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ON THE EFFICIENCY OF A COMPETITOR OF THE TWO-SAMPLE KOLMOGOROV–SMIRNOV AND KUIPER TESTS

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In a paper by Abrahamson [1], it is shown that the Kuiper test generally performs better than the Kolmogorov–Smirnov (K–S) test according to exact Bahadur relative efficiency. The present note concerns the Bahadur efficiency of a related test statistic U_n whose exact null probability distribution is available in the two-sample case with equal sample sizes. It is shown that U_n is often more efficient than the K–S test and may even be as efficient as the Kuiper test.

1. The test statistics. Assume we have two independent samples $x_1, \cdots, x_n$ from a population with distribution function F and $y_1, \cdots, y_n$ from a population with distribution function G. We wish to test the hypothesis H: $F \equiv G$. The test statistics we shall consider are the K–S statistic $K_n = n^{\frac{1}{2}} \sup_x |F_n(x) - G_n(x)| = n^{\frac{1}{2}} \max(D_n^+, D_n^-)$, the Kuiper statistic $V_n = n^{\frac{1}{2}}(D_n^+ + D_n^-)$, and $U_n = n^{\frac{1}{2}} \min(D_n^+, D_n^-)$, where $D_n^+ = \sup_x [F_n(x) - G_n(x)]$, $D_n^- = \sup_x [G_n(x) - F_n(x)]$ and F_n and G_n are the sample distribution functions of x's and y's, respectively.

2. Bahadur relative efficiency. For a complete discussion of Bahadur relative efficiency, see [2]. We only point out here that if $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ are two sequences of test statistics for testing H: $\theta \in \Theta_0 \subset \Theta$, then the efficiency of $\{T_n^{(1)}\}$ relative to $\{T_n^{(2)}\}$, for $\theta \in \Theta - \Theta_0$, is the ratio $c_1(\theta)/c_2(\theta)$, where $c_i(\theta) = 2f_i(b_i(\theta))$, $i = 1, 2$, and f_i and b_i may be obtained as follows.

(i) $b_i(\theta)$ is a function such that $T_n^{(i)}/n^{\frac{1}{2}} \rightarrow b_i(\theta) > 0$ with probability one $[\theta]$ for $\theta \in \Theta - \Theta_0$.

(ii) $f_i(t)$ is a function continuous in an open set containing the range of $b_i(\theta)$ such that $-n^{-1} \log P_0\{T_n^{(i)} \geqq n^{\frac{1}{2}}t\} \rightarrow f_i(t)$, where P_0 denotes the probability distribution of T_n for $\theta \in \Theta_0$.

The function $c_i(\theta)$ is called the *exact slope* of $\{T_n^{(i)}\}$.

3. Exact slopes of the test statistics. We first derive the exact slope of the test statistic $U_n = n^{\frac{1}{2}} \min(D_n^+, D_n^-)$. Since $D_n^+ \rightarrow_{\text{a.s.}} D^+ = \sup_x [F(x) - G(x)]$ and $D_n^- \rightarrow_{\text{a.s.}} D^- = \sup_x [G(x) - F(x)]$, we see $U_n/n^{\frac{1}{2}} \rightarrow \min(D^+, D^-) = b_U(F, G)$. To obtain the function $f_U(t)$ needed in (ii), we employ the results of Gnedenko and Korolyuk [3] as presented by Hájek and Sidák [4], which yield

$$P_0\{D_n^+ \geqq d\} = \binom{2n}{n}^{-1}\binom{2n}{n-h} \tag{1}$$

and

$$P_0\{D_n \geqq d\} = 2\binom{2n}{n}^{-1} \sum_{j=1}^{[n/h]} (-1)^{j+1}\binom{2n}{n-jh} \tag{2}$$

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where $h = -[-nd]$, the least integer greater than or equal to nd. (Equation (2) is valid for n sufficiently large that $h \geqq 2$, i.e., for $n > 1/d$.) Therefore we have, for $0 < d < \frac{1}{2}$,

$$\begin{aligned} P_0\{U_n \geqq n^{\frac{1}{2}}d\} &= P_0\{\min(D_n^+, D_n^-) \geqq d\} \\ &= P_0\{D_n^+ \geqq d\} + P_0\{D_n^- \geqq d\} - P_0\{\max(D_n^+, D_n^-\} \geqq d\} \\ &= 2\tbinom{2n}{n}^{-1}\tbinom{2n}{n-h} - 2\tbinom{2n}{n}^{-1}\textstyle\sum_{j=1}^{[n/h]}(-1)^{j+1}\tbinom{2n}{n-jh} \\ &= 2\tbinom{2n}{n}^{-1}\textstyle\sum_{j=2}^{[n/h]}(-1)^{j}\tbinom{2n}{n-jh}\,. \end{aligned}$$

Upon realizing that $\binom{2n}{n-jh}/\binom{2n}{n-2h} \to 0$ for $j > 2$ and that $[n/h] \leqq 1/d$ we write

$$P_0\{U_n \geqq n^{\frac{1}{2}}d\} = 2\tbinom{2n}{n}^{-1}\tbinom{2n}{n-2h}[1 + o(1)]\,.$$

Application of Stirling's formula to $\binom{2n}{n}^{-1}\binom{2n}{n-2h}$ yields $-n^{-1}\log P_0\{U_n \geqq n^{\frac{1}{2}}d\} \to (1 + 2d)\log(1 + 2d) + (1 - 2d)\log(1 - 2d)$. Thus; since $b_U(F, G) < \frac{1}{2}$ for $F \not\equiv G$, we see that the exact slope of $\{U_n\}$ is $c_U(F, G) = 2g(2\min(D^+, D^-))$, where $g(d) = (1 + d)\log(1 + d) + (1 - d)\log(1 - d)$.

Abrahamson [1] obtained the exact slopes of $\{K_n\}$ and $\{V_n\}$ in the more general setting of unequal sample sizes, which results in rather complicated expressions. In the present situation of equal sample sizes the expressions are quite simple. According to Theorems 3 and 4 of [1], $f_K(d) = \lim -n^{-1}\log P_0\{K_n \geqq n^{\frac{1}{2}}d\} = \lim -n^{-1}\log P_0\{V_n \geqq n^{\frac{1}{2}}d\} = f_V(d)$. By employing (2), one finds $f_K(d) = g(d)$ and therefore, of course, $f_V(d) = g(d)$. (Klotz [5] obtained the result for the K–S statistic.) From this and the fact that $K_n/n^{\frac{1}{2}} \to_{\text{a.s.}} \max(D^+, D^-)$ and $V_n/n^{\frac{1}{2}} \to_{\text{a.s.}} (D^+ + D^-)$, it follows that $c_K(F, G) = 2g(\max(D^+, D^-))$ and $c_V(F, G) = 2g(D^+ + D^-)$.

4. Discussion. We immediately see that c_V is always at least as large as c_K, as was concluded by Abrahamson, and also that c_V is always at least as large as c_U. In situations where $2\min(D^+, D^-) > \max(D^+, D^-)$ we see that U_n performs better than K_n, and in case $D^+ = D^-$, we see that U_n performs as well as V_n. In particular, if for some μ and σ, $G(\mu + t) = 1 - G(\mu - t)$ for all t, and $F(\mu + t) = G(\mu + \sigma t)$ for all t, then $D^+ = D^-$ and therefore we might choose to use U_n because it is more efficient than K_n and its null distribution is somewhat more simply calculated than that of V_n.

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