ON THE CORRELATION COEFFICIENT OF A BIVARIATE,
EQUAL VARIANCE, COMPLEX GAUSSIAN SAMPLE\(^1\)

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Let \( u_n \) denote the sample correlation coefficient for \( n \) observations from a bivariate, equal variance, complex Gaussian distribution. In this note we derive the exact distribution of \( u_n \) by extending a method of Mehta and Gurland to the complex case. The asymptotic behavior of \( E|u_n|^k \) as \( n \to \infty \) is determined via the method of steepest descent. Applicability of the results to the analysis of certain estimators of spectral parameters of stationary time series is discussed.

1. Density of \( u_n \). Let \( \xi_1, \xi_2, \ldots, \xi_n \) be an independent sample from a zero mean, bivariate, equal variance, complex Gaussian distribution with correlation matrix

\[
\Sigma_\xi = E\xi\xi' = \begin{pmatrix} \sigma^2 & \sigma^2 \rho \\ \sigma^2 \rho & \sigma^2 \end{pmatrix}.
\]

The Hermitian sample correlation matrix \( A = (A_{ij}) = n^{-1} \sum_{k=1}^n \xi_k \xi_k' \), \( 1 \leq i, j \leq 2 \), then has the bivariate complex Wishart density

\[
p(A) = |A|^{n-1}[\pi \Gamma(n)\Gamma(n-1)|\Sigma_\xi|^n]^{-1} \exp \left[ -\text{tr}(\Sigma_\xi^{-1}A) \right].
\]

(See Goodman (1963) for a detailed discussion of the complex Gaussian and complex Wishart distributions.) The function \( p(A) \), defined over the domain where \( A \) is Hermitian positive semi-definite, is a compact way of writing the joint density of the four real random variables \( A_{11}, A_{22}, A_{12}, A_{21} \). The usual estimator \( u_n \) of the complex correlation coefficient \( \rho \) is a function of the elements of \( A \), namely

\[
u_n = 2A_{12}[A_{11} + A_{22}]^{-1}.
\]

Throughout the remainder of this section we suppress the subscript \( n \) on \( u_n \). The joint probability density of the real random variables \( u_R \) and \( u_I \), defined by the relation \( u = u_R + iu_I \), may be found by extending the method of Mehta and Gurland (1969) to the complex case as follows. First, two auxiliary variables \( v = A_{11} + A_{22} \) and \( w = A_{12} \) are introduced. A simple calculation reveals that the magnitude of the Jacobian of the transformation \( (A_{11}, A_{22}, A_{12}, A_{21}) \to (u_R, u_I, v, w) \) equals \( (v/2)^4 \). It follows from (1), (2), and some algebra that

\[
p(u_R, u_I, v, w) = K_i(v/2)^4[(v - w)w - (|v|^2/2)^2]^{-1} \exp[-v\sigma_{11} + v\text{Re}(\sigma_{12}u_I)],
\]

where \( K_i^{-1} = \Gamma(n)\Gamma(n-1)\pi\sigma_{11}^n(1 - |\rho|^2) \), \( \sigma_{jk} \) is the \((j, k)\)th element of \( \Sigma_\xi^{-1} \) and the

\[\text{Received October 4, 1971; revised April 12, 1972.}\]

\[\text{AMS 1970 subject classifications. Primary, 62.10; Secondary, 62.15.}\]

\[\text{Key words and phrases. Complex correlation coefficient, bivariate complex Gaussian distribution, complex Wishart distribution.}\]

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density vanishes outside the region \(0 \leq w \leq v < \infty, |u| \leq 2[(w/v) - (w/v)^2]^{1/4}\). Although \(p(u, u, v, w)\) may be obtained by integrating \(p(u, u, v, w)\) over this region with respect to \(v\) and \(w\), it proves more convenient first to introduce the variable \(t\) defined by

\[
w = (v/2)[1 + t(1 - |u|^2)]^3.\]

