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ON BONFERRONI-TYPE INEQUALITIES OF THE SAME DEGREE FOR THE PROBABILITY OF UNIONS AND INTERSECTIONS

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For any collection of exchangeable events $A_1, A_2, \cdots, A_k$ the Bonferroni inequalities are usually stated in the form

$$\max\{N_0, N_2, \cdots, N_{k_e}\} \leqq P\{\bigcup_{i=1}^k A_i\} \leqq \min\{N_1, N_3, \cdots, N_{k_0}\}$$

where $N_0 = 0$, $k_e(k_0)$ is the largest even (odd) integer $\leqq k$,

$$N_\nu = \sum_{\alpha=1}^\nu (-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha \qquad (\nu = 1, 2, \cdots, k)$$

and $P_\alpha = P\{A_{i_1} A_{i_2} \cdots A_{i_\alpha}\}$ for any collection of α events. We may regard N_ν as being of the νth degree because it involves $P_1, P_2, \cdots, P_\nu$; hence the lower and upper bounds above are never of the same degree. In this paper we develop improved lower and upper bounds of the same degree. For degree $\nu = 2$, 3, and 4 these results are given explicitly. A related problem is to get lower and upper bounds for the probability of the intersection of events, P_k, for large k in terms of $P_1, P_2, \cdots, P_\nu$. These are also derived and given explicitly for $\nu = 2$, 3, and 4. Applications of these inequalities to incomplete Dirichlet Type *I*-integrals and to equi-correlated multivariate normal distributions are indicated.

1. Introduction and summary. Let A_i $(i = 1, 2, \cdots, k)$ denote a finite set of events associated with a probability space $(\Omega, \mathscr{F}, P)$ and let $\chi_i(\omega)$ denote the indicator random variable of A_i. Then $\max\{\chi_1(\omega), \chi_2(\omega), \cdots, \chi_k(\omega)\}$ is the indicator variable for the set $\bigcup_{i=1}^k A_i$. In [4] Kounias strengthens the Bonferroni inequalities (cf. Fréchet [2]) by giving both lower and upper bounds for $P\{\bigcup_{i=1}^k A_i\}$ in terms of $P(A_i)$ and $P\{A_i A_j\}$; these bounds are both of the same degree $\nu = 2$. In this paper we consider the fixed collection $A_1, A_2, \cdots, A_k$ with arbitrary ν $(\nu = 1, 2, \cdots, k)$ and derive lower and upper bounds of the same degree ν for $P\{\bigcup_{i=1}^k A_i\}$. Some applications with numerical illustrations are indicated.

For exchangeable random variables $\chi_i(\omega)$ we let $P_\alpha = P\{A_{j_1} A_{j_2} \cdots A_{j_\alpha}\}$. We obtain as a typical example of our bounds for any odd degree $\nu \leqq k$

$$\sum_{\alpha=1}^{\nu-1} (-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha + \binom{k-1}{\nu-1}P_\nu \leqq P\{\bigcup_{i=1}^k A_i\} \leqq \sum_{\alpha=1}^{\nu} (-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha \tag{1.1}$$

and for any even degree $\nu \leqq k$

$$\sum_{\alpha=1}^{\nu} (-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha \leqq P\{\bigcup_{i=1}^k A_i\} \leqq \sum_{\alpha=1}^{\nu-1} (-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha - \binom{k-1}{\nu-1}P_\nu\,. \tag{1.2}$$

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The extra term on the left side of (1.1) and on the right side of (1.2) clearly show the improvement on the Bonferroni bounds. One can easily construct examples where these bounds are attained; in fact, for $\nu = k$ we have equality throughout (1.1) and (1.2). It should be noted that the lower bounds can be negative and the upper bounds can exceed 1. More generally, we obtain for any $\nu \leqq k$ a set of $\nu + 1$ bounds all of the same degree ν (including those in (1.1) or (1.2) depending on the parity of ν) and use the minimum of the upper bounds and the maximum of the lower bounds. Corresponding results are also obtained for bounds on $P(\bigcap_{i=1}^{k} A_i\}$.

2. Basic lemma and its consequences. In order to express the results in a compact notation we will define a Bonferroni indicator random variable and an operation denoted by $*$.

