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ON THE MULTIPLICITY OF A CLASS OF MULTIVARIATE RANDOM PROCESSES[1]

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Some properties of the multiplicity of a class of multivariate random processes are examined with particular emphasis on the derivation of a necessary and sufficient condition in order that the multiplicity of such a process be equal to one. Discussion is limited to processes with mutually orthogonal univariate components, but possible applications of the results to more general situations are suggested.

1. Introduction. Let C be the class of zero mean, second order, left mean square continuous random processes. Let $x(t) \in C$; the closed linear manifold spanned by $x(s)$, $s \leqq t$, is a separable Hilbert space denoted by $H(x, t)$. Cramér [4], [5] and Hida [7] have shown that $x(t)$ can be uniquely decomposed into the orthogonal sum of two random processes $u(t)$ and $v(t)$, such that $u(t)$ is deterministic,[2] $v(t)$ is purely non-deterministic (pnd),[3] $u(t) \perp v(s)$, $\forall s, t \in R$ and $H(x, t) = H(u, t) \oplus H(v, t)$, $\forall t \in R$. Furthermore, the pnd portion $v(t)$ has a canonical decomposition of the form

$$v(t) = \sum_{i=1}^{M} \int_{-\infty}^{t} g_i(t, \tau)\, dz_i(\tau)\,, \qquad \forall t \in R$$

where the $z_i(\tau)$ are mutually orthogonal random processes in C with orthogonal increments, with $E[dz_i(\tau)]^2 = dF_i(\tau)$, and such that

$$H(v, t) = H(z_1, t) \oplus \cdots \oplus H(z_M, t)\,, \qquad \forall t \in R\,.$$

It is possible to choose the $z_i(\tau)$ so that the measures F_i be finite and form a chain of absolute continuity (i.e., $F_1 \gg \cdots \gg F_M$). This is a well-known result [4]; a proof is also suggested by the construction in Section 2 of this paper. The number M, which may be any nonnegative integer or ∞, is uniquely determined by the autocovariance of $x(t)$ and is called the *multiplicity* of the process.

As Mandrekar and Kallianpur [8], [9] have pointed out, the multiplicity of such a random process coincides with the spectral multiplicity of a self-adjoint operator A acting on $H(x)$ and whose resolution of the identity is P_t [the family of projections from $H(x)$ onto $H(x, t)$]. In other words M is the dimension of a minimal generating subspace[4] of A [1].

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[2] That is, $H(u, -\infty) = H(u, t)$, $\forall t \in R$.

[3] That is, $H(v, -\infty) = 0$.

[4] A minimal generating subspace B for a random process $x(t)$ is any subspace of $H(x, \infty)$ with the property that $H(x, \Delta) = G_{\delta \subset \Delta}\{P_x(\delta)[B]\}$, $\forall \Delta = [s, t)$.

Let $X(t) = \{x_1(t), \cdots, x_q(t)\}$ be a multivariate random process whose univariate components $x_i(t)$ belong to C. Then $H(X, t)$ is defined as the closed linear manifold spanned by $x_i(s)$, $s \leqq t$, and $i = 1, \cdots, q$. In this paper we shall be concerned with the case where the $x_i(t)$ are mutually orthogonal, so that

$$H(X, t) = H(x_1, t) \oplus \cdots \oplus H(x_q, t) .$$

It is possible, but considerably more difficult, to treat the general case in a similar way. The process $X(t)$ has a multiplicity representation of the form

$$X(t) = U(t) + \int_{-\infty}^{t} G(t, \tau)\, dZ(\tau)$$

where $U(t)$ is a q-vector deterministic process whose components are the deterministic parts of the $x_i(t)$, $G(t, \tau)$ is a $q \times M$ matrix, and $Z(\tau)$ is an M-vector process whose components are mutually orthogonal processes in C, each with orthogonal increments. Obviously the number M of the components of $Z(\tau)$ is related to the multiplicities M_i of the $x_i(t)$. In this paper we shall prove that

$$\max_{1 \leqq i \leqq q} \{M_i\} \leqq M \leqq \sum_{i=1}^{q} M_i$$

and then we shall give necessary and sufficient conditions for M to be equal to unity. First we shall need some intermediate results developed in the next two sections.