Then \(p(u, u, v, t) = p(u, u, v, w) |\partial w/\partial t|\) is supported on the cylinder \(0 \leq |u| \leq 1, 0 \leq v < \infty, -1 \leq t \leq 1\), wherein it assumes the form

\[
p(u, u, v, t) = K_s(1 - t^2)^{n-3/2}(1 - |u|^2)^{(2n-5)/2}(v/2)^{2n-1} \exp[-v\sigma_{11} + v \Re(\sigma_{12}^u)].
\]

Integrating over \(v\) and \(t\) yields

\[
p(u, u) = K_s(1 - |u|^2)^{(2n-5)/2}\sigma_{11} - \Re(\sigma_{12}^u)]^{-2n}, \quad |u| \leq 1,
\]

where \(K_s = \Gamma(2n)[\pi^{1/2}\Gamma(n)\Gamma(n - \frac{1}{2})\sigma_{22}^u(1 - |\rho|^2)^{2n-1}]^{-1}\).

Subsequent discussion is facilitated by transforming (3) to polar coordinates. Letting \(u = |u|e^{i\theta_u}\) and \(\rho = |\rho|e^{i\theta_0}\), and using the explicit expressions for \(\sigma_{11}\) and \(\sigma_{12}\), we obtain

\[
p(|u|, \theta_u) = C|u|(1 - |u|^2)^{(2n-5)/2}[1 - |u| |\rho| \cos(\theta_u - \theta_0)]^{-2n}
\]

\[
|0 \leq |u| \leq 1, -\pi \leq \theta_u \leq \pi\),
\]

where

\[
C = (1 - |\rho|^2)^n \Gamma(2n)[\pi^{1/2}\Gamma(n)\Gamma(n - \frac{1}{2})2^{2n-1}]^{-1}.
\]

Note that \(|u|\) and \(\theta_u\) are not statistically independent. Extensions of certain of the above results to the unequal variances case appear in Goodman (1957).

2. Asymptotic behavior of \(E|u_n|^k\). The asymptotic behavior of \(E|u_n|^k\) for large \(n\) may be obtained as follows. From (4) we have

\[
E|u_n|^k = C \int_{|u|} \int_{\theta} \int \frac{x^{k+1}}{(1 - x^2)^{3/2}} \left[ \frac{1 - x^2}{(1 - x|\rho| \cos\theta)^2} \right]^n dx d\theta
\]

For \(|\theta| < \pi/2\) the integrand has a high peak centered at \(x = |\rho| \cos \theta\) when \(n\) is large, while for \(\pi/2 < |\theta| < \pi\) the integrand contains the \(nth\) power of a factor that is smaller than 1 for all \(0 \leq x \leq 1\). Accordingly, the error associated with limiting the range of integration to \(|\theta| < \pi/2\) is negligible when \(n\) is large. For each such \(\theta\) the inner integral in (6) is of the form

\[
I = \int g(x) \exp[-nf(x)] dx,
\]

where \(g(x) = x^{k+1}(1 - x^2)^{-3/2}, f(x) = \log[(1 - bx)^0/(1 - x^2)],\) and \(b = |\rho| \cos \theta\).

Since \(f(x)\) has a minimum at \(x = b\) and \(f''(b) = 2(1 - b^2)^{-3},\) the method of steepest descent yields \(I \sim (\pi/n)^{1/2} b^{k+1}/(1 - b^2)^{n+1}\). Therefore

\[
E|u_n|^k \sim \left(\frac{\pi}{n}\right)^{1/2} C \int \frac{((|\rho| \cos\theta)^{k+1}}{(1 - |\rho|^2 \cos^2 \theta)^{n+1}} d\theta.
\]

