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AN UPPER BOUND FOR THE RENEWAL FUNCTION[1]

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In this note we show that the renewal function H corresponding to a random walk with positive mean μ and finite variance σ^2 satisfies the inequality $H(x) < \mu^{-1}x + 3(1 + \mu^{-2}\sigma^2)$.

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables having positive mean μ and finite variance σ^2. Set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geqq 1$. Let H denote the renewal function, defined by

$$H(x) = \sum_{n=0}^{\infty} P(S_n \leqq x) , \qquad -\infty < x < \infty .$$

It is a well-known consequence of Wald's identity that

$$H(x) \geqq \frac{x^+}{\mu} , \qquad -\infty < x < \infty , \tag{1}$$

where $x^+ = x$ for $x > 0$ and $x^+ = 0$ for $x \leqq 0$. In this note we will obtain an inequality in the opposite direction, namely

$$H(x) < \frac{x^+}{\mu} + 3\,\frac{\mu^2 + \sigma^2}{\mu^2} , \qquad -\infty < x < \infty . \tag{2}$$

The constant 3 appearing in (2) could conceivably be made significantly smaller. We could not hope to do better, however, than to sharpen (2) to

$$H(x) \leqq \frac{x^+}{\mu} + \frac{\mu^2 + \sigma^2}{\mu^2} , \qquad -\infty < x < \infty . \tag{3}$$

For if $P(X_i = 1) = 1$, then equality holds in (3) when $x = 0$.

An inequality such as (2) is suggested by the well-known results that

$$\lim_{x\to\infty} \left(H(x) - \frac{x}{\mu} \right) = \frac{\mu^2 + \sigma^2}{2\mu^2} \tag{4}$$

if X_1 has a nonlattice distribution and

$$\lim_{n\to\infty} \left(H(nd) - \frac{nd}{\mu} \right) = \frac{\mu^2 + \sigma^2}{2\mu^2} + \frac{d}{2\mu} \tag{5}$$

if X_1 has a lattice distribution with span d. Observe that in the latter case

$$\begin{aligned} \mu^2 + \sigma^2 &= E|X_1|^2 \\ &= \sum_{n=-\infty}^{\infty} n^2 d^2 P(X_1 = nd) \\ &\geqq d \sum_{n=-\infty}^{\infty} nd P(X_1 = nd) = d\mu , \end{aligned}$$

so the right side of (5) is bounded above by $(\mu^2 + \sigma^2)/\mu^2$.

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We will now prove (2). Let f denote the characteristic function of X_1, defined by

$$f(\theta) = Ee^{i\theta X_1}, \qquad -\infty < \theta < \infty .$$

Since

$$|e^{i\theta} - 1 - i\theta| \leqq \frac{\theta^2}{2}, \qquad -\infty < \theta < \infty ,$$

it follows that

$$\begin{aligned} |f(\theta) - 1 - i\mu\theta| &= |E(e^{i\theta X} - 1 - i\theta X)| \\ &\leqq E|e^{i\theta X} - 1 - i\theta X| \\ &\leqq E\frac{\theta^2 X^2}{2} = \left(\frac{\mu^2 + \sigma^2}{2}\right)\theta^2 . \end{aligned}$$

Set

$$\alpha = \frac{\mu^2 + \sigma^2}{\mu} .$$

Then

(6) $$|f(\theta) - 1 - i\mu\theta| \leqq \frac{\alpha\mu}{2}\theta^2 , \qquad -\infty < \theta < \infty ,$$

and hence

(7) $$|1 - f(\theta)| \geqq \frac{\mu}{2}|\theta| , \qquad -\frac{1}{\alpha} \leqq \theta \leqq \frac{1}{\alpha} .$$

Define $R(x)$, $-\infty < x < \infty$, by

$$\begin{aligned} R(x) &= \int_{-\infty}^{x} P(X_1 \leqq y)\, dy , \qquad & x < 0 , \\ &= \int_{x}^{\infty} P(X_1 \geqq y)\, dy , \qquad & x \geqq 0 , \end{aligned}$$

and define $S(x)$, $-\infty < x < \infty$, by

$$S(x) = \int_{-\infty}^{x} R(y)\, dy .$$

Then $S(x)$ increases to $(\mu^2 + \sigma^2)/2 = \alpha\mu/2$ as $x \to \infty$. Thus