DEFINITION. Let $B_{r,j}$ with $j \leqq k$ and $1 \leqq r \leqq j$ denote the Bonferroni function of degree r on the j sets $A_{i_1}, A_{i_2}, \cdots, A_{i_j}$ defined by

$$B_{r,j} = \sum_{\alpha_1=1}^{j} \chi_{\alpha_1} - \sum_{\alpha_1<\alpha_2} \chi_{\alpha_1}\chi_{\alpha_2} + \cdots + (-1)^{r-1} \sum_{\alpha_1<\alpha_2<\cdots<\alpha_r} \chi_{\alpha_1}\chi_{\alpha_2}\cdots\chi_{\alpha_r} \tag{2.1}$$

where $\chi_\alpha = \chi_\alpha(\omega)$ and $1 \leqq \alpha_i \leqq j$ $(i = 1, 2, \cdots, r)$. We define $B_{0,j}$ to be identically zero for $j \geqq 1$.

DEFINITION. Let $B_{r,i} * B_{s,j}$ denote the function defined by

$$B_{r,i} * B_{s,j} = B_{r,i} + B_{s,j} - B_{r,i}B_{s,j} \,. \tag{2.2}$$

It is implicitly assumed in this definition that the i sets in $B_{r,i}$ do not include any of the j sets in $B_{s,j}$ and vice versa, i.e., the index sets are disjoint, although the sets need not be. It is easily verified that $B_{2,2} = B_{1,1} * B_{1,1}$ and more generally $B_{i,i} * B_{i,j} = B_{i+j,i+j}$. We also have the property

$$B_1 * (B_2 + B_3) = B_1 * B_2 + B_1 * B_3 - B_1 \tag{2.3}$$

which is used in Sections 3 and 4 below.

It is interesting to note for any $*$ product of two B's the sum of the coefficients is identically one. In fact, if Q denotes the operation of setting all χ_i equal to 1

$$\begin{aligned} Q\{B_{r,r} * B_{s,j}\} &= Q\{1 - (1 - B_{r,r})(1 - B_{s,j})\} \\ &= 1 - Q\{1 - B_{r,r}\}Q\{1 - B_{s,j}\} = 1 \end{aligned} \tag{2.4}$$

since $Q\{1 - B_{r,r}\} = \sum_{\alpha=0}^{r} (-1)^\alpha \binom{r}{\alpha} = 0$.

It is easily verified that the $*$ product is both associative and commutative. It should be noted that in general $B_{r,k} * B_{s,j} \neq B_{r+s,k+j}$.

We now state and prove a lemma on certain monotonicities among these Bonferroni functions which will help us to deduce (1.1) and (1.2) when the indicator random variables are exchangeable.

LEMMA 2.1. *For any fixed* $k \geqq 2$ *with* $1 \leqq \nu \leqq k$ *and for any partition of the* *index set* $(1, 2, \cdots, k)$ *into two parts of sizes* 1 *and* $k - 1$, *we have*

$$B_{\nu-1,k} \leqq B_{1,1} * B_{\nu-1,k-1} \leqq B_{k,k} \leqq B_{\nu,k} \quad \text{for } \nu \text{ odd}, \tag{2.5}$$

$$B_{\nu-1,k} \geqq B_{1,1} * B_{\nu-1,k-1} \geqq B_{k,k} \geqq B_{\nu,k} \quad \text{for } \nu \text{ even}. \tag{2.6}$$

PROOF. By direct substitution of (2.1) into the second expression of (2.5) we have for any ν

$$B_{1,1} * B_{\nu-1,k-1} = B_{\nu-1,k} + (-1)^{\nu-1}\chi_1 \sum_{1<\alpha_1<\cdots<\alpha_{\nu-1}} \chi_{\alpha_1} \cdots \chi_{\alpha_{\nu-1}} \tag{2.7}$$

which proves the first inequality in each of (2.5) and (2.6). To prove the second inequality in each of (2.3) and (2.4) we first note that if $B_{r,i} \leqq B_{s,j}$ then $B_{1,1} * B_{r,i} \leqq B_{1,1} * B_{s,j}$ since $B_{1,1} \leqq 1$. It follows that we can iterate the inequality implied by (2.7) for ν odd and even obtaining, respectively,

$$B_{\nu-1,k} \leqq B_{1,1} * B_{\nu-1,k-1} \leqq B_{2,2} * B_{\nu-1,k-2} \leqq \cdots \leqq B_{k,k} \leqq 1, \tag{2.8}$$

$$B_{\nu-1,k} \geqq B_{1,1} * B_{\nu-1,k-1} \geqq B_{2,2} * B_{\nu-1,k-2} \geqq \cdots \geqq B_{k,k} \geqq 0. \tag{2.9}$$

The remaining inequality in each of (2.5) and (2.6) is the well-known Bonferroni bound; this completes the proof of the lemma. The first inequality of (2.9) with $\nu = 2$, namely $B_{1,k} \geqq B_{1,1} * B_{1,k-1}$, is the inequality that appears in Kounias [4]. Of course, this can be further improved, possibly by using $B_{3,k}$, but for the fixed degree 2 neither the Bonferroni bound $B_{3,k}$ nor the improved bound in (2.9), $B_{2,2} * B_{1,k-2}$, are allowed, since both are of degree 3. More generally we can write (2.7), (2.8), and (2.9) with $\nu - 1$ replaced by $\nu - j$ $(j = 0, 1, \cdots, \nu)$ and then (2.8) will hold for $\nu - j$ even and (2.9) will hold for $\nu - j$ odd.

