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ON A CLASS OF ALIGNED RANK ORDER TESTS FOR THE IDENTITY OF THE INTERCEPTS OF SEVERAL REGRESSION LINES[1]

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Based on $k(\geqq 2)$ independent samples, a class of aligned rank order tests for the hypothesis of homogeneity of intercepts of the k (simple) regression lines is considered here. The alignment procedure is similar to the one in Sen [*Ann. Math. Statist.* (1969) **19** 1668-1683], and the theory is developed with the aid of the fundamental results of Jurečková [*Ann. Math. Statist.* (1969) **19** 1889-1900] on the asymptotic linearity of rank statistics in regression parameters. Local asymptotic optimality of the proposed tests is also studied.

1. Introduction. Consider a set of $N(=\sum_{i=1}^k n_i)$ independent random variables Y_{ij}, $1 \leqq j \leqq n_i$, $1 \leqq i \leqq k$, where

$$P\{Y_{ij} \leqq x\} = F_{ij}(x) = F(x - \alpha_i - \beta_i c_{ij}), \tag{1.1}$$

$\mathbf{c}_i = (c_{i1}, \cdots, c_{in_i})$, $i = 1, \cdots, k$ are known vectors of regression constants, $\beta_1, \cdots, \beta_k$ are the slopes, and $\alpha_1, \cdots, \alpha_k$ are the intercepts of the $k(\geqq 2)$ regression lines. F is assumed to be an absolutely continuous cumulative distribution function (cdf). In an earlier paper [Sen (1969)], we considered a class of aligned rank tests for the hypothesis of equality of $\beta_1, \cdots, \beta_k$, treating $\alpha_1, \cdots, \alpha_k$ as nuisance parameters. In the present paper, we consider the null hypothesis

$$H_0: \alpha_1 = \cdots = \alpha_k = \alpha_0 \text{ (unknown)}, \tag{1.2}$$

against the set of alternatives that not all $\alpha_1, \cdots, \alpha_k$ are equal; here we treat $\beta_1, \cdots, \beta_k$ as nuisance parameters (not necessarily all equal). Such a problem often arises in statistical inference; to mention an important area, we refer to indirect quantitative slope-ratio bio-assays [cf. Finney (1952) Chapters 7 and 8], where the equality of the intercepts constitutes the fundamental assumption of the assay and the relative potency is provided by the ratio of the slopes. In passing, we may add that in various situations, $\mathbf{c}_1, \cdots, \mathbf{c}_k$ may be quite different from each other, and any assumption that the average c_{ij} are all equal or are all close to any specified value may not be very realistic.

Since $\beta_1, \cdots, \beta_k$ are unknown, the usual several sample rank tests for location do not work out here. Again, working with the aligned observations $\tilde{Y}_{ij} = Y_{ij} - \hat{\beta}c_{ij}$, $1 \leqq j \leqq n_i$, $1 \leqq i \leqq k$, (where $\hat{\beta}_1, \cdots, \hat{\beta}_k$ are suitable estimates of $\beta_1, \cdots, \beta_k$) vitiates independence and invalidates the basic invariance structure underlying the scope of the usual distribution-free rank tests. Moreover, for a k sample rank statistic based on the $\tilde{Y}_{ij}$, in the resulting model, we introduce k

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random variables $\hat{\beta}_1, \cdots, \hat{\beta}_k$, whose coefficient vectors $\mathbf{c}_1, \cdots, \mathbf{c}_k$ do not necessarily satisfy the basic concordance-discordance condition inherent in the asymptotic linearity of rank statistics in the multiparameter case, as has been studied by Jurečková (1969), (1971) and Hájek (1970), among others. Thus, excepting in some particular cases, several sample (location) rank tests based on the aligned $\tilde{Y}_{ij}$ are not even asymptotically distribution-free (ADF). We overcome this difficulty by considering k one-sample rank statistics for the k sets of $\tilde{Y}_{ij}$, and then, as in Sen (1969), aligning these statistics by a suitable pooled sample estimator of the common (hypothetical) value of α_0. It is in this setup we are able to use results similar to those in Jurečková (1969), (1971) and Hájek (1970) and generate a class of ADF tests for H_0 in (1.2). Asymptotically local optimal properties of the tests are also studied.

