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A NOTE ON SYMMETRIC RANDOM VARIABLES

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There exist independent random variables X_1 and X_2 such that X_1 is symmetric, X_2 is not symmetric, but $X_1 + X_2$ is symmetric. If X_1 and X_2 are i.i.d. random variables with a fractional moment and if for all real α $P[X_1 + \alpha X_2 > 0] = \frac{1}{2}$ then they are symmetric.

1. Theorem. *There exist independent random variables X_1 and X_2 such that X_1 is symmetric, X_2 is not symmetric, but the sum $X_1 + X_2$ is symmetric.*

PROOF. Let $f(y)$ be an odd L_1 function which vanishes inside $[-1, +1]$ such that $\pi i f(y)$ is the Fourier transform of an odd L_1 function $\rho(x)$ on $(-\infty, +\infty)$. The probability density function $\{|\rho(x)| + \rho(x)\}/\int_{-\infty}^{+\infty}\{|\rho(x)| + \rho(x)\}\,dx$ has as imaginary part of its characteristic function $\pi i f(y)/\int_{-\infty}^{+\infty}\{|\rho(x)| + \rho(x)\}\,dx$ which vanishes for $|y| \leqq 1$. Choose any asymmetric random variable X_1 with a Fourier transform that vanishes outside $|y| \leqq 1$ and let X_2 denote an independent random variable with the density function just constructed. Then $X_1 + X_2$ has a real characteristic function and hence its distribution is symmetric. A suitable candidate for $f(y)$ is

$$\begin{aligned} f(y) &= e^{-(y-1)}\sin(y-1) && y \geqq 1 \\ &= 0 && -1 < y < 1 \\ &= -f(-y) && y \leqq -1\,. \end{aligned}$$

Then $\pi i f(y)$ is the Fourier transform of the function $\rho(x)$ where $\rho(x) = \{(2 - x^2)\sin x + 2x\cos x\}/(4 + x^4)$.

2. Theorem. *Let X_1 and X_2 be independent random variables with the common probability density function $g(x)$ possessing a real fractional moment. If for all real α $P[X_1 + \alpha X_2 > 0] = \frac{1}{2}$ then X_1 is a symmetric random variable.*

PROOF. If $P[X_1 + \alpha X_2 > 0] = \frac{1}{2}$ for all real α then by changing to polar coordinates in the x_1, x_2 plane $\frac{1}{2}$ may be written as a function of the angle $\theta(\alpha)$ that the line $x_1 + \alpha x_2 = 0$ makes with respect to the x_1 axis as follows:

$$\tfrac{1}{2} = \int_{\theta(\alpha)-\pi}^{\theta(\alpha)} \left[\int_0^\infty \rho g(\rho\sin\theta) g(\rho\cos\theta)\,d\rho\right] d\theta\,.$$

Differentiating both sides of this equation with respect to $\theta(\alpha)$ and a simple change of variables yields the condition that $\int_{-\infty}^{+\infty} u g(u) g(\beta u)\,du = 0$ for almost all β. Changing to polar coordinates and the subsequent differentiation are justified by the Fubini theorem and the theorem that the derivative of the integral of an L_1 function equals the function almost everywhere.

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Let $|\beta|^s = e^{s\,ln|\beta|}$; for purely imaginary s:

$$\int_{-\infty}^{+\infty} |\beta|^s [\int_{-\infty}^{+\infty} ug(u)g(\beta u)\, du]\, d\beta = 0\,.$$

The order of integration may be interchanged if $ug(u)g(\beta u)$ is integrable as a function of u and β since $|\beta|^s$ has absolute value one if s is purely imaginary. The values of the iterated integrals

$$\int_0^{\infty} [\int_{-\infty}^{+\infty} ug(u)g(\beta u)\, d\beta]\, du \qquad \text{and} \qquad \int_{-\infty}^{0} [\int_{-\infty}^{+\infty} ug(u)g(\beta u)\, d\beta]\, du$$

are $\int_0^{\infty} g(u)\, du$ and $-\int_{-\infty}^{0} g(u)\, du$.

The Tonelli theorem may be applied to the last two integrals to conclude that $ug(u)g(\beta u)$ is integrable. Thus the order of integration may be interchanged yielding:

$$\int_{-\infty}^{\infty} \frac{g(u)}{|u|^s} [\int_{-\infty}^{\infty} |\beta u|^s g(\beta u) u\, d\beta]\, du = 0\,.$$

From this it easily follows:

$$[\int_{-\infty}^{\infty} |v|^s g(v)\, dv] \left[\int_0^{\infty} \frac{g(u)}{|u|^s}\, du - \int_{-\infty}^{0} \frac{g(u)}{|u|^s}\, du \right] = 0\,.$$

If a real fractional moment of X_1 exists, then the values of $\int_{-\infty}^{\infty} |v|^s g(v)\, dv$ are the boundary values of an analytic function along the imaginary axis and cannot vanish on any interval. Thus

$$\int_0^{\infty} \frac{g(u)}{|u|^s}\, du = \int_{-\infty}^{0} \frac{g(u)}{|u|^s}\, du\,.$$

Substituting $ln|u| = v$ and using standard theorems on Fourier transforms yields $g(e^v) = g(-e^v)$ for almost all v; therefore $g(x)$ is even.

It would be interesting to know if the condition that a fractional moment exists could be omitted.

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