The Annals of Mathematical Statistics
1972, Vol. 43, No. 5, 1581-1591

A NOTE ON FINE AND TIGHT QUALITATIVE PROBABILITIES

BY ILKKA NIINILUOTO

Research Council for the Humanities Academy of Finland

Savage (1954) has shown that fine and tight qualitative probabilities are realizable by finitely additive probability measures. His proof for this result is, however, in need of a correction. Fine qualitative probabilities are either atomless or equivalent to the union of n equivalent atoms. Tight qualitative probabilities are always atomless. Qualitative probability structures, which are equivalent to the union of n equivalent atoms, are realizable by a unique probability measure. Fine qualitative probabilities are almost realizable. With these results, the proof for Savage's theorem can be worked out and a theorem of Villegas (1964) can be strengthened.

1. The problem and its solutions. Let I be a Boolean algebra and let $\geqq$ be a binary relation on I. We shall say that the relation $\geqq$ is *almost realizable* if there is a finitely additive probability measure P on I such that

$$P(A) \geqq P(B) \quad \text{if} \quad A \geqq B$$

for all A and B in I. If, in addition,

$$P(A) \geqq P(B) \quad \text{only if} \quad A \geqq B$$

for all A and B in I, we shall say that the relation $\geqq$ is *realizable* (by the probability measure P).

Bruno de Finetti, in 1931, raised the following problem:

(P) Under what conditions is the relation $\geqq$ realizable?

de Finetti interpreted the relation $\geqq$ in the following way: $A \geqq B$ means that the event A is not less probable than the event B for a person X. Similarly, the relations

$$A > B \quad \text{iff} \quad \text{not} \quad B \geqq A$$
$$A \sim B \quad \text{iff} \quad A \geqq B \quad \text{and} \quad B \geqq A$$

mean that X considers the event A to be strictly more probable than the event B and the events A and B to be equally probable (equivalent), respectively. For this reason, the relation $\geqq$ is usually called a *qualitative probability relation*. For de Finetti, a solution to the problem (P) gives an axiom system for subjective probability. If we, however, give an objective interpretation for the relation $\geqq$, then a solution to the problem (P), by a measurement-theoretic representation theorem, is of interest also to proponents of other interpretations of probability.

By Stone's representation theorem for Boolean algebras, we may suppose that the Boolean algebra I is an algebra of sets on a set X. de Finetti suggested the following conditions for $\geqq$:

Received October 19, 1971; revised December 20, 1971.

(C1) if $A \geqq B$ and $B \geqq C$, then $A \geqq C$
(C2) $A \geqq B$ or $B \geqq A$
(C3) $A \geqq \varnothing$
(C4) not $\varnothing \geqq X$
(C5) if $A \cap C = B \cap C = \varnothing$, then

$$A \geqq B \quad \text{iff} \quad A \cup C \geqq B \cup C.$$

The triple $(X, I, \geqq)$, where X is a non-empty set, I is an algebra of sets on X, and $\geqq$ is a binary relation on I such that the conditions (C1)–(C5) hold for all A, B and C in I, is called a *qualitative probability structure* or a QP-*structure*.

de Finetti (1931) showed that a QP-structure $(X, I, \geqq)$ is realizable, if for all n there is an n-fold partition of X into parts which are equivalent with respect to the relation $\geqq$. This condition required X to be an infinite set. A related sufficient condition for the realizability of $\geqq$, when X is infinite, was given by Savage (1954). Sufficient conditions, allowing X to be infinite or finite, were given by Luce (1967). Kraft, Pratt, and Seidenberg (1959) constructed examples of finite QP-structures which are neither realizable nor almost realizable. Moreover, they gave the necessary and sufficient conditions for the realizability of the relation $\geqq$ on a finite Boolean algebra. These conditions were reformulated in a strikingly simple form by Scott (1964). Scott has also found a general solution to the problem (P). His unpublished results can be found in the dissertation of Domotor (1969). The basic idea of this solution is to translate the conditions on the Boolean algebra I to conditions on the Banach space of continuous real-valued functions on the Stone space X of I, and to apply the Mazur-Orlicz theorem to show the existence of a linear functional which will correspond to a finitely additive probability measure P on I. With this method, Domotor solves also problems which are more general than (P) above.[1] In particular, Domotor gives the necessary and sufficient conditions for a binary relation $\geqq$ on $I \times I$ (where I is a finite Boolean algebra) to be realizable by a conditional probability measure. Villegas (1964) has found sufficient conditions for a binary relation $\geqq$ on I to be realizable by a countably additive probability measure.