2. Random and abstract measures. Let $y(t) \in C$ with orthogonal increments. It is known [2], [11] that ψ, defined on the semi-algebra of left-closed, right-open intervals $\delta = [s, t)$ by

$$\psi(\delta) = y(t) - y(s)$$

is an orthogonal random measure which can be uniquely extended to the σ-algebla of Borel sets $\mathscr{B}$. For $A \in \mathscr{B}$, $\psi(A)$ is the lim of $\psi(\delta_n)$ as $F[(A \sim \delta_n) \cup (\delta_n \sim A)] \to 0$ (where $F(\delta) = E[\psi(\delta)]^2$). Furthermore, the Hilbert space $H(y, \delta)$ defined by

$$H(y, \delta) = H(y, t) \ominus H(y, s)$$

is an abstract measure on the same semi-algebra of intervals and can be extended to $\mathscr{B}$. For $A \in \mathscr{B}$, $H(y, A)$ is the span $G\{\psi(\alpha)\}$ where α is any Borel subset of A. The measure properties of $H(y, A)$ follow directly from the fact that

$$\psi(A_1) \perp \psi(A_2) \Leftrightarrow A_1 \cap A_2 = \varnothing$$

for any Borel sets A_1 and A_2; that is from the fact that ψ is an orthogonal random measure.

An obvious property that will be used later is expressed by the following lemma:

LEMMA 1. $H(y, A) = 0 \Leftrightarrow \psi(A) = 0$ *for any* $A \in \mathscr{B}$.

PROOF. If $\psi(A) = 0$, then $\psi(\alpha) = 0$ for any Borel set $\alpha \subset A$. Therefore $H(y, A) = 0$. Conversely if $H(y, A) = 0$, $\psi(\alpha) = 0$ for any Borel set $\alpha \subset A$; consequently $\psi(A) = 0$. □

The real σ-finite measures on $\mathscr{B}$ form a lattice ordered by the relation of absolute continuity. Let m_1 and m_2 be two such measures. By the Lebesgue decomposition theorem [12], the measure m_2 can be written as

$$m_2 = m_2' + m_2''$$

where $m_2' \ll m_1$ and $m_2'' \perp m_1$ (and consequently $m_2' \perp m_2''$). Define $M = m_1 + m_2''$ and $m = m_2'$. Then $M = m_1 \vee m_2$ and $m = m_1 \wedge m_2$. This decomposition holds also for random measures [10] and can be applied to measures ϕ induced on $\mathscr{B}$ by processes $y(t) \in C$ with orthogonal increments.

Let $y_1(t)$ and $y_2(t)$ be two such mutually orthogonal processes. Let ϕ_1 and ϕ_2 be the corresponding random measures and $F_i(\delta) = E[\phi_i(\delta)]^2$, $i = 1, 2$, the associated real measures. The measure F_2 can be written as

$$F_2 = F_2' + F_2'' , \qquad \text{with } F_2' \ll F_1 \text{ and } F_2'' \perp F_1 .$$

Therefore the real line is the disjoint union of two sets A and B such that $F_1(B) = 0$ and $F_2''(A) = 0$. We shall define now ϕ_2' and ϕ_2'' as follows;

$$\phi_2'(\delta) = \phi_2(\delta \cap A)$$

and

$$\phi_2''(\delta) = \phi_2(\delta \cap B)$$

for any $\delta \in \mathscr{B}$. Obviously ϕ_2' and ϕ_2'' are orthogonal random measures with $E[\phi_2'(\delta)]^2 = F_2'(\delta)$ and $E[\phi_2''(\delta)]^2 = F_2''(\delta)$, and in fact

$$\phi_2 = \phi_2' + \phi_2''$$

is the Lebesgue decomposition of ϕ_2 with respect to ϕ_1. The measure Ψ defined by

$$\Psi = \phi_1 + \phi_2''$$

is the lub of ϕ_1 and ϕ_2 and the measure ϕ defined by

$$\phi = \phi_2'$$

is the glb of ϕ_1 and ϕ_2. From these two orthogonal measures, which are also mutually orthogonal, we can form the processes $Y(\tau)$ and $y(\tau)$ with orthogonal increments by placing

$$Y(\tau) = \textstyle\int_{-\infty}^{\tau} \Psi(d\tau) \qquad \text{and} \qquad y(\tau) = \int_{-\infty}^{\tau} \phi(d\tau) .$$