Section 2 deals with the preliminary notions, Section 3 with the optimal parametric tests, while the main results are presented in Section 4. The last section presents the asymptotic relative efficiency results.

2. Preliminary notions and basic assumptions. We assume that $F \in \mathscr{F} = \{F : f(x) = f(-x), \forall x \geqq 0, \text{ and } I(F) < \infty\}$, where

$$I(F) = \int_{-\infty}^{\infty} (f'/f)^2 \, dF \quad \text{and} \quad f = F' . \tag{2.1}$$

Define now

$$\lambda_N^{(i)} = n_i/N , \quad 1 \leqq i \leqq k \quad (\text{so that } \textstyle\sum_{i=1}^k \lambda_N^{(i)} = 1) , \tag{2.2}$$

and assume that as $N \to \infty$

$$\lambda_N^{(i)} \to \lambda^{(i)} , \quad 0 < \lambda^{(i)} < 1 , \qquad i = 1, \cdots, k . \tag{2.3}$$

We may, without any loss of generality, assume that for every i,

$$c_{i1} \leqq \cdots \leqq c_{in_i} , \qquad \text{with at least one strict inequality,} \tag{2.4}$$

$1 \leqq i \leqq k$. Let then

$$\bar{c}_{i,N} = \frac{1}{n_i} \textstyle\sum_{j=1}^{n_i} c_{ij} , \qquad C_{i,N}^2 = \sum_{j=1}^{n_i} (c_{ij} - \bar{c}_{i,N})^2 , \qquad q_{i,N}^2 = n_i \bar{c}_{i,N}^2 / C_{i,N}^2 , \tag{2.5}$$

and assume that $\lim_{N\to\infty} C_{i,N}^2 = \infty$, for all $1 \leqq i \leqq k$. Also, we assume that

$$\liminf n_i^{-1} C_{i,N}^2 \geqq C_0^2 > 0 , \tag{2.6}$$

$$\lim_{N\to\infty} [\max_{1\leqq j \leqq n_i} |c_{ij} - \bar{c}_{i,N}| / C_{i,N}] = 0 , \tag{2.7}$$

$$\limsup q_{i,N}^2 \leqq q_0^2 < \infty , \qquad \text{for all } i = 1, \cdots, k . \tag{2.8}$$

[Note that we are not assuming that $n_i^{-1} C_{i,N}^2 \to C_i^2 > 0$ (as $N \to \infty$); in fact, it may even tend to ∞ (as in the case of $c_{ij} = a_i + jb_i, j = 1, 2, \cdots$) with $N \to \infty$.]

Let now $\phi(u)$, $0 < u < 1$, be an absolutely continuous, non-decreasing and square integrable function inside [0, 1]. We assume that

$$\phi(u) + \phi(1 - u) = 0 , \qquad \text{for all } u : 0 < u < 1 . \tag{2.9}$$

For every positive integer n, let $U_{n1} \leqq \cdots \leqq U_{nn}$ be the order statistics of a sample of size n from the rectangular (0, 1) distribution, and we define a set of n scores by

$$E_n(j) = \phi(EU_{nj}) = \phi(j/(n+1)) \quad \text{or} \quad E_n(j) = E\phi(U_{nj}), \quad 1 \leqq j \leqq n. \tag{2.10}$$

In passing, we remark that the scores in (2.10) for $\phi(u) = 2u - 1$ and $\phi(u) = \Phi^{-1}(u)$, the inverse of the standard normal df, are termed the Wilcoxon and the normal scores. We let

$$\bar{E}_n = \frac{1}{n} \sum_{j=1}^{n} E_n(j), \qquad A_n^2 = \frac{1}{n} \sum_{j=1}^{n} [E_n(j) - \bar{E}_n]^2; \tag{2.11}$$

$$\bar{\phi} = \int_0^1 \phi(u)\, du \quad \text{and} \quad A^2 = \int_0^1 [\phi(u) - \bar{\phi}]^2\, du. \tag{2.12}$$