2. Savage's theorem. The general solution of Scott for the problem (P) is very complex in the case of infinite Boolean algebras, and therefore of limited interest to the theory of subjective probability. The conditions of Savage and Luce, both of which are jointly sufficient but not necessary for the realizability of $\geqq$, can be more easily interpreted in terms of subjective qualitative probability. For Savage, the problem (P) is only a part of a more general problem of finding postulates for the so-called Subjective Expected Utility Model, where a preference relation between acts is a primitive concept and the qualitative proba-

[1] Some of these have also been considered by Koopman (1940), Luce (1968), and Fishburn (1969).

bility relation is defined by a preference between gambles conditioned on events.

Let $(X, I, \geqq)$ be a qualitative probability structure. Following Savage (1954), we shall say that the qualitative probability relation $\geqq$ is *fine* if for all $B > \varnothing$, $B \in I$, there is a partition $\{X_1, \cdots, X_n\}$ of X such that $X_i \in I$ and

$$B \geqq X_i\,, \qquad \text{for all } i = 1, \cdots, n\,. \tag{1}$$

Two events B and C are *almost equivalent* if

$$B \cup E \geqq C \qquad \text{and} \qquad C \cup F \geqq B$$

for all $E > \varnothing$, $F > \varnothing$ in I such that $B \cap E = C \cap F = \varnothing$. This will be denoted by $B \sim^* C$. Trivially, $B \sim C$ implies $B \sim^* C$. The qualitative probability relation $\geqq$ is *tight* if $B \sim^* C$ implies $B \sim C$.

The main result of Savage ((1954) page 38) is the following:

THEOREM 1. *Let $(X, I, \geqq)$ be a fine and tight* QP-*structure, where $I = 2^X$ = the set of all subsets of X. Then the relation $\geqq$ is realizable by a unique probability measure P. Moreover, for this P*

(i) $B \sim^* C$ iff $P(B) = P(C)$

(ii) $P(B) > 0$ *if* $B > \varnothing$

(iii) *for all B in I and for all real numbers a, $0 \leqq a \leqq 1$, there is a set $C \subseteqq B$, C in I, such that*

$$P(C) = aP(B)\,.$$

A proof of this theorem can be found in Savage ((1954) pages 33–38) and in Fishburn ((1970) pages 194–199). The strong consequence (iii) is needed in the construction of a utility function in Savage's system.

In this note we shall show that Savage's proof for his theorem is in need of a minor correction, which can be effected by joining an additional premise to his Theorem 3 ((1954) page 37). In our argument, the following notions are used. An event A in I is an *atom* of the QP-structure $(X, I, \geqq)$ if $A > \varnothing$ and for all $B \subseteqq A$, $B \in I$,

$$B \sim \varnothing \qquad \text{or} \qquad B \sim A\,.$$

The relation $\geqq$ is *atomless* in X if there are no atoms of $(X, I, \geqq)$. A QP-structure $(X, I, \geqq)$ is atomless if the relation $\geqq$ is atomless in X. Two events B and C in I are *almost incompatible* if $B \cap C \sim \varnothing$.

3. Fine, tight, and atomless QP-structures. The following lemma is a simple consequence of the definition of QP-structures.

LEMMA 1. *If $(X, I, \geqq)$ is a* QP-*structure, then for all A, B, C, and D in I*

(a) *if* $B \subseteqq A$, *then* $B \leqq A$

(b) *if* $A \cap B = \varnothing$, $A \geqq C$, *and* $B \geqq D$, *then* $A \cup B \geqq C \cup D$

(c) *if* $A > \varnothing$ *and* $A \cap B = \varnothing$, *then* $A \cup B > B$

(d) *for* $B \subseteqq A$, $A > B$ iff $A - B > \varnothing$.

