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FELLER'S PARAMETRIC EQUATIONS FOR LAWS OF THE ITERATED LOGARITHM

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In this paper the author considers the two methods that Feller discusses in [3] and [4] which find a sequence b_n so that $\limsup S_n(b_n s_n)^{-1} = 1$ a.s. where $S_n = \sum_{i=1}^n X_i$ and X_i are independent random variables with $EX = 0$, $EX^2 < \infty$ and $E[\exp(hX_i)] < \infty$ for all $h < 0$. The more elementary and general method, which is not developed by Feller in [3], is used in a most elementary manner to derive a theorem general enough to include: ($l(n) \equiv (2 ln ln s_n)^{\frac{1}{2}}$).

(A) Kolmogorov's classical law of the iterated logarithm and the result of Egorov [2]: X_i's bounded and $\sup(X_i)l(n)s_n^{-1} = O(1)$ implies $0 < \limsup S_n(l(n)s_n)^{-1} < \infty$.

(B) A slightly different version of a result of Feller [3]: X_i bounded above, $\sup(X_i)l(n)/s_n = O(1)$ and two other conditions then

$$0 < \limsup S_n(l(n)s_n)^{-1} < \infty$$

(the "slightly different version" is to replace one of the "two other conditions" with a different condition).

(C) A generalization of a Thompson [5]: $X_i = a_i Y_i$, where Y_i's are identically distributed with common negative exponential distribution, then $a_i l(n)/s_n = O(1)$ implies $\limsup S_n(s_n l(n))^{-1} = 1$ (the generalization is to require only that Y_i's be identically distributed with $E[\exp(hY_i)] < \infty$ for all $h > 0$). Also under these conditions the theorem includes:

$$a_1 l(n)/s_n = O(1) \quad \text{implies} \quad 0 < \limsup S_n(s_n l(n))^{-1} < \infty .$$

1. Introduction. Let $\{X_i\}$ be a sequence of independent random variables for which:

$$(*) \qquad EX_i = 0 , \qquad \sigma_i^2 \equiv \operatorname{Var}(X_i) < \infty , \qquad s_n^2 \equiv \sum_{i=1}^n \sigma_i^2 \to_n \infty ,$$
$$\Phi_{i,n}(h) \equiv E(\exp(s_n^{-1}hX_i)) < \infty \qquad \text{for all } h > 0 ;$$

and let $S_n \equiv \sum_{i=1}^n X_i$. In two papers ([3], [4]) Feller discusses two methods (to be known here as M_1 and M_2) for the finding of a sequence $b_n \to_n \infty$ so that $\limsup S_n(b_n s_n)^{-1} = 1$ a.s. when $\{X_i\}$ satisfies the additional hypothesis

$$(^+) \qquad s_{n+1}/s_n < (\log s_n)^P$$

for some $P > 0$. These two methods consist of:

(i) finding a sequence h_n, if possible, to solve the parametric equation

$$\lambda_n(h_n) \equiv h_n \Psi_n'(h) - \Psi_n(h_n) = C ln ln s_n$$

where

$$(\text{P.M}_1) \qquad \Psi_n(h) \equiv \sum_{i=1}^n \log \Phi_{i,n}(h)$$

$$(\text{P.M}_2) \qquad \Psi_n(h) \equiv \sum_{i=1}^n (\Phi_{i,n}(h) - 1) ;$$

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(ii) verifying that

(R.M_1) $$h_n \Psi_n''(h_n) = o(\lambda_n(h_n))$$

(R.M_2) A: $\sum_{k=1}^{n} \Phi'_{k,n}(h_n)\Phi_{k,n}^{-1}(h_n)[\Phi_{k,n}(h_n) - 1] = o(\Psi_n'(h_n))$

(R.M_2) B: for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \int_{\varepsilon\Psi_n'(h_n)}^{\infty} xe^{h_n x} F_k(dx) = o(\Psi_n'(h_n))$$

where F_k is the distribution function of X_k,

(R.M_2) C: there is a $c > 0$, such that

$$ch_n \Psi_n'(h_n) \leqq \lambda_n(h_n) \leqq h_n \Psi_n'(h_n) ,$$

and

(iii) setting $b_n \equiv C\Psi'(h_n)$.