For the estimation of $\beta_1, \cdots, \beta_k$, needed for the alignment of the observations, we use the following statistics. Let R_{ij}^0 be the rank of Y_{ij} among $Y_{i1}, \cdots, Y_{in_i}$, and define

$$T_{i,N} = [\sum_{j=1}^{n_i} (c_{ij} - \bar{c}_{i,N}) E_{n_i}(R_{ij}^0)][A_{n_i}^{-1} C_{i,N}^{-1}], \qquad 1 \leqq i \leqq k, \tag{2.13}$$

We also denote by $T_{i,N}(b)$, the (regression-) rank statistic in (2.13) based on the observations $Y_{ij} - bc_{ij}$, $1 \leqq j \leqq n_i$. It follows from Theorem 6.1 of Sen (1969) that

$$T_{i,N}(b) \text{ is } \downarrow \text{ in } b, \quad -\infty < b < \infty, \qquad \text{for all } 1 \leqq i \leqq k. \tag{2.14}$$

Let us now define

$$\phi^*(u) = \phi((1+u)/2), \quad 0 < u < 1, \qquad (A^*)^2 = \int_0^1 [\phi^*(u)]^2\, du = A^2, \tag{2.15}$$

and consider a set of scores

$$E_n^*(j) = E\{\phi^*(U_{nj})\} \quad \text{or} \quad \phi^*(j/(n+1)), \qquad 1 \leqq j \leqq n. \tag{2.16}$$

Then, for the construction of our test, we consider the following type of one-sample statistics:

$$S_{i,N} = \frac{1}{n_i} \sum_{j=1}^{n_i} E_{n_i}^*(R_{ij}^+) \operatorname{Sgn}(Y_{ij}), \qquad 1 \leqq i \leqq k, \tag{2.17}$$

where R_{ij}^+ is the rank of $|Y_{ij}|$ among $|Y_{i1}|, \cdots, |Y_{in_i}|$. Also, if we replace Y_{ij} by $Y_{ij} - a_i - b_i c_{ij}$ in (2.17), the corresponding statistics are denoted by

$$S_{i,N}(a_i, b_i), \qquad 1 \leqq i \leqq k. \tag{2.18}$$

Our test statistic is a quadratic form in the $S_{i,N}(a_i, b_i)$, $1 \leqq i \leqq k$, where the b_i are chosen as the estimators of the β_i derived from $T_{i,N}(b_i)$ in (2.13), and $a_1 = \cdots = a_k = a$ is chosen as some pooled sample estimator of α_0, the (hypothetical) common value of the α_i. Note that for given b_i,

$$S_{i,N}(a_i, b_i) \text{ is } \downarrow \text{ in } a_i, \quad -\infty < a_i < \infty, \qquad 1 \leqq i \leqq k. \tag{2.19}$$

For later use, we let

$$\psi(u) = -[f'(F^{-1}(u))/f(F^{-1}(u))], \quad \psi^*(u) = \psi((1+u)/2), \quad 0 < u < 1, \tag{2.20}$$

so that $\bar{\psi}(u) = 0$ and $\int_0^1 \psi^2(u)\,du = \int_0^1 [\psi^*(u)]^2\,du = I(F) < \infty$. Further, we assume that

$$B(F, \phi) = \int_0^1 \phi(u)\psi(u)\,du = \int_0^1 \phi^*(u)\psi^*(u)\,du > 0\,. \tag{2.21}$$

If F is strongly unimodal and $\phi^*(u)$ is not a constant, then (2.21) holds.

3. Optimal parametric tests. In practice, the commonly used test (under the assumption that F is normal) is based on the variance-ratio criterion

$$Z_N = [\textstyle\sum_{i=1}^k n_i(\tilde{\alpha}_i - \tilde{\tilde{\alpha}})^2/(1 + q_{i,N}^2)]/[(k-1)s_e^2]\,, \tag{3.1}$$

where $\tilde{\alpha}_i = n_i^{-1}\sum_{j=1}^{n_i} Y_{ij} - \bar{c}_{i,N}(\sum_{j=1}^{n_i} Y_{ij}(c_{ij} - \bar{c}_{i,N})/C_{i,N}^2)$, $1 \leqq i \leqq k$, $\tilde{\tilde{\alpha}} = \{\sum_{i=1}^k n_i\tilde{\alpha}_i/(1 + q_{i,N}^2)\}/\{\sum_{i=1}^k n_i/(1 + q_{i,N}^2)\}$, and s_e^2 is the pooled sample mean squares due to error carrying $N - 2k$ degrees of freedom (d.f.). Under H_0 in (1.2), Z_N has the variance-ratio distribution with $(k-1, N-2k)$ d.f., and the test based on Z_N is the most powerful invariant test.