This construction can be applied to the canonical decomposition of $x(t) \in C$ mentioned in the introduction, so that the set of $z_i(\tau)$ obtained has the property that $F_1 \gg \cdots \gg F_M$. This is insured by the following lemma.

LEMMA 2. $$H(y_1, t) \oplus H(y_2, t) = H(Y, t) \oplus H(y, t) , \qquad \forall t \in R .$$

PROOF. Let $u \in H(Y, t) \oplus H(y, t)$; that is

$$u = \alpha\Psi(\delta_1) + \beta\phi(\delta_2)$$

for some $\delta_1, \delta_2 \in \mathscr{B} \cap (-\infty, t]$ and $\alpha, \beta \in R$. We know that $\Psi(\delta_1) = \phi_1(\delta_1) + \phi_2''(\delta_1)$ and $\phi(\delta_2) = \phi_2'(\delta_2)$. But $\phi_2'(\delta_2) = \phi_2(\delta_2 \cap A)$ and $\phi_2''(\delta_1) = \phi_2(\delta_1 \cap B)$;

therefore

$$u = \alpha\psi_1(\delta_1) + \alpha\psi_2(\delta_1 \cap B) + \beta\psi_2(\delta \cap A);$$

that is $u \in H(y_1, t) \oplus H(y_2, t)$. Conversely let $u \in H(y_1, t) \oplus H(y_2, t)$; that is,

$$u = \alpha\psi_1(\delta_1) + \beta\psi_2(\delta_2)$$

for some $\delta_1, \delta_2 \in \mathscr{B} \cap (-\infty, t]$ and $\alpha, \beta \in R$. We have $\psi_1(\delta_1) = \psi_1(\delta_1 \cap A) = \Psi(\delta_1 \cap A)$ and $\psi_2(\delta_2) = \psi_2(\delta_2 \cap A) + \psi_2(\delta_2 \cap B) = \psi_2'(\delta_2 \cap A) + \Psi(\delta_2 \cap B)$. Therefore

$$u = \alpha\Psi(\delta_1 \cap A) + \beta\Psi(\delta_2 \cap B) + \beta\psi(\delta_2 \cap A)$$

that is $u \in H(Y, t) \oplus H(y, t)$. $\square$

3. Principal results and proofs. The multiplicity M of $X(t) = \{x_1(t), \cdots, x_q(t)\}$ is evidently the least number of mutually orthogonal $z_i(\tau)$ with orthogonal increments required in order that

$$H(X, t) = H(z_1, t) \oplus \cdots \oplus H(z_M, t), \qquad \forall t \in R.$$

Since M_i is the least number of such $z_i(\tau)$ required for $H(x_i, t)$ and since by hypothesis $H(X, t) = H(x, t) \oplus \cdots \oplus H(x_q, t)$, it is obvious that M cannot exceed $\sum_{i=1}^q M_i$, the latter being clearly sufficient for the decomposition of $H(X, t)$. This is a property which, by Cramér's assertion [5], holds for any multivariate process with components in C. Mandrekar [9] proves it for wide-sense Markov processes. In the following a rigorous proof of this result will be given for the restricted class of processes examined in this paper. Furthermore several other statements concerning the properties of the multiplicity representation will be proved.

Let us first note that $X(t)$ is pnd if and only if all the multivariate components $x_i(t)$ are pnd. This is the case since

$$H(X, t) = H(x_1, t) \oplus \cdots \oplus H(x_q, t), \qquad \forall t \in R$$

and therefore

$$H(X, -\infty) = H(x_1, -\infty) \oplus \cdots \oplus H(x_q, -\infty).$$

Then the following theorem is true.