(Feller actually works with $C = 1$. However his proof that either method, if it can be completed, will produce the desired b_n (Section 2, [4]), holds for more general C.)

(P.M_2) looks easier to deal with than (P.M_1), and in fact Feller shows (page 7 [3]) that if (P.M_1) has a solution, then so does (P.M_2). (A simple example of a solution existing for (P.M_2) but not for (P.M_1) is furnished by $X_i = \pm a_i$ with probability $\frac{1}{2}$ each, where $\{a_i\}$ is any sequence so that $\exp(e^n) = o(a_n)$. A simple calculation shows that for (P.M_1), $\lambda_n(h) \leqq n$ for all h, and thus the parametric equation has no solution. However for (P.M_2), $\lambda_n(h) \uparrow \infty$ for each n and thus because $\lambda_n(0) = 0$ and $\lambda_n(h)$ is continuous, we see the parametric equation does have a solution.) This may help explain why Feller in [3] abandons M_1 and only develops techniques for solving (P.M_2) and estimating the $\Psi_n'(h_n)$ associated with M_2. On the other hand, note how vastly easier (R.M_1) is to work with than (R.M_2), and in fact as Feller notes on page 5 [3], (R.M_1) is indeed a more general condition than (R.M_2).

Remark. Indeed the solution of (P.M_1) itself has more general aspect, in that such a solution is alone enough to guarantee $\limsup S_n(\Psi_n'(h_n)s_n)^{-1} < C$ a.s. This is seen by an investigation of Feller's proof that M_1, if it can be completed, produces the desired b_n.

Finally M_1 is more attractive than M_2 because the proof that it works is much more elementary and elegant than the proof for M_2.

The purpose of this paper is to show a case of how the generality and simplicity of M_1 can in a very elementary manner lead to a theorem of some scope in relation to laws of the iterated logarithm.

To arrive at this theorem, we first consider a condition that guarantees solutions of both (P.M_1) and (P.M_2). To this end we apply the "General Mean Value Theorem" to $h^{-2}\lambda_n(n)$ and the "Mean Value Theorem" to $\Psi_n'(h)$, and see that there exists h_1, h_2, $0 \leqq h_1, h_2 \leqq h$, so that

$$\lambda_n(h) = (h_2/2)\Psi_n''(h_1) \qquad \text{and} \qquad \Psi_n'(h) = h\Psi''(h_2) .$$

Thus

$$(h^2/2)L_n(h) \leqq \lambda_n(h) \leqq (h^2/2)U_n(h) \quad \text{and} \quad hL_n(h) \leqq \Psi_n'(h) \leqq hU_n(h)$$

where

$$L_n(h) \equiv \inf_{k \leqq n, H \leqq h} \Psi_k''(H) \quad \text{and} \quad U_n(h) \equiv \sup_{k \leqq n, H \leqq h} \Psi_k''(H) .$$

Therefore, the condition

$$(^{++}) \qquad 0 < L \equiv \liminf L_n((2lnlns_n)^{\frac{1}{2}}) < U \equiv \limsup U_n((2lnlns_n)^{\frac{1}{2}}) < \infty$$

guarantees that for $C = L$ and n sufficiently large (P.M$_1$) and (P.M$_2$) have the solution

$$h_n = (2C_n\, lnlns_n)^{\frac{1}{2}} \qquad \text{where} \quad L/U \leqq \liminf C_n \leqq \limsup C_n \leqq 1$$

and for this solution:

(i) $\Psi_n'(h_n) = (2D_n\, lnlns_n)^{\frac{1}{2}}$ where $L^3/U \leqq \liminf D_n \leqq \limsup D_n \leqq U^2$ and
(ii) $\limsup \Psi_n''(h_n) < U < \infty$.