When F is non-normal, the optimality of the Z_N-test is not retained. However, if $\sigma^2(F)$, the variance of F, is finite, some standard computations yield that (i) $s_e^2 \to_p \sigma^2(F)$, as $N \to \infty$, (ii) $n_i^{\frac{1}{2}}(\tilde{\alpha}_i - \alpha_i)(1 + q_{i,N}^2)^{-\frac{1}{2}}/\sigma(F)$ has asymptotically the standard normal distribution, $1 \leqq i \leqq k$, and hence, $(k-1)Z_N$ has asymptotically, under H_0 in (1.2), chi-square distribution with $k - 1$ d.f. Thus, Z_N provides an ADF test for the entire class of F with $\sigma^2(F) < \infty$. If we consider a sequence of (Pitman-) alternatives $\{H_N\}$ specified by

$$H_N: \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_k) = \boldsymbol{\alpha}_N = \alpha_0\mathbf{1} + N^{-\frac{1}{2}}\boldsymbol{\theta}\,, \qquad \mathbf{1} = (1, \cdots, 1)\,, \tag{3.2}$$

where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k) \neq \mathbf{0}$, and it is assumed that

$$\lim_{N\to\infty} q_{i,N} = q_i \quad \text{exists}\,, \qquad 1 \leqq i \leqq k\,, \tag{3.3}$$

then it can be shown that under $\{H_N\}$, Z_N has asymptotically a noncentral chi-square distribution with $k - 1$ d.f. and noncentrality parameter

$$\Delta_Z = [\textstyle\sum_{i=1}^k \lambda^{(i)}(\theta_i - \bar{\theta})^2(1 + q_i^2)^{-1}]/\sigma^2(F)\,, \tag{3.4}$$

where

$$\bar{\theta} = [\textstyle\sum_{i=1}^k \lambda^{(i)}\theta_i(1 + q_i^2)^{-1}][\sum_{i=1}^k \lambda^{(i)}(1 + q_i^2)^{-1}]^{-1}\,. \tag{3.5}$$

Now, for the class of F with $I(F) < \infty$, it follows from the results of Wald (1943) that an asymptotically locally optimal (in the sense of being asymptotically locally most stringent and having best average power over suitable ellipsoidal-surfaces in the parameter space) test for H_0 in (1.2) is based on the likelihood ratio (L_N—) criterion, where $-2\log L_N$ has asymptotically, (i) under H_0 in (1.2), chi-square distribution with $k - 1$ d.f., and (ii) under $\{H_N\}$ in (3.2), a noncentral chi-square distribution with $k - 1$ d.f. and a noncentrality parameter

$$\begin{aligned}\Delta_L &= [\textstyle\sum_{i=1}^k \lambda^{(i)}(\theta_i - \bar{\theta})^2(1 + q_i^2)^{-1}]I(F) \\ &= \Delta_Z[I(F)\sigma^2(F)] \geqq \Delta_Z\,,\end{aligned} \tag{3.6}$$

by the classical Rao-Cramér inequality. Thus, in general, the Z_N-test is not asymptotically optimal. We shall compare both these tests with our proposed one.

4. Aligned rank order tests. For alignment, we consider the following estimator of $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_k)$. Let

$$(4.1)\qquad \hat{\beta}_{i,N}^{(1)} = \sup\{b: T_{i,N}(b) > 0\}, \qquad \hat{\beta}_{i,N}^{(2)} = \inf\{b: T_{i,N}(b) < 0\};$$

$$(4.2)\qquad \hat{\beta}_{i,N} = \tfrac{1}{2}(\hat{\beta}_{i,N}^{(1)} + \hat{\beta}_{i,N}^{(2)}), \quad 1 \leqq i \leqq k, \qquad \hat{\beta}_N = (\hat{\beta}_{1,N}, \cdots, \hat{\beta}_{k,N}).$$