By C3, Lemma 1(a), 1(c), and 1(d), we have

LEMMA 2. *Let* $(X, I, \geqq)$ *be a* QP-*structure. Then*

(a) *an event* $A > \emptyset$ *in* I *is an atom of* $(X, I, \geqq)$ iff *there is no subset* B *of* A *such that* $\emptyset < B < A$.

(b) *the relation* $\geqq$ *is atomless in* X iff *every event* $A > \emptyset$ *in* I *can partitioned into* B *and* C *in* I *such that* $A = B \cup C$, $B \cap C = \emptyset$, $B > \emptyset$ *and* $C > \emptyset$.

LEMMA 3. *Let* $(X, I, \geqq)$ *be a fine atomless* QP-*structure. For every* G *in* I, $G > \emptyset$, *there is a sequence of events* $\{G_n\}$, $n = 1, 2, \cdots$ *in* I *such that* $G_1 = G$, $G_{n+1} \subseteqq G_n$, $G_n > \emptyset$ *for all* n, *and* $G_n \downarrow \emptyset$, *i.e. for all* $H > \emptyset$ *there is* n_H *such that* $G_n < H$ *for* $n \geqq n_H$.

PROOF. Because $G > \emptyset$ is not an atom, there is an event $A_1 \subseteqq G$ such that $\emptyset < A_1 < G_1 = G$. If $A_1 \leqq G_1 - A_1$, we choose $G_2 = A_1$. Otherwise we choose $G_2 = G_1 - A_1$. Continuing in this way we get a sequence $\{G_n\}$, $n = 1, 2, \cdots$ such that $G_1 = G$, $G_{n+1} \subseteqq G_n$, $G_n > \emptyset$ for all n and

$$G_{n+1} \leqq G_n - G_{n+1}\,.$$

Then $G_n \downarrow \emptyset$ when $n \to \infty$. Namely, by the fineness of $\geqq$, for every $H > \emptyset$ there is a partition $\{X_1, \cdots, X_m\}$ of X such that $\emptyset < X_i \leqq H$ for all $i = 1, \cdots, m$. If we now choose k so that $2^{k-1} > m$, then the assumption $G_n \geqq H$ for all n leads to a contradiction $X < G_1$ (with Lemma 1(a)) by using Lemma 1(c), because G_1 contains 2^{k-1} pairwise disjoint events which are at least as probable as G_n.

LEMMA 4. *Let* $(X, I, \geqq)$ *be a fine* QP-*structure. Then the following conditions are equivalent:*

(i) $\geqq$ *is atomless in* X

(ii) *for all* $B > \emptyset$ *and* $C > \emptyset$ *in* I, *there is* $D \subseteqq C$ *in* I *such that* $\emptyset < D < B$

(iii) X *is not equivalent to a union of* n *equivalent atoms.*

PROOF. (i) implies (ii). This is trivial, if $\emptyset < C < B$. If $C \geqq B > \emptyset$, use Lemma 3.

(ii) implies (i). By choosing $B = C$ and Lemma 2(a).

(i) trivially implies (iii).

(iii) implies (i). This is proved by showing that if there is one atom in $(X, I, \geqq)$ then X is equivalent to the union of n equivalent atoms, for some n. Let A be the only atom of $(X, I, \geqq)$. If $A < X$, then $\geqq$ is atomless in $X - A > \emptyset$. Take $B \subseteqq X - A$ such that $\emptyset < B < A$ (by the last part of the proof of Lemma 3). Then there cannot exist a partition $\{X_1, \cdots, X_m\}$ of X such that $X_i \leqq B$ for all $i = 1, \cdots, m$, because

$$X_i \cap A > \emptyset \qquad \text{for some } i = i_0$$

whence

$$X_{i_0} \geqq X_{i_0} \cap A \sim A > B$$

(A is an atom). Because $\geqq$ was supposed to be fine, we have a contradiction. Hence, $A \sim X$. If there is more than one atom, then by the fineness of $\geqq$ they must all be equivalent. Thus, by the fineness of $\geqq$, X must be equivalent to the union of n atoms, for some n.

