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ON A CERTAIN CLASS OF LIMIT DISTRIBUTIONS

BY R. SHANTARAM AND W. HARKNESS

State University of New York at Stony Brook and The Pennsylvania State University

Suppose that G is the distribution function (df) of a (non-negative) rv Z satisfying the integral-functional equation $G(x) = b^{-1} \int_0^{lx} [1 - G(u)]\, du$, for $x > 0$, and zero for $x \leqq 0$, with $l \geqq 1$. Such a df G arises as the limit df of a sequence of iterated transformations of an arbitrary df of a rv having finite moments of all orders. When $l = 1$, G must be the simple exponential df and is unique. It is shown, for $l > 1$, that there exists an infinite number of df's satisfying this equation. Using the fact that any df G which satisfies the given equation must have finite moments $\nu_k = K!\, b^k l^{k(k-1)/2}$ for $k = 0, 1, 2, \cdots$, it is shown that the df of the rv $Z = UV$, where U and V are independent rv's having log-normal and simple exponential distributions, respectively, satisfies the integral functional equation. It is then easy to exhibit explicitly a family of solutions of the equation.

1. Introduction. The class of distribution functions (df's) of nonnegative random variables (rv's) defined by the solutions of the integral equation

$$G(x) = b^{-1} \int_0^{lx} [1 - G(u)]\, du\,, \qquad x > 0$$

and zero elsewhere is discussed. Here, $l \geqq 1$ is a fixed constant and $b \equiv \int_0^\infty [1 - G(u)]\, du < \infty$. Distribution functions (df's) G_0 such that for $X > 0$, $G_0(x) = b^{-1} \int_0^x [1 - G(y)]\, dy$, occur in renewal theory and as the limit of certain residual waiting times ([2] page 354–356). For $l = 1$, or when $G_0 = G$, it is readily seen that the exponential distribution, defined by $G(x) = 1 - e^{-x/b}$ for $x > 0$, is the unique solution to this equation. For $l > 1$, these distributions arise as limits of certain sequences of iterated transformations of arbitrary df's of nonnegative rv's with finite moments of all orders. This is discussed immediately below. Steutel ([5] page 74) proved that any df G satisfying this integral equation must be infinitely divisible. It can be shown that the convolution $G = F_1 * F_2$ of an exponential df F_1 and a log-normal df F_2 is a solution of the equation. Finally, Robson ([4]) has shown that certain random phenomena associated with the study of transect sampling problems have df's of the form considered here and which satisfy the above equation.

Let $F(x)$ be the df of a nonnegative rv (i.e., Pr $(X \geqq 0) = 1$) with finite moments $\mu_n = E(X^n)$ of all orders. Consider the sequence of absolutely continuous df's $G_n(x)$ defined recursively as follows. Put

$$\begin{aligned} G_1(x) &= \mu_1^{-1} \int_0^x [1 - F(y)]\, dy\,, && \text{for} \quad x > 0 \\ &= 0\,, && \text{for} \quad x \leqq 0\,. \end{aligned}$$

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For $n > 1$, let

$$G_n(x) = \mu_{1,n-1}^{-1} \int_0^x [1 - G_{n-1}(y)]\, dy\,, \qquad \text{for} \quad x > 0$$
$$= 0\,, \qquad \text{for} \quad x \leqq 0$$

where $\mu_{1,n-1} = \int_0^\infty [1 - G_{n-1}(y)]\, dy$. It has been shown ([3]) that under certain conditions, the class of limit distributions (as $n \to \infty$) is obtained as the solutions of the integral equation

$$G(x) = b^{-1} \int_0^{lx} [1 - G(u)]\, du\,, \qquad x > 0 \tag{1}$$

and zero elsewhere. Here, $l > 1$ is a fixed constant and $b = \int_0^\infty [1 - G(u)]\, du < \infty$.

2. Result. In this paper we shall characterize this class of df's. To begin with, let us consider the case $b = 1$; the general case can be derived from this case. Then (1) becomes

$$G(x) = \int_0^{lx} [1 - G(u)]\, du\,, \qquad x > 0 \tag{2}$$

with $G(+\infty) = 1$. Let $g(x) = 1 - G(x)$ for all x, so that $g(x) = 1$ for $x \leqq 0$, and

$$1 - g(x) = \int_0^{lx} g(u)\, du\,, \qquad \text{for} \quad x > 0\,. \tag{3}$$

In view of the definition of g and (3) it is evident that g is nonnegative, continuous for $x \geqq 0$ and monotonically non-increasing with $g(0) = 1$ and $g(+\infty) = 0$. Further, g is infinitely differentiable and $(-1)^k g^{(k)}(x) \geqq 0$ for all $k = 0, 1, 2, \cdots$ and $x > 0$. Thus, g is completely monotonic, so that by Bernstein's characterization ([6] page 160) of such functions we have

$$g(x) = \int_0^\infty e^{-tx}\, d\alpha(t)\,, \qquad \text{for} \quad x > 0 \tag{4}$$

where $\alpha(t)$ is non-decreasing function of bounded variation. Note that (4) implies that $g(x)$ is analytic for $x > 0$. Further, since (3) (or equivalently, $-g'(x) = lg(lx)$) is valid for $x > 0$ we have

$$\int_0^\infty te^{-tx}\, d\alpha(t) = l \int_0^\infty e^{-tlx}\, d\alpha(t)\,. \tag{5}$$