Then ([1], [9]), $\hat{\boldsymbol{\beta}}_N$ is a translation-invariant robust and consistent estimator of $\boldsymbol{\beta}$. Consider then the aligned (one-sample) rank order statistics

$$(4.3)\qquad S_{i,N}(a, \hat{\beta}_{i,N}), \qquad i = 1, \cdots, k,$$

where for every real a $(-\infty < a < \infty)$, we let

$$(4.4)\qquad S_N^*(a) = \textstyle\sum_{i=1}^k w_{i,N} S_{i,N}(a, \hat{\beta}_{i,N}).$$

$$(4.5)\qquad w_{i,N} = [n_i/(1 + q_{i,N}^2)]/[\textstyle\sum_{i=1}^k n_i/(1 + q_{i,N}^2)], \qquad i = 1, \cdots, k.$$

We now estimate the common (hypothetical) value of α_0 by $\hat{\alpha}_N$, where

$$(4.6)\qquad \hat{\alpha}_N^{(1)} = \sup\{a: S_N^*(a) > 0\}, \qquad \hat{\alpha}_N^{(2)} = \inf\{a: S_N^*(a) < 0\};$$

$$(4.7)\qquad \hat{\alpha}_N = \tfrac{1}{2}(\hat{\alpha}_N^{(1)} + \hat{\alpha}_N^{(2)}).$$

Our proposed test is then based on the statistic

$$(4.8)\qquad Q_N = [\textstyle\sum_{i=1}^k n_i S_{i,N}^2(\hat{\alpha}_N, \hat{\beta}_{i,N})/(1 + q_{i,N}^2)]A^{-2},$$

and Theorem 4.1 establishes that Q_N provides an ADF test for H_0 in (1.2).

THEOREM 4.1. *Under* (1.2) *and the assumptions of Section* 2, Q_N *has asymptotically chi-square distribution with* $k - 1$ d.f.

The proof follows directly from Lemmas 4.4, 4.5 and 4.6 (to follow).

LEMMA 4.2. *Under the regularity conditions of Section* 2, *as* $N \to \infty$,

$$(4.9)\qquad |T_{i,N}(\beta_i) + B(F, \phi)C_{i,N}(\hat{\beta}_{i,N} - \beta_i)/A| \to_p 0, \qquad \textit{for all } 1 \leqq i \leqq k.$$

The proof follows directly from the definitions of $T_{i,N}$ and $\hat{\beta}_{i,N}$ in (2.13) and (4.1)–(4.2), asymptotic linearity (in regression parameter) of $T_{i,N}$ [cf. Theorem 3.1 of Jurečková (1969)], Theorem 2.1 of Hájek (1970), and the Bonferroni inequality. □

Since $T_{i,N}(\beta_i)$ is asymptotically $\mathscr{N}(0, 1)$ when β_i holds, it follows from (4.9) that

$$(4.10)\qquad |C_{i,N}(\hat{\beta}_{i,N} - \beta_i)| = O_p(1) \qquad \text{for all } 1 \leqq i \leqq k.$$

Consider now a finite interval $I = \{x: |x| \leqq K\}$, where $K(<\infty)$ is given. Then, by an extension of Theorem 3.1 of Jurečková (1969) to signed rank statistics [viz., van Eeden (1972)], it can be shown that

$$(4.11)\qquad \lim_{N\to\infty}\{\sup_{a,b\in I}|n_i^{\frac12}\{S_{i,N}(\alpha_i+N^{-\frac12}a,\beta_i+N^{-\frac12}b)-S_{i,N}(\alpha_i,\beta_i)\} + (\lambda_N^{(i)})^{\frac12}[aB(F,\phi)+b\bar{c}_{i,N}B(F,\phi)]|\}=0\,,\qquad \text{in probability,}$$

for all $1 \leqq i \leqq k$. Hence, from (2.6), (2.8), (4.10), (4.11), Lemma 4.2 and Theorem 2.1 of Hájek (1970), we have the following.

