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# ON MARKOV PROCESSES WITH RIGHT-DETERMINISTIC GERM FIELDS

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Given a Hunt process $X(t)$, we investigate the consequences of the assumption that $\mathscr{G}(T+) = \sigma(X(T))$ for every finite stopping time $T$, where $\mathscr{G}(T+) = \bigcap_{\varepsilon>0} \mathscr{F}^0[T, T+\varepsilon)$. Such processes constitute a simple extension of the right-continuous Markov chains without instantaneous states.

**0. Introduction.** Let $(\Omega, \mathscr{F}, \mathscr{F}_t, X(t), \theta_t, P^x)$ be a Hunt process on a locally compact space with countable base $(E, \mathscr{E})$, where $\mathscr{E}$ denotes the Borel sets. The notation is that of [1] page 45. For each $t$, let $\mathscr{F}_t^0$ denote the sigma-field generated by $X(s)$, $s \leqq t$, and let $T$ be a finite, $\mathscr{F}_{t+}^0$-stopping time (i.e. $\{T < t\} \in \mathscr{F}_t^0$ for each $t > 0$). The influence of chance in such a process in general occurs continuously in time, but the work of K. L. Chung, P. A. Meyer, and others has shown that there is a considerable difference between the operation of chance "from the past" at time $T$ and "to the future." To be more precise, let $\mathscr{F}^0(T, T+\varepsilon)$ be the sigma-field generated by $X(T+s)$, $0 \leqq x < \varepsilon$, and let $\mathscr{G}^+(T) = \bigcap_{\varepsilon>0} \mathscr{F}^0[T, T+\varepsilon]$ be the "right germ field" at time $T$, containing but not necessarily equaling the sigma-field $\sigma(X(T))$ generated by $X(T)$. The operation of chance to the future at $T$ is only made possible by the non-equality of $\mathscr{G}^+(T)$ and $\sigma(X(T))$, while that from the past is still more problematical due to the lack of a really satisfactory concept of left germ field (except at constant times). However, to study the distinction between these two local effects a natural idea is to exclude one and then determine what remains of the other. The idea of the present paper is simply to exhibit the role of chance from the past by assuming that it does not exist to the future.

DEFINITION 0. The process $X(t)$ is said to have right-deterministic germ fields if, after elimination from $\Omega$ of a fixed negligible set (where we use "negligible" to describe a set of $P^x$ completion-measure 0 for all $x \in E$), one has for all finite $\mathscr{F}_{t+}^0$-stopping times $T$,

$$\mathscr{G}^+(T) = \sigma(X(T))\,. \tag{0.1}$$

REMARK. We warn the reader against confusing (0.1) with "$\mathscr{G}^+(T)$ is the least $\sigma$-field containing $\sigma(X(T))$ and the negligible sets of $\mathscr{G}^+(T)$." The latter is merely an extension of the Blumenthal $0-1$ Law, and can be shown to always hold for $X(t)$. Although the difference lies in negligible sets for each $T$, it crucially affects the scope of processes considered, with the result that the remaining ones behave rather like processes without instantaneous states.

We assume henceforth that the Hunt process $(\Omega, \mathscr{F}, \mathscr{F}_t, X(t), \theta_t, P^x)$ has

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right-deterministic germ fields, and proceed to analyze it into certain elementary analytical components, from which an equivalent process is then reconstructed in a standard way. The hypothesis can be given reformulations which are sometimes more clear-cut and readily verifiable. We shall state three here (postponing the proof of equivalence to the next section).

THEOREM 0. *The condition of right-deterministic germ fields is equivalent to each of the following*:

(i) *After elimination of a negligible set,*

$$\mathscr{G}^+(0) = \sigma(X(0)) \qquad (\textit{i.e.}\ (0.1)\ \textit{holds with}\ T = 0)\,.$$

(ii) *There is a* $B^+ \times \mathscr{E}$*-measurable function* $f(t, x)$ *such that for all* $x$,

$$P^x\{X(t) = f(t, x),\ 0 \leqq t < \varepsilon \ \textit{for some}\ \varepsilon > 0\} = 1\,.$$

(iii) *After elimination of a negligible set, for all* $t$ *and* $w$, $w_0$ *in* $\Omega$, $X(t, w) = X(0, w_0)$ *implies* $X(t + s, w) = X(s, w_0)$ *for all* $0 \leqq s < \varepsilon(t, w, w_0)$ *sufficiently small.*

**1. Analysis and construction of the processes.** The following concept sets the stage for analyzing the evolution of the process.

DEFINITION 1.1. A family $w_x(t)$, $x \in E$, $t \geqq 0$, of $E$-valued, $\mathscr{E}$-measurable functions is called a canonical path system for $X(t)$ if, for all $x \in E$,

(a) $w_x(0) = x$
(b) $w_{w_x(s)}(t) = w_x(s + t)$ for all $s$, $t \geqq 0$, and
(c) $P^x\{X(s) = w_x(s),\ 0 \leqq s < \varepsilon$ for some $\varepsilon > 0\} = 1$.