LEMMA 5. *If $(X, I, \geqq)$ is a* QP-*structure such that X is equivalent to a union of n equivalent atoms, then the relation $\geqq$ is realizable by a unique probability measure P.*

PROOF. Let X be equivalent to the union of n equivalent atoms X_i, $i = 1, \cdots, n$. Without a loss in generality, we may assume that these atoms are almost incompatible. Then for every A in I there is a unique number k_A such that

$$k_A = \text{ the number of indexes } i \text{ in } \{1, \cdots, n\} \text{ such that } X_i \cap A > \varnothing .$$

Then $k_X = n$, $k_{X_i} = 1$ for all $i = 1, \cdots, n$, and

$$k_{A \cup B} = k_A + k_B , \qquad \text{if } A \cap B = \varnothing .$$

Moreover,

$$A \geqq B \quad \text{iff} \quad k_A \geqq k_B .$$

The probability measure P on I can, then, be defined by

$$P(A) = k_A/n , \qquad \text{for all } A \text{ in } I .$$

In the QP-structure $(X, I, \geqq)$ of Lemma 5, each atom is almost equivalent to $\varnothing$. This shows that the relation $\geqq$ is not tight. More generally, we can prove the following lemma.

LEMMA 6. *If a* QP-*structure $(X, I, \geqq)$ contains an atom A, then $\geqq$ is not tight.*

PROOF. Let $A > \varnothing$ be an atom of $(X, I, \geqq)$. Then by Lemma 1(d) $X-A < X$. However, $X-A \sim^* X$. Namely, if E is an event in I such that $E > \varnothing$ and $E \cap (X-A) = \varnothing$, then $E \subseteqq A$ and (because A is an atom) $E \geqq A$. Hence, by C5

$$(X-A) \cup E \geqq (X-A) \cup A = X .$$

Therefore, $\geqq$ is not tight.

By joining Lemmas 4 and 6, we have the following result.

THEOREM 2. *Let $(X, I, \geqq)$ be a* QP-*structure. If $\geqq$ is tight, then $\geqq$ is atomless in X. If $\geqq$ is fine, then either $\geqq$ is atomless in X or X is equivalent to the union of n equivalent atoms.*

4. The proof of Savage's theorem. In Theorem 3 of Savage ((1954) page 37), it is assumed that $\geqq$ is a fine qualitative probability relation, and claimed that the condition (ii) of Lemma 4 holds. This is, however, equivalent, by Lemma 3, to the claim that $\geqq$ is atomless in X. This is a mistake, as we see from

Theorem 2.

Instead of Savage's Theorem 3, we can prove the following Lemma.[2]

LEMMA 7. *Let* $(X, I, \geqq)$ *be a fine* QP-*structure, where* X *is not equivalent to* n *equivalent atoms. Then*

(i) *if* $B \sim^* G$, $C \sim^* H$, *and* $B \cap C = G \cap H = \emptyset$, *then* $B \cup C \sim^* G \cup H$

(ii) *every finite partition of* X *into almost equivalent events is an almost uniform partition of* X

(iii) *every event* B *in* I *can be partitioned into two almost equivalent events*

(iv) *every event* B *in* I *can be partitioned into* 2^n *almost equivalent events for all* n.

PROOF. (i) Suppose that $B \cup C < X$ and let $E > \emptyset$ be an event such that $E \cap (B \cup C) = \emptyset$. Because $\geqq$ is atomless in X (Theorem 2), we can partition E into disjoint parts $E_1 > \emptyset$ and $E_2 > \emptyset$ (Lemma 2(b)). By assumption, $B \cup E_1 \geqq G$ and $C \cup E_2 \geqq H$. Because of $(B \cup E_1) \cap (C \cup E_2) = \emptyset$, we have by Lemma 1(b) $(B \cup E_1) \cup (C \cup E_2) \geqq G \cup H$, that is $(B \cup C) \cup E \geqq G \cup H$.

(ii) Let $X_1, \cdots, X_n$ be a n-fold partition of X into almost equivalent parts. By (i), all unions of r elements of this partition are almost equivalent. Hence, no union of r elements is more probable than any union of $r + 1$ elements, which means that $\{X_1, \cdots, X_n\}$ is an almost uniform partition of X.