So by the definition of $\lambda_n(h)$, we see that if $(^{++})$ holds then

$$\lambda_n(h_n) < h_n \Psi'(h_n) < Uh_n(lnlns_n)^{\frac{1}{2}}$$

which combined with (ii) means (R.M$_1$) holds! Thus the more general nature of (R.M$_1$) and the above Remark show without any further considerations that

THEOREM. *If $\{X_i\}$ is a sequence of independent random variables so that $(^*)$ and $(^{++})$ holds then*

$$\limsup S_n(2s_n{}^2 lnlns_n)^{-\frac{1}{2}} \leqq UL .$$

If $(^+)$ also holds then

$$L^{5/2}/U^{\frac{1}{2}} \leqq \limsup S_n(2s_n{}^2 lnlns_n)^{-\frac{1}{2}} \leqq UL .$$

In the rest of this paper, we will consider X_i's of the form $a_i Y_i$, with the requirement that for some $K \in (0, \infty)$, $r \equiv \max_{1 \leqq i \leqq n} a_i(lnlns_n)/s_n \leqq K$ for n sufficiently large. Under this restriction, we will consider some of the many situations (see (A)—(D) below) where $(^{++})$ holds, and thereby indicate some of the scope of the theorem.

Regarding the conditions $(^*)$ and $(^+)$ we only remark:

1. that since $r_n = O(1)$ implies $a_{n+1}/s_{n+1} \to 0$ and since $1 = (s_n{}^2/s_{n+1}^2 + EY_{n+1}^2 a_{n+1}^2/s_{n+1}^2)$, $(^+)$ clearly holds if $\sup_i EY_i{}^2 < \infty$ (this is pertinent to (A), (B), and (C) below), and
2. that $(^*)$ is clearly satisfied if the X_i's are bounded above (this is pertinent to (A) and (B)).

We will show using further elementary techniques that $(^{++})$ holds:

(A) if X_i's are bounded random variables where $a_i \equiv \sup |X_i|$. Thus the theorem contains a result of Egorov (Theorem 4, [2]). (His proof is different and not as elementary.) We will further see that $r_n = o(1)$ implies $U = L = 1$

and so the theorem contains Kolmogorov's classical "Law of the Iterated Logarithm."

(B) if the X_i's are (i) bounded above with $a_i \equiv \sup(X_i)$, (ii) $\sum_{i=1}^{n} a_i^2 E(Y_i^{+2})/s_n^2 > d > 0$ for all n, and (iii) there is an ε_0 and $\beta > 0$ so that for n sufficiently large, $E^2(Y_n^+)/E(Y_n^{+2}) > e^{-2K}$ implies $E[X_n^- I_{[X_n^- < \varepsilon]}]/EX_n^- > \beta$. Thus the theorem contains a slightly different version of a result of Feller (Section 10 [3]), which says that if $r_n = O(1)$ and (i), (ii) and (iii)

$$\textstyle\sum_{i \in K_n} EY_i^2 \rightarrow_n 0 \qquad \text{where} \quad K_n \equiv \{k \leqq n \colon P[|Y_i| > \varepsilon a_n] > \varepsilon\},$$

are satisfied, then $0 < \limsup S_n(2s_n^2 \ln\ln s_n)^{-\frac{1}{2}} < \infty$.

(C) if Y_i's are identically distributed and satisfy (*).

We further show that $r_n = o(1)$ implies $U = L = 1$ and thus the theorem contains a result of Thompson [5] which only deals with the special case of Y_i's being negatively exponential.

(D) if Y_i's satisfy (*), $\max_n \phi_n(h) < \infty$ for all h (where $\phi_n(h) \equiv E[\exp(hY_n)]$),

(a) the Y_i's are symmetric, (this gives $0 < L$); and

(b) $EY_i^4/(E(Y_i^2))^2 \leqq c < \infty$ for all i or $\Psi'''(h) < 0$ for all h (this give $U < \infty$).

(*Note*: $EY_i^4 > (E(Y_i^2))^2$ is of course always true.)