THEOREM 1. *Let $X(t) = \{x_1(t), \cdots, x_q(t)\}$ with $x_i(t)$, $x_j(t)$ mutually orthogonal for $i \neq j$ and with $x_i(t) \in C$, $i = 1, \cdots, q$. Let M_i be the multiplicity of $x_i(t)$ and M be the multiplicity of $X(t)$. Then*

$$\max_{1 \leqq i \leqq q} \{M_i\} \leqq M \leqq \sum_{i=1}^q M_i.$$

PROOF. Let B be a minimal generating subspace for $X(t)$. Then $H(X, \Delta) = G_{\delta \subset \Delta}\{P_X(\delta)[B]\}$, $\forall \Delta = [s, t)$, where $P_X(\delta)$ denotes the projection on $H(X, \delta)$. But for the class of processes under consideration $H(x_i, \delta) = P_i[H(X, \delta)]$ where P_i denotes the projection on $H(x_i)$. Also $P_i \cdot P_X(\delta) = P_{x_i}(\delta)$, where $P_{x_i}(\delta)$ is the projection on $H(x_i, \delta)$. Therefore

$$H(x_i, \Delta) = G_{\delta \subset \Delta}\{P_i P_X(\delta)[B]\} = G_{\delta \subset \Delta}\{P_{x_i}(\delta)[B]\} = G_{\delta \subset \Delta}\{P_{x_i}(\delta) \cdot P_i[B]\}.$$

This means that $B_i = P_i[B]$ is a generating subspace for $x_i(t)$. The dimension of B_i is certainly less than or equal to M, the dimension of B. Therefore M_i, the dimension of a minimal generating subspace for $x_i(t)$, does not exceed M either; and since this is true $\forall i$, we have

$$\max_{1 \leqq i \leqq q} \{M_i\} \leqq M . \tag{1}$$

If B_i is a minimal generating subspace for $x_i(t)$, then $B_1 \oplus \cdots \oplus B_q$ is a generating subspace for $X(t)$, since $P_X(\delta) = P_{x_1}(\delta) + \cdots + P_{x_q}(\delta)$. Therefore, as was stated earlier, M, the dimension of a minimal generating subspace for $X(t)$, cannot exceed $\sum_{i=1}^q M_i$; that is

$$M \leqq \sum_{i=1}^q M_i . \tag{2}$$ □

It is interesting to know under what conditions the equality holds in (1) or in (2). In the remainder of this paper a necessary and sufficient condition will be derived in order that $M = 1$; that is, in order for the equality to hold in (1) for a particular value of M. The case $M = 1$, in addition to its theoretical significance in the problem of the mathematical representation of random processes [3], plays an important role in linear estimation and detection as well as in the modeling of signals by causal and invertible linear systems [6]. Before we proceed let us mention a corollary of the preceding theorem following directly from (1).

COROLLARY 1. *In order that $X(t)$ be a* pnd *process with multiplicity one it is necessary that each $x_i(t)$, $i = 1, \cdots, q$ be a* pnd *process with multiplicity one.*

Now suppose that $M_i = 1$, $i = 1, \cdots, q$ and let $y_i(\tau)$ be the corresponding process with orthogonal increments in the canonical decomposition of each $x_i(t)$. Let F_i be its associated real measure [i.e., $E[dy_i(\tau)]^2 = dF_i(\tau)$]. Under these circumstances and provided that the $x_i(t)$, and consequently the $y_i(\tau)$, are mutually orthogonal, the following is true:

THEOREM 2. *The multiplicity M of $X(t) = \{x_1(t), \cdots, x_q(t)\}$ is equal to unity if and only if the measures F_i are pairwise mutually singular.*