Since \(|\rho| < 1\) in nondegenerate cases, the integrand in (7) peaks sharply at \(\theta = 0\)
for large \( n \). Steepest descent therefore is applicable again. In this regard we note that, to second order in \( \theta \),

\[
(1 - |\rho|^2 \cos^2 \theta)^{-n} = \exp[-n \log(1 - |\rho|^2 \cos^2 \theta)] \sim (1 - |\rho|^2)^{-n} \exp\left(-\frac{n|\rho|^2 \theta^2}{1 - |\rho|^2}\right).
\]

It follows that

\[
E|u_n|^k \sim \left(\frac{\pi}{n}\right)^{\frac{k}{2}} C \frac{|\rho|^{k+1}}{(1 - |\rho|^2)^{n+\frac{k}{2}}} \int_{-\infty}^{\infty} \exp\left(-\frac{n|\rho|^2 \theta^2}{1 - |\rho|^2}\right) d\theta = \frac{\pi C|\rho|^k}{n(1 - |\rho|^2)^n}.
\]

Substituting for \( C \) from (5) yields the desired result,

\[
E|u_n|^k \sim \frac{\pi^{\frac{1}{2}} (2n)|\rho|^k}{n\Gamma(n)\Gamma(n - \frac{1}{2})2^{2n-1}}.
\]

Asymptotic expansion of the gamma functions in (8) verifies the intuitively obvious fact that \( E|u_n|^k \rightarrow |\rho|^k \) as \( n \rightarrow \infty \), whereupon the \( L_r \)-convergence theorem (Loève (1963)) implies that the \( |u_n| \) converge in \( r \)th mean to the constant \( |\rho| \) for all \( r > 0 \).

**3. Applicability to estimation of spectral parameters.** The above results find direct application to the analysis of certain radar estimates of spectral parameters of distributed-velocity media such as storm clouds and clear air turbulence. Specifically, the in-phase and quadrature signals returned from the portion of such a medium that is located at a fixed range from the transmitter may be modeled as the real and imaginary parts, respectively, of a zero mean complex Gaussian random process \( \{Z_t\} \). It follows that range gating of the returns from a pair of narrow radar pulses spaced \( T \) seconds apart produces a bivariate complex Gaussian random variable \( \xi \). If there is no clutter from ambiguous range cells, then \( \xi = (Z_t, Z_{t+T}) \). Moreover, if \( \{Z_t\} \) is wide-sense stationary, which usually is the case, then \( Z_t \) and \( Z_{t+T} \) have equal variances. Their correlation matrix \( \Sigma_\xi \) then is of the form (1) with \( \sigma^2 = \int dF(\gamma) \) and \( \sigma^2 \rho = \int e^{i2\pi \gamma T} dF(\gamma) \), where \( F \) is the spectral distribution function of \( \{Z_t\} \). An independent sample \( \xi_j = (Z_{t_j}, Z_{t_j+T}) \), \( 1 \leq j \leq n \), of such bivariate, equal variance, complex Gaussian random variables may be obtained either by frequency-stepping the radar carrier frequency or by inserting sufficiently long delays between successive pulse pairs. In many applications it is of interest to estimate the centroid \( \gamma_0 \) and spread \( \sigma_x^2 \) of \( F \), which are defined by \( \int (\gamma - \gamma_0) dF(\gamma) = 0 \) and \( \sigma^2 \sigma_x^2 = \int (\gamma - \gamma_0)^2 dF(\gamma) \). Reasonable estimators of these quantities proposed by Rummel (1968a) are expressible as functions of the complex statistic \( u = |u|e^{i\theta_u} \), specifically \( \hat{\gamma}_0 = (2\pi T)^{-1} \theta_u \) and \( \hat{\sigma}_x^2 = (2\pi^2 T^2)^{-1} (1 - |u|) \). Some properties of these estimators and of certain extensions of them to cases in which \( \{Z_t\} \) is corrupted by additive, independent, “white” receiver noise have been explored by Rummel (1968b), Hofstetter (1970), and Berger (1971).
REFERENCES


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