2. Proof of A—D. For convenience we let $\psi_i(h) = \log \phi_i(h)$, $F_i(x)$ be the distribution function of Y_i, and when no confusion can arise we suppress the i. Throughout the rest of this paper we will need to keep in mind the following relations:

(R.1) $$s_n^2 = \textstyle\sum_{i=1}^{n} a_i^2 EY_i^2,$$

(R.2) $$\Psi_n''(h) = \textstyle\sum_{i=1}^{n} a_i^2/s_n^2 \Psi_i''(a_i h/s_n)$$

(R.3) $$\Psi''(h) = (\phi''(h)\phi(h) - (\phi'(h))^2/\phi^2(h)$$

(R.4) $$\phi^{(k)}(h) = \textstyle\int y^k \exp(hy)F(dy) \qquad \text{for } k = 0, 1, 2, \cdots.$$

Without loss of generality we will assume $a_n \uparrow \infty$.

A *and* B. In both cases $Y_i \leqq 1$, thus since $\phi(h) \geqq 1$ for all h we have by (R.3), $\Psi''(h) \leqq \phi''(h) \leqq EY_i^2 e^h$ and thus by (R.2) and (R.1)

$$\Psi_n''((2\ln\ln s_n)^{\frac{1}{2}}) < [\textstyle\sum_{i=1}^{n} (a_i^2 EY_i^2)/s_n^2] \exp(K) = \exp(K)$$

for all n sufficiently large, and so $U \leqq \exp(K) < \infty$.

(R.3) and (R.4) show for $\varepsilon \geqq 0$

(l_1) $$\psi_i''(h) \geqq \phi''(h)e^{-h} - (e^h EY_i^+ - e^{-\varepsilon h} E(Y_i^- I_{[Y_i^- < \varepsilon]})^2 e^{-2h}$$

(l_2) $$\begin{aligned}\psi_i''(h) &= V(Z^+) + V(Z^-) + 2EZ^+EZ^- \\ &\geqq E(Y_i^- I_{[Y_i^- < \varepsilon]})E(Y_i^+)e^{-h(2+c)}\end{aligned}$$

where Z is a random variable whose distribution is given by $e^{hy}\,dF_{Y_i}(y)/\phi_i(h)$ and V means the variance.

Let $v_n \equiv E^2(Y_n^+)/EY_i^2$, and $v_n^+ \equiv E^2(Y_i^+)/E(Y_i^{+2})$,

(A): (R.4) further shows $\phi''(h) \geqq e^{-h}EY_i^2$ and so (l_1) with $\varepsilon = 1$ becomes

$$(l_1') \qquad \Psi_i''(h) \geqq [(e^h - e^{-h})^2 v_i]EY_i^2 e^{-2h}\,.$$

By hypothesis, $EY_i^+ = EY_i^-$, and so taking $\varepsilon = 1$ in (l_2) we have

$$(l_2') \qquad \Psi_i''(h) \geqq v_i\, EY_i^2 e^{-3h}\,.$$

Now if $v_n < e^{-2K}$, then use (l_1'), and if $v_n > e^{-2K}$ use (l_2') to obtain for sufficiently large n:

$$L_n((2 ln ln s_n)^{\frac{1}{2}}) \geqq \min((e^{-K} - (e^K - e^{-K})^2 e^{-2K})e^{-2K},\, e^{-4K}) \equiv v(K)\,.$$

Thus $L \geqq v(K) > 0$.

If $r_i = 0\,(1)$, then one can take K as close to 0 as desired and we see by the upper and lower limits of U and L that $U = L = 1$.

(B): (R.4) show that (l_1) with $\varepsilon = 0$ becomes

$$(l_1'') \qquad \Psi_i''(h) \geqq (1 - (v_i + e^h)^2)e^{-2h}E(Y_i^{+2})\,.$$

Letting $\varepsilon = \varepsilon_0$ from (iii) of the hypothesis, we see (l_2) becomes

$$(l_2'') \qquad \text{if} \quad v_i^+ > e^{-2k}\,, \qquad \Psi_i''(h) \geqq \beta e^{-2k}e^{h(2+\varepsilon_0)}E(Y_i^{+2})\,,$$

We now proceed as in (A), i.e., if $v_n^+ < e^{-2k}$ we use (l_1'') and if $v_n^+ > e^{-2k}$ we use (l_2'') to obtain by hypothesis (ii): $L_n((2 ln ln s_n)^{\frac{1}{2}}) \geqq \alpha \min((1 - e^{-2K})e^{-2K}, \beta e^{-4K}) \equiv v > 0$ for n sufficiently large. Thus $L \geqq v > 0$.