(8) $$0 \leqq S(x) \leqq \frac{\alpha\mu}{2} , \qquad -\infty < x < \infty .$$

Let $K(x)$, $-\infty < x < \infty$, be the probability density function

$$K(x) = \frac{1}{2\pi}\left[\frac{\sin(x/2)}{x/2}\right]^2 , \qquad -\infty < x < \infty .$$

Its characteristic function

$$k(\theta) = \int_{-\infty}^{\infty} e^{i\theta x} K(x)\, dx , \qquad -\infty < \theta < \infty ,$$

is given by

(9) $$\begin{aligned} k(\theta) &= 1 - |\theta| , \qquad & |\theta| \leqq 1 , \\ &= 0 , \qquad & |\theta| > 1 . \end{aligned}$$

It is easy but a little tedious to use tables of

$$\int_0^x \frac{\sin y}{y}\, dy \qquad \text{and} \qquad \int_\infty^x \frac{\cos y}{y}\, dy$$

to verify that

$$\int_0^3 (3 - y)K(-y)\,dy > \frac{1 + K(0)}{2}\,. \tag{10}$$

According to an identity of M. Dubman ([1] page 43)

$$\begin{aligned}\int_{-\infty}^{\infty} \alpha^{-1}K(\alpha^{-1}(x - y))\left(H(y) - \frac{y^+}{\mu} - \frac{S(y)}{\mu^2}\right) \\ = \frac{-i}{2\pi\mu^2}\int_{-\alpha^{-1}}^{\alpha^{-1}} e^{-ix\theta}k(\alpha\theta)\,\frac{(f(\theta) - 1 - i\mu\theta)^2}{\theta^3(1 - f(\theta))}\,d\theta\,.\end{aligned} \tag{11}$$

It follows from (6), (7), and (9) that the absolute value of the right side of (11) is bounded above by

$$\frac{\alpha^2}{2\mu}\int_{-\alpha^{-1}}^{\alpha^{-1}} \frac{1}{2\pi}\,k(\alpha\theta)\,d\theta = \frac{\alpha K(0)}{2\mu}\,.$$

We now conclude from (8) and (11) that

$$\int_{-\infty}^{\infty} \alpha^{-1}K(\alpha^{-1}(x - y))\left(H(y) - \frac{y^+}{\mu}\right)dy \leqq \frac{\alpha}{2\mu}\,(1 + K(0))\,. \tag{12}$$

Choose $x \geqq 0$. Since $H(y)$ is non-decreasing in y it follows from (1) and (12) that

$$\begin{aligned}\frac{1 + K(0)}{2} &\geqq \int_x^{\mu H(x)} \alpha^{-2}K(\alpha^{-1}(x - y))(\mu H(x) - y)\,dy \\ &= \int_0^{(\mu H(x) - x)/\alpha} K(-y)\left(\frac{\mu H(x) - x}{\alpha} - y\right)dy\,.\end{aligned}$$

Thus by (10)

$$\frac{\mu H(x) - x}{\alpha} < 3$$

or equivalently

$$H(x) < \frac{x}{\mu} + \frac{3\alpha}{\mu} = \frac{x^+}{\mu} + 3\,\frac{\mu^2 + \sigma^2}{\mu^2}\,.$$

This completes the proof of (2) for $x \geqq 0$ and hence for all x, since if $x < 0$

$$H(x) \leqq H(0) < 3\,\frac{\mu^2 + \sigma^2}{\mu^2} = \frac{x^+}{\mu} + 3\,\frac{\mu^2 + \sigma^2}{\mu^2}\,.$$

REFERENCE

[1] Dubman, M. (1970). Estimates of the renewal function when the second moment is infinite. Ph. D. dissertation, Univ. of California, Los Angeles.

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