LEMMA 4.3. *Under the regularity conditions of Section 2, as $N \to \infty$,*

$$(4.12)\qquad \sup_{a\in I}\{|n_i^{\frac12}[S_{i,N}(\alpha_i+N^{-\frac12}a,\hat\beta_{i,N})-S_{i,N}(\alpha_i,\beta_i)] + aB(F,\phi)[\lambda_N^{(i)}]^{\frac12}+\bar{c}_{i,N}AT_{i,N}(\beta_i)|\}\to_p 0\,,\qquad 1\leqq i\leqq k\,;$$

$$(4.13)\qquad \sup_{a,b\in I}\{|n_i^{\frac12}[S_{i,N}(\alpha_i+N^{-\frac12}a,\hat\beta_{i,N})-S_{i,N}(\alpha_i+N^{-\frac12}b,\hat\beta_{i,N})] + [\lambda_N^{(i)}]^{\frac12}(a-b)B(F,\phi)|\}\to_p 0\,,\qquad 1\leqq i\leqq k\,.$$

Now, $S_{i,N}(\alpha_i, \beta_i)$ and $T_{i,N}(\beta_i)$ are mutually stochastically independent, and as $n_i \to \infty$,

$$(4.14)\qquad \mathscr{L}(n_i^{\frac12}S_{i,N}(\alpha_i,\beta_i)/A)\to\mathscr{N}(0,1)\,,\qquad \mathscr{L}(T_{i,N}(\beta_i))\to\mathscr{N}(0,1)\,,$$

$1 \leqq i \leqq k$. Hence, from (4.12), we conclude that for every finite a, $n_i^{\frac12}S_{i,N}(\alpha_i + N^{-\frac12}a, \hat\beta_{i,N})$ has asymptotically a normal distribution with mean $-[\lambda_N^{(i)}]^{\frac12}aB(F, \phi)$ and variance $A^2 + q_{i,N}^2A^2 = A^2(1 + q_{i,N}^2)$, $1 \leqq i \leqq k$. Thus, from (4.4)–(4.7) and the above result, we readily obtain that

$$(4.15)\qquad |N^{\frac12}(\hat\alpha-\alpha_0)| = O_p(1)\,,$$

under H_0 in (1.2) or under $\{H_N\}$ in (3.2). Let then

$$(4.16)\qquad \hat\alpha_{i,N}^{(1)}=\sup\{a: S_{i,N}(a,\hat\beta_{i,N})>0\}\,,\qquad \hat\alpha_{i,N}^{(2)}=\inf\{a: S_{i,N}(a,\hat\beta_{i,N})<0\}\,;$$

$$(4.17)\qquad \hat\alpha_{i,N}=\tfrac12(\hat\alpha_{i,N}^{(1)}+\hat\alpha_{i,N}^{(2)})\,,\qquad 1\leqq i\leqq k\,.$$

Then, from (4.12), (4.13), (4.16) and (4.17), we obtain, by some standard computations, the following.

LEMMA 4.4. *Under the assumptions of Section 2, as $N \to \infty$,*

$$(4.18)\qquad \mathscr{L}(n_i^{\frac12}[\hat\alpha_{i,N}-\alpha_i]B(F,\phi)A^{-1}(1+q_{i,N}^2)^{-\frac12})\to\mathscr{N}(0,1)\,,\qquad \textit{for all } 1\leqq i\leqq k\,.$$

We now use the above lemmas to prove the following basic lemmas.

LEMMA 4.5. *Under (1.2) or (3.2)–(3.3), and the regularity conditions of Section 2, as $N \to \infty$,*

$$(4.19)\qquad |N^{\frac12}(\hat\alpha_N-\textstyle\sum_{i=1}^k w_{i,N}\hat\alpha_{i,N})|\to_p 0\,.$$

PROOF. By (2.8) and (4.18), as $N \to \infty$,

$$(4.20)\qquad |N^{\frac12}(\hat\alpha_{i,N}-\alpha_i)| = O_p(1)\qquad \text{for all } 1\leqq i\leqq k\,.$$

Now, by (4.4), $S_N^*(a)$ is a monotone step-function of a for any fixed sample.