REMARK 2.1. We note from (2.8) and (2.9) that the degree increases by steps of 1 as we approach $B_{k,k}$ from either side. Hence, for any star product of degree $\nu < k$ (say, $Z = B_{j,k} * B_{\nu-j,k-j}$) there is always another function or star product (either $B_{k,k}$ if $j = k - 1$ or $B_{j+1,j+1} * B_{\nu-j,k-j-1}$ if $j < k - 1$) whose degree is $\nu + 1$ and which is in the same string of inequalities as Z and hence closer to $B_{k,k}$.

3. Bounds for the probability of a union.

In this section we derive lower and upper bounds of the same degree ν for the probability of a union of exchangeable events $A_1, A_2, \cdots, A_k$. These bounds are expressed in terms of P_j $(j = 1, 2, \cdots, \nu)$ and we define the degree of either bound to be ν if it is the largest subscript to appear. It will be convenient to state most of our results in symbolic form involving powers of P and $1 - P$; this means that, after a formal expansion in powers of P we replace P^α by P_α for all α and P_0 by 1.

LEMMA 3.1. *Symbolically, for any* $j \leqq \nu \leqq k$

$$E\{B_{j,j} * B_{\nu-j,k-j}\} = 1 - (1 - P)^j \textstyle\sum_{i=0}^{\nu-j} (-1)^i \binom{k-j}{i} P^i = L_{\nu-j}^{(\nu)} \text{ (say)}. \tag{3.1}$$

PROOF. Using the fact that

$$B_{\nu-j,k-j} = B_{k-j,k-j} + (-1)^{\nu-j+1} \sum_{\alpha_1<\cdots<\alpha_{\nu-j+1}} \chi_{\alpha_1} \cdots \chi_{\alpha_{\nu-j+1}} + \cdots + (-1)^{k-j}\chi_{\alpha_1} \cdots \chi_{\alpha_{k-j}} \tag{3.2}$$

and the property (2.3), we find after taking expectations that we have symbolically

$$
\begin{aligned}
&E\{B_{j,j} * B_{\nu-j,k-j}\} \\
(3.3)\qquad &= 1-(1-P)^k + (1-P)^j \textstyle\sum_{\alpha=1}^{k-\nu} (-1)^{\nu-j+\alpha}\binom{k-j}{\nu-j+\alpha} P^{\nu-j+\alpha} \\
&= 1-(1-P)^k + (1-P)^j\{(1-P)^{k-j} - \textstyle\sum_{i=0}^{\nu-j}(-1)^i\binom{k-j}{i}P^i\} \\
&= 1-(1-P)^j \textstyle\sum_{j=0}^{\nu-j}(-1)^i\binom{k-j}{i}P^i .
\end{aligned}
$$

We use below the notation $L_{\nu-j}$ for $L_{\nu-j}^{(\nu)}$ when there is no danger of confusion.

Our bounds of the νth degree for the probability of a union are succinctly given in the following

THEOREM 1. *For any fixed degree* $\nu \leqq k$

$$(3.4)\qquad \max\{L_0, L_2 \cdots, L_{\nu_e}\} \leqq P\{\textstyle\bigcup_{i=1}^k A_i\} \leqq \min\{L_1, L_3, \cdots, L_{\nu_0}\},$$

where $\nu_e(\nu_0)$ *is the largest even (odd) integer* $\leqq \nu$,

$$(3.5)\qquad L_{\nu-j} = \textstyle\sum_{\alpha=1}^{m}(-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha + \sum_{\alpha=m+1}^{\nu}(-1)^{\alpha-1}a_\alpha P_\alpha \qquad (0 \leqq j \leqq \nu \leqq k),$$

$a_\alpha = \sum_{\beta \geqq \alpha+j-\nu}\binom{j}{\beta}\binom{k-j}{\alpha-\beta}$ *and* $m = \min\{j, \nu-j\}$.

PROOF. From (2.8) and (2.9) it follows that $L_{\nu-j} = E\{B_{j,j} * B_{\nu-j,k-j}\}$ is a lower bound for $\nu - j$ even and is an upper bound for $\nu - j$ odd. The remainder of the proof lies in the fact that the expression for $L_{\nu-j}$ in (3.5) can be obtained by direct expansion of (3.1).

