i^{-1}(-v)) \right) \, dv \right\}
\]

which converges as \( n \to \infty \) to

\[
\exp\left\{ \int_0^\infty \left( \exp\left\{ iuN_i^{-1}(-v) \right\} - 1 - \frac{iuN_i^{-1}(-v)}{1 + N_i^{-1}(-v)^2} n_i(N_i^{-1}(-v)) \right) \, dv \right\}
\]

\[
= \exp\left\{ \int_0^\infty \left( e^{iux} - 1 - \frac{iu}{1 + x^2} \right) n_i(x) \, dN_i(x) \right\}.
\]

Since \( n_i \) is the Radon-Nikodym derivative of \( N_i \) with respect to \( N_i \), this is the characteristic function (11).
REFERENCES


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