(iii) Fishburn ((1970) pages 195–198), following Savage's suggestion, has given a detailed proof for a similar theorem according to which every event can be partitioned into two equivalent events, if $\geqq$ is fine and tight. His proof with some modifications, which are due to the fact that $\geqq$ in our case may be non-tight, applies also to our claim (iii).[3]

Corresponding to Fishburn's theorems C6 and C7 ((1970) page 195) we have

---

[2] This paper grew out of an unsuccessful attempt to construct a proof for the first step of Theorem 3 about fine qualitative probabilities in L. J. Savage's *The Foundations of Statistics* ((1954) page 37). This step could be easily justified by changing the original definition of fineness with a stronger one requiring that for every event $B > \emptyset$ there exists a partition $\{X_1, \cdots, X_n\}$ of X such that $B > X_i$, for all $i = 1, \cdots, n$, i.e. that each element of the partition is strictly less probable than B. The main point of this note is that this change is not necessary. Assuming the original definition of fineness, the main theorem of Savage can be proved, but then his Theorem 3 has to be reformulated.

[3] Villegas ((1964) page 1791) proves that every event B can be partitioned into two equivalent events if the relation $\geqq$ is atomless and monotonely continuous. His idea of using Zorn's lemma to show the existence of a minor incomplete partition of B, which is maximal with respect to inclusion, fails in our case, however, as was pointed out to me by Professor Savage. Namely, let $X = \{a_i \mid i = 1, 2, \cdots\}$ be a denumerable set such that $\geqq$ is fine in X and for all $B \subseteq X$, $B \sim \emptyset$ iff B is finite. If B is an event for which $\emptyset < B \leqq X - B$, then the sequence of events, $X_1 = B$; $X_2 = X_1 \cup \{a_1\}$; $\cdots$; $X_{n+1} = X_n \cup \{a_n\}$; $\cdots$, defines a chain of minor incomplete partitions of X. However, every upper bound of it is equivalent to X and therefore, is not a minor incomplete partition of X. This gives us, in effect, an example of a fine and atomless QP-structure where $\geqq$ is not monotonely continuous.

(A) If A, B, and C are pairwise disjoint events such that $A \leqq B$, $B < A \cup C$, and not $B \sim^* A \cup C$, then there is a $D \subseteqq C$ for which $D > \varnothing$ and $B \cup D < A \cup (C-D)$.

(B) If $A > \varnothing$, $B > \varnothing$, and $A \cap B = \varnothing$, then B can be partitioned into C and D for which $C \leqq D \leqq A \cup C$.

The proof of (B) is the same as in Fishburn ((1970) page 196). For the proof of (A), note first that from $B < A \cup C$ and not $B \sim^* A \cup C$ it follows that there is an event $D_1 > \varnothing$ for which $D_1 \cap B = \varnothing$ and $B \cup D_1 < A \cup C$. Because $\geqq$ is fine and atomless (Theorem 2), there is a $D_2 \subseteqq C$ such that $\varnothing < D_2 < D_1$ (Lemma 4 (ii)). Hence, by C5

$$B \cup D_2 < B \cup D_1 < A \cup C .$$

By Lemma 2(a), the event D_2 can be partitioned into D and D' such that $\varnothing < D \leqq D'$. Hence, from

$$B \cup D \cup D' < A \cup (C-D) \cup D$$

it follows by (C5) that

$$B \cup D \leqq B \cup D' < A \cup (C-D) .$$

Let $A > \varnothing$ be an event in I. Because $\geqq$ is atomless, there is a partition $\{B_1, C_1, D_1\}$ of A such that $B_1 \leqq C_1 \cup D_1$ and $C_1 \leqq B_1 \cup D_1$. If one of these two $\leqq$ is $\sim$ or $\sim^*$, our proof is complete. Assume then that

$$B_1 < C_1 \cup D_1 , \quad C_1 < B_1 \cup D_1 , \quad \text{not } B_1 \sim^* C_1 \cup D_1 , \text{ not } C_1 \sim^* B_1 \cup D_1 .$$