As a corollary to either Lemma 3.1 or Theorem 1 we write out one of our bounds explicitly corresponding to $j = 1$.

COROLLARY. *For any integer* $\nu \leqq k$

$$(3.6)\qquad L_{\nu-1} = \textstyle\sum_{\alpha=1}^{\nu-1}(-1)^{\alpha-1}\binom{k}{\alpha}P_\alpha + (-1)^{\nu-1}\binom{k-1}{\nu-1}P_\nu$$

which is a lower (upper) bound on $P\{\bigcup_{i=1}^k A_i\}$ *for* ν *odd (even).*

PROOF. This follows from (3.4) after expanding (3.1) with $j = 1$. □

REMARK 3.1. It follows from Remark 2.1 that for any one of our νth order bounds with $\nu < k$ there will always be a $\nu + 1$st order bound which is closer to the true value; here we assume that the collection $A_1, A_2, \cdots, A_k$ remains fixed (cf. Remark 5.2).

The result (3.6) gives the two new bounds in (1.1) and (1.2); the other two bounds in (1.1) and (1.2) correspond to $j = 0$ in (3.1).

Since these bounds are especially useful for small values of ν, we write them explicitly for $\nu = 2$, 3, and 4. For degree $\nu = 2$ the bounds are for $k \geqq 2$

$$(3.7)\qquad L_0^{(2)} = 2P_1 - P_2; \qquad L_2^{(2)} = kP_1 - \binom{k}{2}P_2; \qquad L_1^{(2)} = kP_1 - (k-1)P_2 .$$

For degree $\nu = 3$ the bounds are for $k \geqq 3$

$$
\begin{aligned}
(3.8)\qquad &L_0^{(3)} = 3P_1 - 3P_2 + P_3; &&L_2^{(3)} = kP_1 - \binom{k}{2}P_2 + \binom{k-1}{2}P_3, \\
&L_1^{(3)} = kP_1 - (2k-3)P_2 + (k-2)P_3; &&L_3^{(3)} = kP_1 - \binom{k}{2}P_2 + \binom{k}{3}P_3 .
\end{aligned}
$$

For degree $\nu = 4$ the bounds are for $k \geqq 4$

$$
\begin{aligned}
L_0^{(4)} &= 4P_1 - 6P_2 + 4P_3 - P_4\,; \\
L_2^{(4)} &= kP_1 - \tbinom{k}{2}P_2 + (k-2)^2 P_3 - \tbinom{k-2}{2}P_4\,, \\
L_4^{(4)} &= kP_1 - \tbinom{k}{2}P_2 + \tbinom{k}{3}P_3 - \tbinom{k}{4}P_4\,, \\
L_1^{(4)} &= kP_1 - 3(k-2)P_2 + (3k-8)P_3 - (k-3)P_4\,, \\
L_3^{(4)} &= kP_1 - \tbinom{k}{2}P_2 + \tbinom{k}{3}P_3 - \tbinom{k-1}{3}P_4\,.
\end{aligned} \tag{3.9}
$$

For $k = \nu$ we note that all the $\nu + 1$ bounds are identically the same; hence the common value must be the correct value.

It follows from the property (2.4) that for any $L_{\nu-j}$ obtained as the expectation of a product in (3.1), i.e., with $0 < j \leqq \nu$, the sum of the coefficients is 1. For $j = 0$ the sum is easily seen to be $1 - (-1)^\nu\binom{k-1}{\nu}$, which is again 1 for $\nu = k$.

4. Bounds for the probability of a k-fold intersection.

In this section we obtain bounds for the probability of a k-fold intersection of exchangeable events in terms of j-fold intersections with $1 \leqq j \leqq \nu$, where ν is the fixed degree of our bounds ($\nu \leqq k$).

Let $\tilde{A}_i$ denote the complement of A_i ($i = 1, 2, \cdots, k$) and $\tilde{B}_{r,j}$ with $j \leqq k$ and $1 \leqq r \leqq k$ denote the same Bonferroni function as in (2.1) with all A_i replaced by $\tilde{A}_i$. We now prove a lemma corresponding to (3.1) and then obtain a result similar to Theorem 1.