Setting $t = lu$ on the left side yields

$$\int_0^\infty e^{-ltx}\, [d\alpha(t) - t\, d\alpha(lt)] \equiv 0 \qquad \text{for} \quad x > 0\,. \tag{6}$$

Thus (4) is a solution of (3) as soon as the non-decreasing, bounded function $\alpha(t)$ satisfies (6) and $\int_0^\infty d\alpha(t) = 1$.

We now obtain explicitly the subset of solutions (4) with $\alpha'(t)$ continuous everywhere ($t > 0$) and satisfying (6) in a trivial manner, viz.,

$$h(t) = lth(lt)\,, \qquad t > 0 \tag{7}$$

where, for simplicity, we have used $h(t)$ for $\alpha'(t)$. (7) implies that $h(\lambda) = h(1)$, where $\lambda = 1/l$. Conversely, if $\Phi(t)$ is arbitrary (but continuous) on $[\lambda, 1]$ with $\Phi(\lambda) = \Phi(1)$ and $h(t)$ is defined to be $\Phi(t)$ on $[\lambda, 1]$ and elsewhere by means of

(7) then, explicitly,

$$\begin{aligned} h(t) &= l^{n(n-1)/2} t^{n-1} \Phi(l^{n-1} t) \qquad \text{for} \quad l^{-n} \leqq t \leqq l^{-(n-1)} \\ &= l^{n(n-1)/2} \Phi(t/l^n)/t^n \qquad \text{for} \quad l^{n-1} \leqq t \leqq l^n \end{aligned}$$

for $n = 1, 2, \cdots$. Clearly, $h(t)$ is bounded on $[0, \infty]$; in fact,

$$\begin{aligned} h(t) &\leqq M l^{-(n-1)(n-2)/2} \qquad \text{for} \quad l^{-n} \leqq t \leqq l^{-(n-1)} \\ &\leqq M l^{-n(n-1)/2} \qquad \text{for} \quad l^{n-1} \leqq t \leqq l^n . \end{aligned}$$

Here, $M = \max_{\lambda \leqq t \leqq 1} \Phi(t)$. This guarantees that $\alpha(t)$ is non-decreasing and bounded if $\Phi(t) \geqq 0$ on $[\lambda, 1]$. The $h(t)$ thus obtained provides a solution

$$(8) \qquad g(x) = \int_0^\infty e^{-tx} h(t)\, dt = \sum_{k=-\infty}^{\infty} \int_{l^k}^{l^{k+1}} e^{-tx} h(t)\, dt$$

of (3). We now proceed to express this solution explicitly in terms of the arbitrary, nonnegative, continuous function $\Phi(t)$ with $\Phi(\lambda) = \Phi(1)$, defined on $[\lambda, 1]$. For $k = 0, 1, 2, \cdots$

$$\begin{aligned} \int_{l^k}^{l^{k+1}} e^{-tx} h(t)\, dt &= l \int_{l^{k-1}}^{l^k} e^{-tlx} h(t)\, dt \\ &= \int_{l^{k-1}}^{l^k} e^{-tlx} h(t)\, dt/t \end{aligned}$$

in view of (7). Proceeding similarly (or by induction)

$$\int_{l^k}^{l^{k+1}} e^{-tx} h(t)\, dt = c_k \int_\lambda^1 \exp(-txl^{k+1}) h(t)\, dt/t^{k+1}$$

for $k \geqq 0$, where $c_k^{-1} = l^{k(k+1)/2}$. Further, for $k \geqq 0$

$$\int_{\lambda^{k+1}}^{\lambda^k} e^{-tx} h(t)\, dt = c_k \int_\lambda^1 \exp(-tx\lambda^k) h(t) t^k\, dt .$$

Hence, from (8)

$$\begin{aligned} (9) \qquad g(x) &= \sum_{k=0}^{\infty} \left\{ \int_{l^k}^{l^{k+1}} + \int_{\lambda^{k+1}}^{\lambda^k} \right\} e^{-tx} h(t)\, dt \\ &= \sum_{k=0}^{\infty} c_k \int_\lambda^1 \{ t^{-k-1} \exp(-txl^{k+1}) + t^k \exp(-tx\lambda^k) \} \Phi(t)\, dt . \end{aligned}$$

Since $g(0) = 1$, $\Phi(t)$ has to satisfy the normalization

$$(10) \qquad \sum_{k=0}^{\infty} c_k \int_\lambda^1 (t^k + t^{-k-1}) \Phi(t)\, dt = 1$$

but we still cannot express the integral in (8) in the form given there unless $h(t) = \alpha'(t)$ is continuous at $t = l^k$, $k = 0, \pm 1, \pm 2, \cdots$.