(C) Note (R.1) and (R.3) show since ϕ'' is continuous,

$$0 < \inf_{h \leqq k} \phi''(h) \leqq L \leqq U \leqq \sup_{h \leqq k} \phi''(h) < \infty\,.$$

Further note that if $r_n = o(1)$, then K is as close to 0 as desired and so since $\phi''(0) = 1$ we have $L = U = 1$.

(D) (a) By (R.1) and (R.2) we see that in order to show $L > 0$, it suffices to show $(m)\ \phi''(h) \geqq EY_i^2/\phi^2(h)$. (This need not be true if Y_i is not symmetric, and in fact is clearly false if $EY_i^3 = \phi'''(0) < 0$.)

We will need the following lemma.

LEMMA. *Let $f'(y) \geqq 0$ for all $y \geqq 0$, then for all $x, y \geqq 0$, $xf(x) + yf(y) \geqq yf(x) + xf(y)$.*

PROOF. Without loss of generality let $\phi(0) = 0$ and $x \geqq z$. Holding x fixed we allow z to run between 0 and x. Let $g(z) \equiv xf(x) + zf(z) - zf(x) - xf(z)$ for $0 \leqq z \leqq x$. Note $g'(z) = f(z) - f(x) + f'(z)\,(z - x) \leqq 0$ since $f'(x) \geqq 0$ and $x \geqq z$, i.e., $g(z)$ is monotone between 0 and x. But $g(0) = xf(x) > 0$ and $g(x) = 0$ and thus $g(z) \geqq 0$ for $x \geqq z$, and the proof of the lemma is complete.

Since Y_i is symmetric, we see

$$\phi(h) = 2 \int_0^\infty \cosh(hy)F(dy)$$

and thus

$$z(h) \equiv \phi''(h)\phi(h) - (\phi'(h))^2 = 4 \int_0^\infty \int_0^\infty v(h, x, y)F(dx)F(dy)$$

where

$$v(h, x, y) \equiv (x^2 + y^2) \cosh hx \cosh hy - 2xy \sinh hy \cdot \sinh hx .$$

It will suffice to show $v'(h, x, y) \geqq 0$ ($'$ means derivative with respect to h) since it implies $z'(h) \geqq 0$ and this combined with $z(0) = EY_i^2$ establishes (m).

Now

$$v'(h, s, y) = h^{-3} \cosh hy \cosh hx[(hz)^2f(hz)^2 + (hy)^2f(hy)^2 - (hy)^2f(hx)^2 - (hx)^2f(hy)^2]$$

where $f(z) \equiv z \tanh z^{\frac{1}{2}}$. Noting $f'(z) = \tanh z^{\frac{1}{2}} + (\frac{1}{2}z)^{\frac{1}{2}} \operatorname{sech}^2 z \geqq 0$ for $z \geqq 0$ we have appealing to our lemma the desired fact that $v'(h, x, y) \geqq 0$.

(b) The first condition implies $U < \infty$ by noting that by Schwarz's inequality

$$\psi''(h) < \phi''(h) \leqq (EY^4)^{\frac{1}{2}}\phi^{\frac{1}{2}}(2h) \leqq (EY^2)[\phi(2h)]^{\frac{1}{2}}$$

and then appealing to (R.2). The second condition implies $U < \infty$ by noting $\Psi''(0) = EY_i^2 < \infty$, and $\Psi''(h)$ is continuous at 0.

FINAL REMARK. It should be noted that under M_2, $\Psi_n''(h) = \sum_{k=1}^n s_n^{-2}\phi_k''(hs_n^{-1})$ and so since $\phi_k''(h)$ increases in h and $\phi_k''(0) = EX_k^2$, we have $\psi_n''(h) \geqq 1$ for all h; i.e., the lower bound of $(^{++})$ is always true for M_2. However $(R.M_2)$ seems to obscure any advantage this might afford. (The upper bound seem as tractable in one method as the other.)

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