LEMMA 4.1. *For any $j \leqq \nu \leqq k$ all of our* νth *degree bounds can be written in the form*

$$1 - E\{\tilde{B}_{j,j} * \tilde{B}_{\nu-j,k-j}\} = \textstyle\sum_{\alpha=0}^{\nu-j} (-1)^\alpha \binom{k-\nu-1+\alpha}{\alpha}\binom{k-j}{\nu-j-\alpha} P_{\nu-\alpha} = M_{\nu-j}^{(\nu)} \; (say)\,. \tag{4.1}$$

PROOF. Let $\tilde{Q}_i = P\{\tilde{A}_{\alpha_1} \tilde{A}_{\alpha_2} \cdots \tilde{A}_{\alpha_i}\}$. From (3.1) we take complements and write symbolically

$$M_{\nu-j}^{(\nu)} = (1 - \tilde{Q})^j \textstyle\sum_{i=0}^{\nu-j} (-1)^i \binom{k-j}{i} \tilde{Q}^i \tag{4.2}$$

where, after expanding in powers of $\tilde{Q}$, we replace $\tilde{Q}^i$ by $\tilde{Q}_i$. Using inclusion-exclusion, we find that

$$\tilde{Q}_i = \textstyle\sum_{j=0}^{i} (-1)^j \binom{i}{j} P_j = (1 - P)^i \tag{4.3}$$

where $P_j = P\{A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_j}\}$ for any subset of size j as in previous sections and $P_0 = 1$. It follows that

$$(1 - \tilde{Q})^j = \textstyle\sum_{\alpha=0}^{j} (-1)^\alpha \binom{j}{\alpha} (\tilde{Q})^\alpha = \sum_{\alpha=0}^{j} (-1)^\alpha \binom{j}{\alpha} (1 - P)^\alpha = P^j\,. \tag{4.4}$$

Hence (4.2) can be written as

$$
\begin{aligned}
M_{\nu-j}^{(\nu)} &= \textstyle\sum_{\alpha=0}^{\nu-j} (-1)^\alpha P^{\alpha+j} \sum_{i=\alpha}^{\nu-j} (-1)^i \binom{i}{\alpha}\binom{k-j}{i} \\
&= \textstyle\sum_{\alpha=0}^{\nu-j} (-1)^{\nu-j-\alpha} \binom{k-j}{\alpha}\binom{k-j-\alpha-1}{\nu-j-\alpha} P^{\alpha+j}\,.
\end{aligned} \tag{4.5}
$$

This reduces to (4.1) and proves the lemma if we set $\nu - j - \alpha = \gamma$ and replace P^i by P_i. It should be noted that the $M_{\nu-j}^{(\nu)} = M_{\nu-j}$ ($j = 0, 1, \cdots, \nu$) are all of degree ν and we drop the superscript when there is no danger of confusion.

Our bounds of the νth degree for the probability of a k-fold intersection are succinctly given in the following

THEOREM 2. *For any fixed degree* $\nu \leqq k$

$$\max\{M_1, M_3, \cdots, M_{\nu_0}\} \leqq P\{\bigcap_{i=1}^k A_i\} \leqq \min\{M_0, M_2, \cdots, M_{\nu_e}\}, \tag{4.6}$$

where $M_{\nu-j}$ *is defined by* (4.1).

PROOF. Since $\tilde{L}_{\nu-j} = E\{\tilde{B}_{j,j} * \tilde{B}_{\nu-j,k-j}\}$ are bounds on $P\{\bigcup_{i=1}^k \tilde{A}_i\}$ we obtain from (3.4)

$$\begin{aligned}\max\{1 - \tilde{L}_1, 1 - \tilde{L}_3, \cdots, 1 - \tilde{L}_{\nu_0}\} \\ \leqq 1 - P\{\bigcup_{i=1}^k \tilde{A}_i\} \leqq \min\{1 - \tilde{L}_0, 1 - \tilde{L}_2, \cdots, 1 - \tilde{L}_{\nu_e}\}.\end{aligned} \tag{4.7}$$

Since $\widetilde{\bigcup_{i=1}^k \tilde{A}_i} = \bigcap_{i=1}^k A_i$ and $1 - \tilde{L}_{\nu-j} = M_{\nu-j}$, the result (4.6) follows.

COROLLARY. *For the special case* $k - \nu = 1$ *we can write symbolically all our bounds in the form*

$$P^{\nu-\beta}(1 - P)^{\beta+1} \geqq 0 \qquad (\beta = 0, 1, \cdots, \nu). \tag{4.8}$$

PROOF. From (4.1) for $k = \nu + 1$ and $\beta = \nu - j$ we have

$$\begin{aligned}M_\beta = \sum_{\alpha=0}^{\beta} (-1)^\alpha \binom{\beta+1}{\beta-\alpha} P^{\nu-\alpha} = P^{\nu-\beta} \sum_{i=0}^{\beta} (-1)^{\beta+i} \binom{\beta+1}{i} P^i \\ = (-1)^\beta P^{\nu-\beta}(1 - P)^{\beta+1} + P^{\nu+1}.\end{aligned} \tag{4.9}$$