Then following Fishburn, but using (A) and (B) in the place of his C6 and C7, we can show that there is a sequence of partitions $\{B_n, C_n, D_n\}$ of A such that for all $n = 1, 2, \cdots$

(i) $B_n < C_n \cup D_n$, $C_n < B_n \cup D_n$
(ii) not $B_n \sim^* C_n \cup D_n$, not $C_n \sim^* B_n \cup D_n$
(iii) $B_n \subseteqq B_{n+1}$, $C_n \subseteqq C_{n+1}$, $D_{n+1} \subseteqq D_n$
(iv) $D_{n+1} \leqq D_n - D_{n+1}$, $D_n > \varnothing$.

As in the proof of Lemma 3, Condition 4 implies that $D_n \downarrow \varnothing$. Let

$$B = \bigcup_1^\infty B_n , \qquad C = (\bigcup_1^\infty C_n) \cup (\bigcap_0^\infty D_n) .$$

Then $A = B \cup C$, $B \cap C = \varnothing$, and $B \sim^* C$. Namely, following Fishburn's proof with (A) in the place of C6, the assumptions $B < C$ and not $B \sim^* C$ lead to a contradiction with $C_n < B_n \cup D_n$. (Note that not $B \sim^* C$ implies that not $B \sim^* \cup C_n$, because $\cap D_n \sim \varnothing$.) Hence,

$$B \geqq C \qquad \text{or} \qquad B \sim^* C .$$

Similarly, the assumptions $\cup C_n < B \cup (\cap D_n)$ and not $B \sim^* C$ lead to a contradiction with $B_n < C_n \cup D_n$. Hence,

$$B \leqq C \qquad \text{or} \qquad B \sim^* C .$$

These two conclusions imply

$$B \sim C \quad \text{or} \quad B \sim^* C,$$

which concludes the proof of (iii). (iv) follows then immediately.

From Theorem 2, Lemma 7 (ii) and (iv), and a theorem of Savage ((1954) page 34), Theorem 3 follows immediately:

THEOREM 3. *If* $(X, I, \geqq)$ *is a fine* QP-*structure, then* $(X, I, \geqq)$ *is almost realizable by a unique probability measure P.*

This theorem can be strengthened to the following form:

THEOREM 4. *If* $(X, I, \geqq)$ *is a fine* QP-*structure, then* $(X, I, \geqq)$ *is almost realizable by a unique probability measure P. If, in addition, X is not equivalent to the union of n equivalent atoms, then for this probability measure P*

(i) $B \sim^* C$ iff $P(B) = P(C)$

(ii) $P(B) > 0$ *if* $B > \varnothing$

(iii) *for B in I and for all real numbers a*, $0 \leqq a \leqq 1$, *there is* $C \subseteq B$ *such that* $P(C) = aP(B)$.

PROOF. (a) The existence of P and (iii) follow from Theorem 3 and a theorem of Savage ((1954) page 34).

(b) $P(B) = P(C)$ if $B \sim^* C$: Because $\geqq$ is almost realizable by P and $B \sim^* C$, we have

$$P(B) + P(G) \geqq P(C)$$
$$P(C) + P(H) \geqq P(B)$$

for all $G > \varnothing$, $H > \varnothing$, $G \cap B = H \cap C = \varnothing$. Let $H > \varnothing$ and $\varepsilon > 0$. By (iii) we can choose G such that $G \leqq H$ (whence $P(G) \leqq P(H)$) and $P(G) = \varepsilon P(X) = \varepsilon$. Hence

$$|P(B) - P(C)| \leqq P(G) = \varepsilon .$$

(c) $P(B) > 0$, if $B > \varnothing$: By lemma 7 (iv) there is a partition of X into 2^n almost equivalent events X_i, $i = 1, \cdots, 2^n$, for all n. Choose n so great that $X_i \leqq B$ for all $i = 1, \cdots, 2^n$. By (b) of this proof, $P(X_i) = P(X_j)$ for all i, j. Hence $P(X_i) = 1/2^n > 0$ for all $i = 1, \cdots, 2^n$. Therefore, $P(B) \geqq P(X_i) > 0$.

