The Annals of Mathematical Statistics
1972, Vol. 43, No. 6, 2095-2098

PAIRS OF ONE DIMENSIONAL RANDOM WALK PATHS

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We first show a way of constructing a path of length $2n$ from a pair of paths of length n by means of which one may arrive at many results on pairs of paths of length n, simply by examining properties of paths of length $2n$. Secondly, for two random walk paths of length n, A and B, with vertical coordinates $A(i)$ and $B(i)$ respectively, at times $i = 0, 1, \cdots, n$, and such that for some m $A(m) > B(m)$ but $A(i) = B(i)$ when $i < m$, we define $d_{A,B}(i) = \frac{1}{2}(A(i) - B(i))$. For obvious reasons $A(i) - B(i)$ is always even, which incidentally, implies that the intersection of two paths are points with integral coordinates. We find that $d_{A,B}$ can be graphed against time by a three-valued random walk path, i.e. a path which may have horizontal steps. Questions about the pair consisting of A and B may then be answered by observing the path described by $d_{A,B}$. Results in the theory of three-valued random walk paths can thus be translated into results about pairs of random walk paths of equal length.

1. Method 1. For a random walk path Z we let $Z(i)$ be its vertical coordinate at $t = i$. Given the paths A and B, of length n, we construct the key path of A and B, denoted by $K_{A,B}$, thus: $K_{A,B}(2r) = A(r) - B(r)$, $K_{A,B}(2s+1) = A(s+1) - B(s)$, for $0 \leqq r \leqq n$ $0 \leqq s \leqq n-1$. By the preceding construction, for each pair of paths of length n there are two key paths, and for each path of length $2n$ such that not all of its points occurring at even times lie on the axis, the inverse of this construction gives a unique pair of paths which have it as a key path.

THEOREM 1. *The number of pairs of paths of length n which meet for the* rth *time at* $t = n$, $r < n$, *is* $2^{r-1}r(2n-r)^{-1}\binom{2n-r}{n}$.

PROOF. By the above construction we see that the number of pairs of paths of length n which meet for the rth time at $t = n$ is one half of the number of paths of length $2n$ which have their rth return to the t-axis at $t = 2n$, $2^r r(2n-r)^{-1}\binom{2n-r}{n}$. (See [1], page 90.)

In the same manner we may obtain:

COROLLARY 2. *There are* $\frac{1}{2}\binom{2s}{s}\binom{2n-2s}{n-s}$ *pairs of paths of length n which meet for the last time at* $t = s < n$. (See [1], page 79.)

DEFINITION. Given two paths of length n, A and B, if $A(m) = B(m)$ and $A(m+1) \neq B(m+1)$, then the union of the segments from $(m, A(m))$ to $(m+1, A(m+1))$ and $(m, A(m))$ to $(m+1, B(m+1))$ is called a V-section of the pair consisting of A and B. Similarly if $A(m) = B(m)$ and $A(m-1) \neq B(m-1)$.

THEOREM 3. *The number of pairs of paths of length n which have* $2r$ V*-sections,* $r > 0$, $2r \leqq n$, *and begin and end with a* V*-section is* $2^{r-1}r(n-r)^{-1}\binom{2n-2r}{n}$.

Received June 28, 1971.

PROOF. In a random walk path, let a segment consisting of two steps of the same slope and which either begins or ends on the axis be called a key-V-segment. For paths A and B, $A(m) = B(m)$ and $A(m+1) \neq B(m+1)$ iff $K_{A,B}$ has a key-V-segment which begins at $(2m, 0)$. Also $A(m) = B(m)$ and $A(m-1) \neq B(m-1)$ iff $K_{A,B}$ has a key-V-segment which ends at $(2m, 0)$.

We shall consider the question (a): How many paths of length $2n$ are there which have $2r$ key-V-segments, $2r \leqq n$, $r > 0$, one of which begins at $t = 0$ and another ends at $t = 2n$? Given a path of the kind described in (a), we locate all the sections in it which begin and end with a key-V-segment and do not contain crossings of the axis. We shall call them key-V-sections, and the paths, of the kind described in (a), in which the key-V-sections have no points above the axis, characteristic. Given a characteristic path, we erase the r key-V-segments with negative slope, and piece the remainder together in the obvious way, thus obtaining a path of length $2n - 2r$ with a first passage through $2r$ at $t = 2n - 2r$. The construction is clearly invertible. The number of characteristic paths is thus seen to be equal to the number of paths of length $2n - 2r$ making their first passage through $2r$ at $t = 2n - 2r$, $r(n-r)^{-1}\binom{2n-2r}{n}$. (See [1], page 89.) Moreover by individual reflections of the key-V-sections of a characteristic path across the t-axis we obtain 2^r paths of the type described in (a). Any path of the type required in (a) may be so obtained. Hence the answer to question (a) is $2^r r(n-r)^{-1}\binom{2n-2r}{n}$, and the number of pairs of paths satisfying the hypotheses of the theorem is one half of this number.

REMARK. Method 1 can be applied to answer questions on points of coincidence of a pair of paths of arbitrary dimension.