PROOF. (i) *Sufficiency.* If $F_i \perp F_j$, $i \neq j$, then by successively applying the construction developed in Section 3 and with the aid of Lemma 2 it is obvious that $M = 1$ with $y(\tau) = y_1(\tau) + \cdots + y_q(\tau)$. This is the case since $y_1(\tau)$ and $y_2(\tau)$ can be replaced by their sum $y_1(\tau) + y_2(\tau)$ due to their mutual singularity. Then the processes $y_3(\tau)$ and $y_1(\tau) + y_2(\tau)$ can be replaced by their sum, etc. Lemma 2 ensures that

$$H(y, \tau) = H(y_1, \tau) \oplus \cdots \oplus H(y_q, \tau) .$$

(ii) *Necessity.* If $M = 1$, let $y(\tau)$ be the unique component in the canonical decomposition of $X(t)$. Since $H(y, t) = H(X, t) = H(y_1, t) \oplus \cdots \oplus H(y_M, t)$, it follows that $\phi_i \ll \phi$, $\forall i$ by Lemma 1. Furthermore it follows [8] that each ϕ_i

has a Radon–Nikodym derivative with respect to ψ, that is

$$\psi_i(A) = \int_A g_i(\tau)\psi(d\tau)\,, \qquad i = 1, \cdots, q$$

for any Borel set A. Since the ψ_i are mutually orthogonal we have

$$E[\psi_i(A)\psi_j(A)] = \int_A g_i(\tau)g_j(\tau)F(d\tau) = 0$$

for $i \neq j$ and every $A \in \mathscr{B}$; therefore

$$g_i(\tau)g_j(\tau) = 0 \quad \text{a.e.} \quad [F]$$

which implies that ψ_i and ψ_j have disjoint supports, that is

$$F_i \perp F_j\,. \qquad \square$$

Alternatively, it is possible to note that $F_i \ll F$ and then define

$$\bar{\psi}_i(A) = \int_A |\rho_i(\tau)|^{\frac{1}{2}}\psi(d\tau)$$

where $\rho_i(\tau) = dF_i(\tau)/dF(\tau)$. Then, for the process $\bar{y}_i(\tau)$ defined by

$$\bar{y}_i(\tau) = \int_{-\infty}^{\tau} \bar{\psi}_i(d\tau)$$

we have

$$E[d\bar{y}_i(\tau)]^2 = dF_i(\tau)\,.$$

Therefore up to an isomorphism the processes $\bar{y}_i(\tau)$ can be considered as components in the canonical decomposition of $X(t)$. By the mutual orthogonality of the univariate components it follows that

$$E[\bar{\psi}_i(A)\bar{\psi}_j(A)] = 0 \qquad A \in \mathscr{B},\ i \neq j$$

and, therefore, that the functions $\rho_i(\tau)$ have disjoint supports (i.e., that $F_i \perp F_j$, $i \neq j$). $\square$

4. A generalization. Clearly all of the preceding results are true in a more general context, namely whenever a random process $X(t)$ is related to the processes $x_i(t)$ in such a way that $H(X, t) = H(x_1, t) \oplus \cdots \oplus H(x_q, t)$. In order for this to be the case it is not necessary that $X(t)$ be a multivariate, "vector" process. For example, it is possible to have $X(t) = x_1(t) + \cdots + x_q(t)$. In particular consider the case

$$X(t) = x_1(t) + x_2(t)$$

where $X(t)$ is interpreted as some observation process in a signal transmission situation, $x_1(t)$ being the signal and $x_2(t)$ some form of uncorrelated noise. If the basic relationship expressed by $H(X, t) = H(x_1, t) \oplus H(x_2, t)$ is assumed to be true, then some important statements can be deduced from the preceding results. A broad sufficient condition for the applicability of the prewhitening method in linear, least-mean square estimation is known to be that the observation process's multiplicity be one [6]. But then it follows that both the signal and noise processes must have unit multiplicity and, if they have, a necessary and sufficient condition for the observation to have $M = 1$ is that the random measures spanning signal and noise spaces respectively be mutually singular.

Certainly the estimation problem described is rather severely restricted, since the relation $H(X, t) = H(x_1, t) \oplus H(x_2, t)$, $\forall t \in R$, implies that zero-error estimation of the signal is possible. It indicates, however, that the multiplicity properties discussed earlier are valid in a more general context than the one of multivariate random processes with mutually orthogonal univariate components. Furthermore, it may be conjectured that some of these properties may hold even when the orthogonality assumption is relaxed.

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