(d) $B \sim^* C$ if $P(B) = P(C)$: Suppose that there is an event $G > \varnothing$, $G \cap B = \varnothing$, such that $B \cup G < C$. Then by (iii) of this theorem, $P(G) > 0$. Hence,

$$P(C) \geqq P(B \cup G) = P(B) + P(G) > P(B),$$

which contradicts our assumption $P(B) = P(C)$.

Suppose now that $(X, I, \geqq)$ is a fine and tight QP-structure. Then it is almost realizable by a unique probability measure P. Then

$$P(B) = P(C) \quad \text{iff} \quad B \sim^* C \qquad \text{(by Theorem 4 (i))}$$
$$\text{iff} \quad B \sim C \qquad (\geqq \text{ is tight}).$$

Therefore $(X, I, \geqq)$ is realizable by P, and the main theorem of Savage (Theorem 1 above) follows immediately from Theorem 4.

Savage ((1954) page 40) claims that no fine and non-tight QP-structure can be realizable. This claim needs the following qualification, as can be seen from the case studied in Lemma 5.

THEOREM 5. *Let $(X, I, \geqq)$ be a fine and non-tight* QP-*structure, where X is not equivalent to the union of n equivalent atoms. Then $(X, I, \geqq)$ is not realizable.*

PROOF. Because $\geqq$ is not tight, there are events B and C such that $B < C$ but $B \cup E \geqq C$ for all $E > \emptyset$, $E \cap B = \emptyset$. If $(X, I, \geqq)$ were realizable by P, then we should have

$$P(B) < P(C) \qquad \text{and} \qquad P(B) + P(E) \geqq P(C)\,,$$

that is

$$P(E) \geqq P(C) - P(B) > 0\,,$$

for all $E > \emptyset$, $E \cap B = \emptyset$. Then by Lemma 3, $\geqq$ could not be atomless in X, which contradicts Theorem 2.

5. Qualitative probability σ-algebras. Following Villegas (1964), we shall say that the pair $(I, \geqq)$ is a *qualitative probability algebra*, if $(X, I, \geqq)$ is a QP-structure, where I is a σ-algebra. Then the relation $\geqq$ is *monotonely continuous* if for every sequence $\{A_n\}$, $n = 1, 2, \cdots$, in I, where $A_1 \subseteqq A_2 \subseteqq \cdots$ and $A_n \uparrow A$, and for event B in I such that $B \geqq A_n$ for all n, we have $B \geqq A$. If the relation $\geqq$ is monotonely continuous, we shall say that $(I, \geqq)$ is a *qualitative probability σ-algebra.*

The main theorems of Villegas follow:

THEOREM 6. *Let $(I, \geqq)$ be a qualitative probability algebra which is realizable by a probability measure P. Then P is σ-additive if and only if $\geqq$ is monotonely continuous.*

THEOREM 7. *If a qualitative probability σ-algebra is atomless, it is realizable by a unique σ-additive probability measure.*

Villegas proves also that monotonely continuous and atomless qualitative probability relations are fine and tight.

LEMMA 8. *If a qualitative probability σ-algebra $(I, \geqq)$ is atomless, then $\geqq$ is fine and tight.*

PROOF. $\geqq$ is fine and tight iff for all B and C such that $B < C$ there is a partition $\{X_1, \cdots, X_m\}$ of X such that $X_i \cup B < C$ for all $i = 1, \cdots, m$. (See Savage (1954) Theorem 4, page 38.) By Theorem 7, $\geqq$ is realizable by P. Hence $P(B) < P(C)$. Choose n so that

$$0 < 1/2^n < P(C) - P(B)\,.$$

By Theorem 4 of Villegas (1964), there an 2^n-fold uniform partition $\{X_i\}$ of X, that is

$$P(X_i) = 1/2^n , \qquad \text{for all } i = 1, \cdots, 2^n .$$

Then

$$P(B \cup X_i) \leqq P(B) + P(X_i) = P(B) + 1/2^n$$
$$< P(B) + P(C) - P(B) = P(C) .$$

Hence $B \cup X_i < C$ for all $i = 1, \cdots, 2^n$.