REMARK 1. In view of the above remark we may state the results obtained in the following generalized form.

THEOREM 1. *Let* $\Phi(t) \geqq 0$ *defined on* $[\lambda, 1]$ *be bounded there and let* $\Phi(\lambda) = \Phi(1)$. *Further, let* (10) *be satisfied. Then* (9) *is a solution of* (3).

The assumed conditions imply that Φ is continuous at all the points l^k, k integer, and this is all we need to justify (8).

REMARK 2. We recall that the above theorem does not necessarily exhaust *all* the solutions, since a general solution is (4) under the condition (6). In the case that $\alpha'(t)$ exists and is continuous (6) implies

$$d\alpha(t) = t\, d\alpha(lt)\,, \qquad t > 0 \text{ (a.e.)} \tag{11}$$

REMARK 3. We have considered only the case $b = 1$ explicitly. For b arbitrary it is easily seen that (4) is still a solution with $\alpha(t)$ non-decreasing and bounded and

$$\int_0^\infty e^{-tx}\,[d\alpha(t) - b^{-1}t\, d\alpha(lt)] \equiv 0\,, \qquad x > 0\,. \tag{12}$$

This reduces to (6) by letting first $t = bu$ and then $a^*(u) = \alpha(bu)$.

3. An explicit solution. In this section we look at the solution of (1) by probabilistic considerations. We shall obtain a "natural" solution as well as exhibit an infinity of solutions. We first note ([3] page 411) that the df defined by (1) has finite moments ν_k of all orders and that $\nu_k = k!\, b^k l^{k(k-1)/2}$. We shall first obtain a df with this moment sequence. Noting that the moment generating function of a standard normal distribution is equal to $l^{t^2/2}$, we find, by setting $t = k(\ln l)^{\frac{1}{2}}$ and $x = (\ln v)/a$, where $a = (\ln l)^{\frac{1}{2}}$, that $\{l^{k^2/2}\}$ is the moment sequence of the log-normal distribution with density function (pdf)

$$f(v) = v^{-1}(2\pi \ln l)^{\frac{1}{2}} \exp\{-(\ln v)^2/2 \ln l\}\,, \qquad v > 0\,.$$

Further, $k!\, b^k l^{-k/2}$ is the kth moment of the gamma density

$$f_1(u) = bl^{-\frac{1}{2}}e^{-ul^{\frac{1}{2}}/b}\,.$$

Hence, ν_k is the kth moment of a rv Z where $Z = UV$ with U and V independent and with pdf's respectively $f_1(u)$ and $f(v)$. Z has the df

$$\begin{aligned} G(z) = \Pr(Z \leqq z) &= \int_0^\infty \int_0^{z/v} f_1(u) f(v)\, du\, dv \\ &= 1 - \int_0^\infty f(v) e^{-z/v\theta}\, dv \\ &= 1 - \int_0^\infty e^{-z/v\theta}(2\pi \ln l)^{-\frac{1}{2}} v^{-1} \exp\{-(\ln v)^2/2 \ln l\}\, dv \end{aligned}$$

where $\theta = b/l^{\frac{1}{2}}$. Bernstein's characterization ensures that this df satisfies (1).

However, the solution is by no means unique. The moment sequence $\{l^{k^2/2}\}$ generates an indeterminate Stieltjes moment problem. In fact, Stieltjes has shown ([1] page 88) that for $k = 0, 1, 2, \cdots$

$$\int_0^\infty u^k u^{-\ln u} \sin(2\pi \ln u)\, du = 0\,.$$

(The fact that this result is also true for $k = -1$ can be seen by putting $v = \ln u$. One sees that the resulting integral is that of an odd function of v over the interval $(-\infty, \infty)$). Noting that $u^{-\ln u} = \exp\{-(\ln u)^2\}$, and using, instead of $f(v)$, the pdf $f^*(v) = [1 + \alpha \sin(2\pi \ln v)]f(v)$, where $|\alpha| < 1$, we obtain

$$\begin{aligned} G^*(z) &= \Pr(Z^* \leqq z) = \Pr(UV^* \leqq z) \\ &= 1 - \int_0^\infty e^{-z/v\theta} f^*(v)\, dv \\ &= 1 = \int_0^\infty e^{-tz} f^*(1/t\theta)\, dt/t^2\theta \end{aligned}$$

and Bernstein's theorem again shows that $G^*(z)$ is a solution of (1) for every α, $-1 < \alpha < 1$. This explicitly exhibits an infinity of solutions.

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Department of Statistics
Pennsylvania State University
University Park, Penn. 16802