Hence, by (4.6)–(4.7), (4.13) and (4.18), we obtain by some standard steps that

$$|N^{\frac{1}{2}}S_N^*(\hat{\alpha}_N)| = o_p(1) \qquad \text{as } N \to \infty, \tag{4.21}$$

and similarly, by (4.3), monotonicity of $S_{i,N}$, (4.16)–(4.17), (4.13) and (4.18), as $N \to \infty$,

$$|N^{\frac{1}{2}}S_{i,N}(\hat{\alpha}_{i,N}, \hat{\beta}_{i,N})| = o_p(1) \qquad \text{for all } 1 \leqq i \leqq k. \tag{4.22}$$

Further, under (3.2)–(3.3) and (2.3),

$$\sum_{i=1}^k w_{i,N}(\theta_i - \bar{\theta}) \to 0 \qquad \text{as } N \to \infty, \tag{4.23}$$

where $\bar{\theta}$ is defined by (3.5). Hence from (4.4), (4.5), (4.13), (4.15), (4.20), (4.21), (4.22) and (4.23), we have

$$\begin{aligned} N^{\frac{1}{2}}S_N^*(\hat{\alpha}_N) &= \sum_{i=1}^k w_{i,N}\{N^{\frac{1}{2}}[S_{i,N}(\hat{\alpha}_N, \hat{\beta}_{i,N}) - S_{i,N}(\hat{\alpha}_{i,N}, \hat{\beta}_{i,N})]\} + o_p(1) \\ &= \sum_{i=1}^k w_{i,N}(\hat{\alpha}_{i,N} - \hat{\alpha}_N)B(F, \phi) + o_p(1). \end{aligned} \tag{4.24}$$

Since, by (4.21), the left-hand side of (4.24) is $o_p(1)$, the lemma follows.

LEMMA 4.6. *Under H_0 in (1.2) or $\{H_N\}$ in (3.2)–(3.3), and the regularity conditions of Section 2, as $N \to \infty$*

$$Q_N \sim_p [B^2(F, \phi)/A^2][\sum_{i=1}^k w_{i,N}[\hat{\alpha}_{i,N} - \hat{\alpha}_N]^2(1 + q_{i,N}^2)^{-1}]. \tag{4.25}$$

PROOF. By (4.13), (4.15), (4.20) and (4.22), as $N \to \infty$,

$$\begin{aligned} n_i^{\frac{1}{2}}S_{i,N}(\hat{\alpha}_N, \hat{\beta}_{i,N}) &= n_i^{\frac{1}{2}}[S_{i,N}(\hat{\alpha}_N, \hat{\beta}_{i,N}) - S_{i,N}(\hat{\alpha}_{i,N}, \hat{\beta}_{i,N})] + o_p(1) \\ &= n_i^{\frac{1}{2}}(\hat{\alpha}_{i,N} - \hat{\alpha}_N)B(F, \phi) + o_p(1), \qquad 1 \leqq i \leqq k, \end{aligned} \tag{4.26}$$

and hence, the lemma follows from (4.8) and (4.26).

Now, under (3.2)–(3.3), it follows from Lemma 4.4 that as $N \to \infty$,

$$\mathscr{L}(n_i^{\frac{1}{2}}[\hat{\alpha}_{i,N} - \alpha_0] \mid H_N) \to \mathscr{N}([\lambda^{(i)}]^{\frac{1}{2}}\theta_i, A^2(1 + q_i^2)/B^2(F, \phi)), \tag{4.27}$$

$1 \leqq i \leqq k$, and hence, from Lemmas 4.5 and 4.6, we obtain the following.

THEOREM 4.7. *Under the regularity conditions of Section 2 and (3.2)–(3.3), Q_N has asymptotically a noncentral chi-square distribution with $k - 1$ d.f. and noncentrality parameter*

$$\Delta_Q = [B^2(F, \phi)/A^2][\sum_{i=1}^k \lambda^{(i)}(\theta_i - \bar{\theta})^2/(1 + q_i^2)]. \tag{4.28}$$

5. Asymptotic relative efficiency (ARE) **results.** Let us define first

$$\rho(\phi, \psi) = [B(F, \phi)]/\{AI(F)\}; \tag{5.1}$$

by definition in (2.21), $0 \leqq \rho(\phi, \psi) \leqq 1$, and $\rho(\phi, \psi) = 1$ iff $\phi(u) = \psi(u): 0 < u < 1$. It follows from (3.6) and (4.28) that the ARE of the Q_N-test with respect to the likelihood ratio test is equal to

$$e_{Q,L} = \Delta_Q/\Delta_L = \rho^2(\phi, \psi), \tag{5.2}$$

and hence, if $\phi(u) \equiv \psi(u)$, we obtain that the Q_N-test has the same asymptotically local optimal properties as of the likelihood ratio test. For related results on $\rho^2(\phi, \psi)$ for various (ϕ, ψ), we refer to Hájek (1962), who considered the simple regression problem.