Villegas proves then that every fine and tight probability algebra can be extended to a qualitative probability σ-algebra. He concludes that "there is no loss in generality if we consider only qualitative probabilities which are monotonely continuous" (page 1787). Nevertheless, his conclusion seems to be in need of a qualification. In fact, we can prove a stronger result than Villegas.

THEOREM 8. *If a qualitative probability algebra is fine and tight, then it can be extended to an atomless qualitative probability σ-algebra.*

PROOF. Let $(I_0, \geqq_0)$ be a fine and tight qualitative probability algebra. By Savage's theorem it is realizable by a unique finitely additive probability measure P_0. Then the probability algebra (I_0, P_0) can be extended to a probability σ-algebra (I, P), where I is a σ-algebra and P is a σ-additive probability measure on I. Define now

$$A > B \quad \text{iff} \quad P(A) \geqq P(B)$$

for all A and B in I. Because P is σ-additive, then by Theorem 6 the relation $\geqq$ is monotonely continuous. By Theorem 2, the relation $\geqq_0$ is atomless. Therefore P_0, P, and $\geqq$ are also atomless.

According to Theorem 8, any fine and tight qualitative probability algebra can be extended to a σ-algebra which is uniquely realizable by a σ-additive probability measure. It does not, however, guarantee that this extension can be made in a natural and straightforward way. It is known that not every finitely additive probability measure P_0 defined on an algebra of subsets I_0 of a set X can be extended to a σ-additive probability measure P on the σ-algebra I generated by I_0. The reason for this fact may be that X has too few points. Therefore, the extension may be possible only by adding new points to the set X (see Sikorski (1964) page 203). Let, for example, X be the closed interval $[0, 1]$. There are finitely additive extensions of Lebesgue measure to the set 2^X of all subsets of X, giving the same measure to congruent sets. If we define for all B and C in 2^X

$$B \geqq C \quad \text{iff} \quad P(B) \geqq P(C) ,$$

where P is such an extension of Lebesgue measure, then $(X, 2^X, \geqq)$ is a fine and tight QP-structure. However, similar σ-additive extensions of Lebesgue measure to 2^X do not exist (see Savage (1954) page 40). Therefore $(x, 2^X, \geqq)$ cannot be extended to a qualitative probability σ-algebra without adding points to $X = [0, 1]$.

Acknowledgment. I am greatly indebted to the late Professor Leonard J. Savage who, before his untimely death, kindly read an earlier version of this note and had a friendly and encouraging discussion with me during The IVth International Congress for Logic, Methodology and Philosophy of Science in Bucharest, 1971.

REFERENCES

DOMOTOR, Z. (1969). Probabilistic relational structures and their applications. Technical Report No. 144, Department of Statistics, Stanford Univ.

DE FINETTI, B. (1931). Sul significato soggettivo della probabilita. *Fund. Math.* **17** 298–329.

FISHBURN, P. (1969). Weak qualitative probability on finite sets. *Ann. Math. Statist.* **40** 2118–2126.

FISHBURN, P. (1970). *Utility Theory for Decision Making.* Wiley, New York.

KRAFT, C. H., PRATT, J. and SEIDENBERG, A. (1959). Intuitive probability on finite sets. *Ann. Math. Statist.* **30** 408–419.

LUCE, R. D. (1967). Sufficient conditions for the existence of a finitely additive probability measure. *Ann. Math. Statist.* **38** 780–786.

LUCE, R. D. (1968). On the numerical representation of qualitative conditional probability. *Ann. Math. Statist.* **39** 481–491.

SAVAGE, L. J. (1954). *The Foundations of Statistics.* Wiley, New York.

SCOTT, D. (1964). Measurement structures and linear inequalities. *J. Math. Psychol.* **1** 233–247.

SIKORSKI, R. (1964). *Boolean Algebras*, (2nd ed.). Springer-Verlag, Berlin.

VILLEGAS, C. (1964). On qualitative probability σ-algebras. *Ann. Math. Statist.* **35** 1787–1796.

MANNGRHEIMINTIG 25 A 17
HELSINKI 25, FINLAND