2. Method 2. We shall use random walk paths which may differ from the random walk paths of the coin-tossing type in having horizontal steps. Such a path will be called an $\binom{n}{k}$ path if it is of length n and has exactly k horizontal steps. T_m denotes the vertical coordinate of such a path at $t = m$. The number of $\binom{n}{n-p-q}$ paths with p steps of positive slope and q steps of negative slope is $n!/(k!\,p!\,q!)$ as can be easily verified. If an $\binom{n}{k}$ path ends at $T_n = x$, then p and q are determined by $p - q = x$ and $p + q + k = n$. We will denote the number of $\binom{n}{k}$ paths which end at $T_n = x$ by $M_{n,k,x}$. If $M_{n,k,x} \neq 0$, then $M_{n,k,x} = n!/(k!\,p!\,q!)$.

LEMMA 4. *There are* $(x/n)M_{n,k,x}$ $\binom{n}{k}$ *paths which end at* $T_n = x$ *and for which* $T_i > 0$ *when* $1 \leqq i \leqq n$, *if* $x > 0$.

PROOF. A straightforward generalization of the proof for the case $k = 0$; the latter may be found in [1] page 73, (the ballot theorem).

DEFINITION. Given two $\binom{n}{0}$ paths A and B for which $d_{A,B}$ is defined as in the Abstract, then the graph of $d_{A,B}$ against t is called the associated path of A and B.

Properties of pairs of $\binom{n}{0}$ paths can be discovered by observing the behavior of three-valued random walk paths whose first non-horizontal step is up, and then constructing the pairs of $\substack{n\\0}$ paths having them as associated paths. We illustrate this in the following theorem.

THEOREM 5. *The number of pairs of* $\substack{n\\0}$ *paths which are such that if one of the paths is at some point above the other it is never below it, and such that the paths end at* (n, x) *and* $(n, x-2z)$ *respectively, is, letting* $p = \frac{1}{2}(n+x)$ *and* $q = \frac{1}{2}(n-x)$:

(a) $(z+1)(n+1)^{-1}\binom{n+1}{p-z}\binom{n+1}{p+1}$ *when* $x - z \leqq 0$ *and* $z > 0$.
(b) $(z+1)(n+1)^{-1}\binom{n+1}{q}\binom{n+1}{q+z+1}$ *when* $x - z > 0$ *and* $z > 0$.
(c) $(n+1)^{-1}\binom{n+1}{p}\binom{n+1}{p+1} - \binom{n}{p}$ *when* $x \leqq 0$ *and* $z = 0$.
(d) $(n+1)^{-1}\binom{n+1}{q}\binom{n+1}{q+1} - \binom{n}{q}$ *when* $x > 0$ *and* $z = 0$.

PROOF OF (a). The associated path of a pair of $\substack{n\\0}$ paths of the kind described in the statement of part (a) is an $\substack{n\\k}$ path, for some k, which ends with $T_n = z$ and never goes below the axis. The number of such paths is the same as the number of $\substack{n+1\\k}$ paths which end with $T_{n+1} = z + 1$ and stay strictly above the axis, which is, by Lemma 4, $(z+1)/(n+1)M_{n+1,k,z+1}$. Given an $\substack{n\\k}$ path which ends with $T_n = z$, never going below the axis, we can construct all the pairs of $\substack{n\\0}$ paths which have it as associated path and fulfill the conditions of part (a). Since the associated path ends at (n, z), $z > 0$, z units go toward the total number of vertical steps gained by the upper $\substack{n\\0}$ path between $t = 0$ and $t = n$. Its other vertical gains must be the number of times, u, such that if between $t = i$ and $t = i + 1$ the associated path has a horizontal step then between $t = i$ and $t = i + 1$ the upper $\substack{n\\0}$ path has an up step. A similar argument applies for its vertical losses.

We see that $u = \frac{1}{2}(k + x - z)$. The number of ways u can be achieved is $\binom{k}{u}$. Then the number of pairs of $\substack{n\\0}$ paths satisfying the hypotheses of part (a) is

$$\begin{aligned}\textstyle\sum_{k=x-z}^{n-z} \binom{k}{u}(z+1)/(n+1)M_{n+1,k,z+1} &\\ = (z+1)/(n+1) &\\ \times \textstyle\sum_{u=0}^{p-z} (n+1)!/(u!\,(p-u-z)!\,(p-u+1)!\,(n-2p+u+z)!)\,. &\end{aligned}$$

The second sum represents the coefficient of $t^{p-z}y^{z+1}$ in $(t + ty^{-1} + y + 1)^{n+1} = (ty^{-1}+1)^{n+1}(y+1)^{n+1}$, which is $\binom{n+1}{p-z}\binom{n+1}{p+1}$.

PROOF OF (b). We reflect every pair of paths satisfying the hypotheses of part (b) across the t-axis. We then apply the result of part (a) with x replaced by $2z - x$.

PROOF OF (c) AND (d). The proof of (c) is a slight modification of the proof of (a) and the result (d) is obtainable from (c) in the same way that (b) was derived from (a).

Acknowledgment. The author wishes to thank Professor G. C. Rota and Doctor A. D. De Acosta of M.I.T. for their suggestions with regard to the presentation of this manuscript.

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