Again from (3.4) and (4.28), we obtain that the ARE of the Q_N-test with respect to the variance-ratio test is equal to

$$e_{Q,Z} = \sigma^2(F)B^2(F, \phi)/A^2 , \tag{5.3}$$

which agrees with the ARE of the several sample rank test for location [cf. Puri (1964)]. As such, we have that for the Q_N-test based on normal scores, $e_{Q,Z}$ is bounded below by 1, where the lower bound is attained iff F is normal. Also, for Wilcoxon scores, (5.3) is bounded below by 0.864 for all F, while for many non-normal F, it is greater than one.

REMARK. We are able to claim asymptotic optimality of Q_N (when $\phi = \psi$) by using the same $\phi(u)$ to generate the scores for the $T_{i,N}$ and the $S_{i,N}$. If the ϕ^* for the $S_{i,N}$ were not derived from ϕ, say, we had $\phi_1(u)$ and $\phi_2{}^*(u) = \phi_2((1 + u)/2)$, where $\phi_1 \neq \phi_2$, then writing $A_1{}^2$ and $A_2{}^2$ as in (2.12) for $\phi = \phi_1$ and ϕ_2, we would have in (4.12) $AT_{i,N}(\beta_i)$ replaced by $A[B(F, \phi_2)/B(F, \phi_1)]T_{i,N}(\beta_i)$. Since, $B(F, \phi_i)$, $i = 1, 2$ are unknown and not necessarily equal, in Lemma 4.4, we require to change $A^2(1 + q_{i,N}^2)/B^2(F, \phi)$ by $A_2{}^2/B^2(F, \phi_2) + A_1{}^2 q_{i,N}^2/B^2(F, \phi_1)$, and hence, Lemmas 4.5 and 4.6 do not hold. Hence, we are not in a position to use Q_N in (4.8). Of course, it is possible to estimate $B(F, \phi_i)$, $i = 1, 2$ [viz., Sen (1969) Section 4], and use an appropriate quadratic form in the $\hat{\alpha}_{i,N}$. However, the optimality, as claimed for $\phi \equiv \psi$, will not be retained.

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REFERENCES

[1] ADICHIE, J. N. (1967). Estimates of regression parameters based on rank tests. *Ann. Math. Statist.* **38** 894–904.

[2] FINNEY, D. J. (1952). *Statistical Methods in Biological Assays.* Griffin, London.

[3] HÁJEK, J. (1962). Asymptotically most powerful rank order tests. *Ann. Math. Statist.* **33** 1124–1147.

[4] HÁJEK, J. (1968). Asymptotic normality of simple linear statistics under alternatives. *Ann. Math. Statist.* **39** 325–346.

[5] HÁJEK, J. (1970). Miscellaneous problems of rank test theory. *Nonparametric Techniques in Statistical Inference*, ed. M. L. Puri. Cambridge Univ. Press, 1–17.

[6] JUREČKOVÁ, J. (1969). Asymptotic linearity of a rank statistic. *Ann. Math. Statist.* **40** 1889–1900.

[7] JUREČKOVÁ, J. (1971). Nonparametric estimate of regression coefficient. *Ann. Math. Statist.* **42** 1328–1338.

[8] PURI, M. L. (1964). Asymptotic efficiency of a class of c-sample tests. *Ann. Math. Statist.* **35** 102–121.

[9] SEN, P. K. (1969). On a class of rank order tests for the parallelism of several regression lines. *Ann. Math. Statist.* **40** 1668–1683.

[10] VAN EEDEN, C. (1972). An analogue, for signed rank statistics, of Jurečková's asymptotic linearity theorem for rank statistics. *Ann. Math. Statist.* **43** 791–802.

[11] WALD, A. (1943). Tests of statistical hypotheses concerning several parameters when the number of observations is large. *Trans. Amer. Math. Soc.* **54** 426–482.

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