ON THE EFFICIENCY OF A COMPETITOR OF THE TWO-SAMPLE KOLMOGOROV–SMIRNOV AND KUIPER TESTS

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In a paper by Abrahamson [1], it is shown that the Kuiper test generally performs better than the Kolmogorov–Smirnov (K–S) test according to exact Bahadur relative efficiency. The present note concerns the Bahadur efficiency of a related test statistic $U_n$ whose exact null probability distribution is available in the two-sample case with equal sample sizes. It is shown that $U_n$ is often more efficient than the K–S test and may even be as efficient as the Kuiper test.

1. The test statistics. Assume we have two independent samples $x_1, \ldots, x_n$ from a population with distribution function $F$ and $y_1, \ldots, y_n$ from a population with distribution function $G$. We wish to test the hypothesis $H: F \equiv G$. The test statistics we shall consider are the K–S statistic $K_n = n^2 \sup_x |F_n(x) - G_n(x)| = n^2 \max (D_n^+, D_n^-)$, the Kuiper statistic $V_n = n^2 (D_n^+ + D_n^-)$, and $U_n = n^2 \min (D_n^+, D_n^-)$, where $D_n^+ = \sup_x [F_n(x) - G_n(x)]$, $D_n^- = \sup_x [G_n(x) - F_n(x)]$ and $F_n$ and $G_n$ are the sample distribution functions of $x$'s and $y$'s, respectively.

2. Bahadur relative efficiency. For a complete discussion of Bahadur relative efficiency, see [2]. We only point out here that if $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ are two sequences of test statistics for testing $H: \theta \in \Theta_0 \subset \Theta$, then the efficiency of $\{T_n^{(1)}\}$ relative to $\{T_n^{(2)}\}$ for $\theta \in \Theta - \Theta_0$, is the ratio $c_i(\theta)/c_2(\theta)$, where $c_i(\theta) = 2f_i(b_i(\theta))$, $i = 1, 2$, and $f_i$ and $b_i$ may be obtained as follows.

(i) $b_i(\theta)$ is a function such that $T_n^{(i)}/n^2 \to b_i(\theta) > 0$ with probability one $[\theta]$ for $\theta \in \Theta - \Theta_0$.

(ii) $f_i(t)$ is a function continuous in an open set containing the range of $b_i(\theta)$ such that $-n^{-1} \log P_\theta[T_n^{(i)} \geqslant nt] \to f_i(t)$, where $P_\theta$ denotes the probability distribution of $T_n$ for $\theta \in \Theta_0$.

The function $c_i(\theta)$ is called the exact slope of $\{T_n^{(i)}\}$.

3. Exact slopes of the test statistics. We first derive the exact slope of the test statistic $U_n = n^2 \min (D_n^+, D_n^-)$. Since $D_n^+ \to D^+$, $D^- = \sup_x [F(x) - G(x)]$ and $D_n^- \to D^-$, $D = \sup_x [G(x) - F(x)]$, we see $U_n/n^2 \to \min (D^+, D^-) = b_0(F, G)$. To obtain the function $f_0(t)$ needed in (ii), we employ the results of Gnedenko and Korolyuk [3] as presented by Hájek and Sidák [4], which yield

(1) $P_0[D_n^+ \geqslant d] = (2^n)^{-1}(-2^n)$

and

(2) $P_0[D_n \geqslant d] = 2(-2^n)^{-1} \sum_{j=1}^{[\log_2 d]} (-1)^{j+1}(-2^n)$

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where \( h = \lceil -nd \rceil \), the least integer greater than or equal to \( nd \). (Equation (2) is valid for \( n \) sufficiently large that \( h \geq 2 \), i.e., for \( n > 1/d \).) Therefore we have, for \( 0 < d < \frac{1}{2} \),

\[
P_d[U_n \geq n^d] = P_d[\min(D_n^+, D_n^-) \geq d] = P_d[D_n^+ \geq d] + P_d[D_n^- \geq d] - P_d[\max(D_n^+, D_n^-) \geq d] = 2(\frac{2^n}{n^h})^{-1} \left( \frac{2^n}{n^h} \right) - 2(\frac{2^n}{n^h})^{-1} \sum_{j=1}^{[n/h]} (-1)^{j+1} \left( \frac{2^n}{n^h} \right)^j.
\]

Upon realizing that \( \frac{(n-2j)}{(n-2h)} \rightarrow 0 \) for \( j > 2 \) and that \( [n/h] \leq 1/d \) we write

\[
P_d[U_n \geq n^d] = 2(\frac{2^n}{n^h})^{-1} \left( \frac{2^n}{n^h} \right)[1 + o(1)].
\]

Application of Stirling's formula to \( (\frac{2^n}{n^h}) \) yields \(-n^{-1} \log P_d[U_n \geq n^d] \rightarrow (1 + 2d) \log(1 + 2d) + (1 - 2d) \log(1 - 2d)\). Thus, since \( b_0(F, G) < \frac{1}{2} \) for \( F \neq G \), we see that the exact slope of \( U_n \) is \( c_F(F, G) = 2g(2 \min(D^+, D^-)) \), where \( g(d) = (1 + d) \log(1 + d) + (1 - d) \log(1 - d) \).

Abrahamson [1] obtained the exact slopes of \( K_n \) and \( V_n \) in the more general setting of unequal sample sizes, which results in rather complicated expressions. In the present situation of equal sample sizes the expressions are quite simple. According to Theorems 3 and 4 of [1], \( f_K(d) = \lim -n^{-1} \log P_d[K_n \geq n^d] = -n^{-1} \log P_d[V_n \geq n^d] = f_V(d) \). By employing (2), one finds \( f_K(d) = g(d) \) and therefore, of course, \( f_V(d) = g(d) \). (Klotz [5] obtained the result for the K-S statistic.) From this and the fact that \( K_n/n^d \rightarrow_{a.s.} \max(D^+, D^-) \) and \( V_n/n^d \rightarrow_{a.s.} \max(D^+ + D^-) \), it follows that \( c_F(F, G) = 2g(\max(D^+, D^-)) \) and \( c_V(F, G) = 2g(D^+ + D^-) \).

4. Discussion. We immediately see that \( c_V \) is always at least as large as \( c_K \), as was concluded by Abrahamson, and also that \( c_V \) is always at least as large as \( c_F \). In situations where \( 2 \min(D^+, D^-) > \max(D^+, D^-) \) we see that \( U_n \) performs better than \( K_n \), and in case \( D^+ = D^- \) we see that \( U_n \) performs as well as \( V_n \). In particular, if for some \( \mu \) and \( \sigma \), \( G(\mu + t) = 1 - G(\mu - t) \) for all \( t \), and \( F(\mu + t) = G(\mu + \sigma t) \) for all \( t \), then \( D^+ = D^- \) and therefore we might choose to use \( U_n \) because it is more efficient than \( K_n \) and its null distribution is somewhat more simply calculated than that of \( V_n \).

REFERENCES