Hence for both odd and even β

$$(-1)^\beta (M_\beta - P_k) = P^{\nu-\beta}(1 - P)^{\beta+1}. \tag{4.10}$$

From (4.6) we know that the left side of (4.10) is nonnegative and this proves the corollary. □

These bounds (4.6) are especially useful for small values of ν and we therefore write them explicitly for $\nu = 2$, 3, and 4. For degree $\nu = 2$ the bounds are for $k \geqq 2$

$$\begin{aligned}M_1^{(2)} &= (k-1)P_2 - (k-2)P_1, \\ M_0^{(2)} &= P_2, \\ M_2^{(2)} &= \binom{k}{2}P_2 - k(k-2)P_1 + \binom{k-1}{2}.\end{aligned} \tag{4.11}$$

For degree $\nu = 3$ the bounds are for $k \geqq 3$

$$\begin{aligned}M_1^{(3)} &= (k-2)P_3 - (k-3)P_2, \\ M_3^{(3)} &= \binom{k}{3}P_3 - (k-3)\binom{k}{2}P_2 + k\binom{k-2}{2}P_1 - \binom{k-1}{3}, \\ M_0^{(3)} &= P_3, \\ M_2^{(3)} &= \binom{k-1}{2}P_3 - (k-1)(k-3)P_2 + \binom{k-2}{2}P_1.\end{aligned} \tag{4.12}$$

For degree $\nu = 4$ the bounds are for $k \geqq 4$

$$\begin{aligned}M_1^{(4)} &= (k-3)P_4 - (k-4)P_3, \\ M_3^{(4)} &= \binom{k-1}{3}P_4 - (k-4)\binom{k-1}{2}P_3 + (k-1)\binom{k-3}{2}P_2 - \binom{k-2}{3}P_1, \\ M_0^{(4)} &= P_4, \\ M_2^{(4)} &= \binom{k-2}{2}P_4 - (k-2)(k-4)P_3 + \binom{k-3}{2}P_2, \\ M_4^{(4)} &= \binom{k}{4}P_4 - (k-4)\binom{k}{3}P_3 + \binom{k}{2}\binom{k-3}{2}P_2 - k\binom{k-2}{3}P_1 + \binom{k-1}{4}.\end{aligned} \tag{4.13}$$

For $k = \nu$ all the $\nu + 1$ bounds are identical and hence this must be the correct value.

REMARK 4.1. With the help of (4.8) and the Bonferroni inequality it can easily be verified that for the bounds (3.7), (3.8), and (3.9) as well as for the bounds (4.11), (4.12), and (4.13), the result in Remark 3.1 holds. Thus we need not consider all the bounds of degree at most ν, but can restrict our attention to bounds of degree exactly ν.

5. Applications and numerical illustration. Suppose n balls are dropped independently into $k + 1$ cells, k of which have a common single-trial cell probability $p \leqq 1/k$ (and one with probability $1 - kp$). Let $\min(k, n)$ denote the observed minimum of the k cells frequencies in this multinomial problem. Then it is shown by Olkin and Sobel [5] that for $r \geqq 1$ and $n \geqq kr$

$$P\{\min(k, n) \geqq r\} = I_p^{(k)}(r, n) , \tag{5.1}$$

where

$$I_p^{(k)}(r, n) = \frac{\Gamma(n+1)}{\Gamma^k(r)\Gamma(n - kr + 1)} \int_0^p \cdots \int_0^p (1 - \textstyle\sum_{i=1}^k x_i)^{n-kr} \prod_{\alpha=1}^k x_\alpha^{\,r-1} dx_\alpha. \tag{5.2}$$

For $n < kr$ we define $I_p^{(k)}(r, n)$ to be zero. Similarly we let $\max(k, n)$ denote the observed maximum of the k cell frequencies. For $k = 1$ it reduces to the incomplete beta function, $I_p^{(1)}(r, n) = I_p(r, n - r + 1)$.

If we let the set A_i denote the event that the frequency in the ith cell is at least r, then $\bigcap_{i=1}^k A_i$ denotes the event that $\max(k, n) \geqq r$ and the event $A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_j}$ (for any collection of size j) denotes the event that $\min(j, n) \geqq r$. Hence by using the inclusion-exclusion principle, it is easy to see that

$$\begin{aligned} P\{\max(k, n) \geqq r\} = E\{B_{k,k}\} &= \textstyle\sum_{\alpha=1}^k (-1)^{\alpha-1}\binom{k}{\alpha} P\{\min(\alpha, n) \geqq r\} \\ &= \textstyle\sum_{\alpha=1}^k (-1)^{\alpha-1}\binom{k}{\alpha} I_p^{(\alpha)}(r, n) . \end{aligned} \tag{5.3}$$

If $n < kr$ then some of the terms on the right side of (5.3) will vanish so that we need only sum up to $[n/r]$, the integer part of n/r. The distribution of $\max(k, n)$ can be obtained from (5.3) using tables for $I_p^{(\alpha)}(r, n)$ $(\alpha \leqq k)$. However for higher values of k some of these tables are not available and it becomes useful to obtain bounds for the left side of (5.3), preferably of the same degree. Thus we utilize the results of Sections 3 and 4 as illustrated below.

Suppose, for example, we wish to find lower and upper bounds of degree $\nu = 3$ for $P\{\max(k, n) \geqq r\}$ when $k = 10$, $r = 2$, $n = 8$, and $p = \frac{1}{10}$. Using (3.8) for the upper bound we obtain .98715 since

$$E\{B_{3,10}\} = .98715 ; \qquad E\{B_{2,2} * B_{1,8}\} > 1 . \tag{5.4}$$

For the lower bound we use (3.8) or (1.1) and obtain

$$E\{B_{1,1} * B_{2,9}\} = 10P_1 - 45P_2 + 36P_3 = .85132 . \tag{5.5}$$

The exact value, using (5.3), is .98186.

For many values of k, r, and n it should be noted that "degeneracies" will enter into our problem in different possible ways. For example, if $n \geqq kr$ then

$$P\{\max(k, n) \geqq r\} = E\{B_{k,k}\} = 1 , \tag{5.6}$$

since for any distribution of n balls in k cells the maximum frequency is at least $[n/k] \geqq r$. Another type of "degeneracy" is when $E\{B_{\nu,k}\}$ already gives the exact answer. For example, if $[n/r] \geqq \nu$ then $I_p^{(j)}(r, n) = 0$ for $j > \nu$ and hence we see from (5.3) that

$$P\{\max(k, n) \geqq r\} = E\{B_{k,k}\} = E\{B_{\nu,k}\} , \tag{5.7}$$

so that the νth degree bounds give exact answers. In this sense the bounds in (1.1) and (1.2) cannot be further sharpened. In the above numerical illustration $[n/r] = 4 = \nu$ and hence the 4th degree bounds will be exact.

To illustrate numerically the third degree ($\nu = 3$) bounds for the probability of an intersection in (4.12), consider the multinomial example with $k = 10$, $r = 2$, $n = 40$ and $p = .09$. In this example the exact value of $P\{\min(k, n) \geqq r\}$ is

$$P_{10} = I_{.09}^{(10)}(2, 40) = ,24434 . \tag{5.8}$$

From (4.12) the lower bound is $\max(.02826, .23809) = .23809$ and the upper bound is $\min(.29487, .68801) = .29487$.

We can use the same example with a higher ν to see if the bounds get closer. For $\nu = 4$ from (4.13) the lower bound is $\max(.09040, .24070) = .24070$ and the upper bound is $\min(.60264, .27682, .24456) = .24456$. Thus in our example we note that both bounds are closer to the exact value (5.8) for $\nu = 4$ than for $\nu = 3$.

Another application of these bounds is to the case of k equi-correlated multivariate normal chance variables X_i, where we can assume that $E\{X_i\} = 0$ and $\sigma^2(X_i) = 1$ ($i = 1, 2, \cdots, k$). Let A_i denote the event that $X_i < h$ so that $\bigcap_{i=1}^{\alpha} A_i$ is the event that $\max_{1 \leqq i \leqq \alpha} X_i < h$ and $\bigcup_{i=1}^{k} A_i$ is the event that $\min_{1 \leqq i \leqq k} X_i < h$. Again we have the problems of finding bounds of common degree ν on (1) the cdf of $\min_{1 \leqq i \leqq k} X_i$ in terms of $P_\alpha = P_\alpha(h) =$ the cdf of $\max_{1 \leqq i \leqq \alpha} X_i$ for $\alpha = 1, 2, \cdots, \nu$ and on (2) the cdf of $\max_{1 \leqq i \leqq k} X_i$ in terms of the same quantities $P_\alpha(h)$.

For Problem 1 we show below that existing tables can be used to obtain the exact answer if the common correlation ρ between X_i and X_j (for $i \neq j$) is nonnegative. For $\rho \geqq 0$, if we let $X_i = Y_i(1 - \rho)^{\frac{1}{2}} - Y_0 \rho^{\frac{1}{2}}$ ($i = 1, 2, \cdots, k$) where the Y_i are independent standardized normal, then it is easily seen that

$$P_\alpha(h) = \int_{-\infty}^{\infty} \Phi^\alpha\left(\frac{x\rho^{\frac{1}{2}} + h}{(1 - \rho)^{\frac{1}{2}}}\right) d\Phi(x) \qquad (\alpha = 1, 2, \cdots, k) . \tag{5.9}$$

Similarly, the cdf of $H = \min_{1 \leqq i \leqq k} X_i$ is easily seen to be

$$\begin{aligned} G_k(h) &= 1 - \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{x\rho^{\frac{1}{2}} + h}{(1 - \rho)^{\frac{1}{2}}}\right)\right]^k d\Phi(x) \\ &= 1 - \int_{-\infty}^{\infty} \Phi^k\left(\frac{x\rho^{\frac{1}{2}} - h}{(1 - \rho)^{\frac{1}{2}}}\right) d\Phi(x) . \end{aligned} \tag{5.10}$$

To illustrate the use of the bounds consider the case $\rho = \frac{1}{2}$, $h = 1$ and $k = 5$. The exact answers of problems 1 and 2 are .98506 and .58608, respectively. For problem 1 the lower and upper bounds for degree $\nu = 4$ from (3.9) are max(.97859, .97461, .39899) = .97859 and min(.99095, 1.02569) = .99095, respectively. For problem 2 the lower and upper bounds for degree $\nu = 4$ from (4.13) are max(.57562, .57959) = .57959 and min(.62670, .59196, .60101) = .59196, respectively.

REMARK 5.1. In the interchangeable case the Chung–Erdös lower bound [1] for the probability of a union, the matrix inequality of Whittle [6] and Gallot [3] and the extension to the singular case by Kounias [4] all coincide and can be written in the form

$$P\{\bigcup_{i=1}^{k} A_i\} \geqq \frac{kP_1^2}{(k-1)P_2 + P_1}, \tag{5.11}$$

where $P_1 = P\{A_i\}$ and $P_2 = P\{A_i \cap A_j\}$ for all i, j. As pointed out by the referee this bound (5.11), which depends only on P_1 and P_2, is better than our second degree ($\nu = 2$) lower bounds given in (3.7) above when $(k-1)P_2 \leqq (k-2)P_1 \leqq (k-1)(k-2)P_2$. However, it should be noted that (5.11) has (as yet) no generalization to higher degree and for sufficiently high ν ($\leqq k$) our bounds will improve on (5.11). The reader is referred to the paper by Kounias [3] for discussion of still another improvement on (5.11).

REMARK 5.2. It has been noted by E. Kounias that one can further improve our lower bounds when $\nu < k$ by considering all possible subcollections of the interchangeable events $A_1, A_2, \cdots, A_k$. A similar improvement may be possible for the upper bound if we are allowed to adjoin events to our collection to form a larger set of interchangeable events. For example, for the lower bound in Theorem 1 our result is improved by considering a subcollection of size s where $\nu \leqq s \leqq k$ and replacing the left side of (3.4) by

$$\max_{\nu \leqq s \leqq k} \max\{L_{0,s}; L_{2,s}; \cdots; L_{\nu_e,s}\} \leqq P\{\bigcup_{i=1}^{s} A_i\} \leqq P\{\bigcup_{i=1}^{k} A_i\}, \tag{5.12}$$

where $L_{i,s}$ is defined as in (3.5) in terms of the s events $A_1, A_2, \cdots, A_s$. For the left side of (1.1) with $\nu = 3$ this gives

$$\max_{3 \leqq s \leqq k} \{\binom{s}{1}P_1 - \binom{s}{2}P_2 + \binom{s-2}{2}P_3\} \leqq P\{\bigcup_{i=1}^{k} A_i\}. \tag{5.13}$$

From (5.13) with $k = 5$ we find that when

$$\frac{P_1 + 3P_3}{4} < P_2 < \frac{P_1 + 2P_3}{3}, \tag{5.14}$$

holds, $s = 4$ gives a larger (and hence better) result in (5.13) than $s = 3$ or $s = 5$, namely $4P_1 - 6P_2 + 3P_3$. Similar modifications can be made to the bounds in Section 4 (and possibly in the illustrations of Section 5).

It is not known whether our bounds or the improved bounds resulting from the above modification are the best linear bounds in any sense; this is an interesting question that needs further research.

REMARK 5.3. It should be emphasized that our bounds have not been shown in any setting more general than exchangeable events. Actually many interesting and useful applications for such bounds already appear in this setting. In particular, a simple-minded replacement of $\binom{k}{\alpha}P_\alpha$ by S_α (= the sum of the probabilities of all possible intersections of the A_i taken α at a time) in our bounds is not justified since our symbolic methods do make use of exchangeablility (e.g., in the proof of Lemma 3.1).

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