THEOREM 1.1. *There exists a canonical path system.*

PROOF. Assume that the negligible set of Definition 0 has been discarded, and let $\{w_x', x \in E\}$ be any remaining family such that $X(0, w_x') = x$ for all $x \in E$. Then from $\mathscr{G}^+(0) = \sigma(X(0))$ it follows that if $\varepsilon_x > 0$ is sufficiently small we have $P^x\{X(s) = X(s, w_x'),\ 0 \leqq s < \varepsilon_x\} > 0$. Indeed $\{X(0) = x\}$ is an atom of $\mathscr{G}^+(0)$, so that it is impossible that for two paths $w_1$, $w_2$ and $s_n \to 0$ one has $X(0, w_1) = X(0, w_2) = x$ and $X(s_n, w_1) \neq X(s_n, w_2)$ for all $n$, for this would imply that $\limsup_n \{w : X(s_n, w) = X(s_n, w_1)\}$ would separate $w_1$ and $w_2$ in $\mathscr{G}^+(0)$. This shows in particular that $P^x\{X(s) = X(s, w_x'),\ 0 \leqq s \leqq \varepsilon$ for some $\varepsilon > 0\} = 1$, and the existence of $\varepsilon_x$ is clearly a consequence. For such $\varepsilon_x$ the quasi-left continuity of $X(t)$ implies that $X(s, w_x')$ is continuous, $0 \leqq s < \varepsilon_x$, for if $s_n$ increases to $s < \varepsilon_x$ then $\lim_{n\to\infty} X(s_n, w_x') = X(s, w_x') = X(s+, w_x')$. We now need the following lemmas.

LEMMA 1.1. *Let* $f(t)$ *and* $g(t)$ *satisfy* $f(0) = g(0)$ *and, for each* $t \geqq 0$, $f(t + s) = X(s, w'_{f(t)})$, $g(t + s) = X(s, w'_{g(t)})$ *for* $0 \leqq s < \varepsilon(t)$. *Then if* $f$ *and* $g$ *are continuous in* $[0, c)$ *they are identical in* $[0, c)$.

PROOF. Let $t' = \sup\{t : f(s) = g(s),\ 0 \leqq s \leqq t\}$. If $t' < c$, then $f(t') = g(t')$ by continuity. But since $X(s, w'_{f(t')})$ is continuous for small $s$, this contradicts the definition of $t'$.

As a consequence of this lemma we have

LEMMA 1.2. *For each $x \in E$, let $t_x = \sup\{t'$: there exists an $f(s)$ continuous in $[0, t')$ with $f(0) = x$ and, for each $t < t'$, $f(t+s) = X(s, w'_{f(t)})$ for $0 \leqq s < \varepsilon$ small$\}$. Then there exists a unique function $f(s)$, $0 \leqq s < t_x$, having these properties. Furthermore, either $t_x = \infty$, and then $f(t_1 + s) = f(t_2 + s)$ whenever $f(t_1) = f(t_2)$, or else $t_x < \infty$, and then $f(s)$ is one-to-one on $[0, t_x)$ and $\lim_{s \to t_x -} f(s)$ does not exist.*

PROOF. Except for the statement about $t_x < \infty$ the results are obvious from Lemma 1.1. Suppose that $t_x < \infty$ and $f(s)$ is not one-to-one. If $f(t_1) = f(t_2)$, $t_1 < t_2$, we could extend $f$ cyclically for all $t$ by setting, for $t > t_1$,

$$f(t) = f\left(t - \left[\frac{t - t_1}{t_2 - t_1}\right](t_2 - t_1)\right)$$

where $[t]$ is the greatest integer $\leqq t$. All of the requirements would be met, contradicting $t_x < \infty$. Similarly, if $\lim_{s \to t_x -} f(s)$ exists we could let this be $f(t_x)$ and then extend $f$ beyond by using $w'_{f(t_x)}$, contradicting the definition of $t_x$.

To define our canonical path system we have only to set

$$\begin{aligned} w_x(t) &= f(t) && \text{if} \quad t_x = \infty \\ &= f(t - [t/t_x]t_x) && \text{if} \quad t_x < \infty, \end{aligned}$$

where, of course, $f$ depends on $x$. It is easy to see from Lemma 1.2 that the required consistency properties hold.

We remark for future reference that only the condition of Theorem 0, (i) has been used in this construction. The probabilities for $X(t)$ will be introduced by means of the system $w_x(t)$.

THEOREM 1.2. *The random time $R = \inf\{t: X(t) \neq w_{X(0)}(t)\}$ is an exact terminal time for $X(t)$. It is the first jump time in the sense that $X(R) \neq X(R-)$ over $\{R < \infty\}$ and $X(t)$ is continuous in $[0, R)$.*

PROOF. Since $X(t)$ has left limits, it is clear from Lemma 1.2 that $R < t_{X(0)}$ if $t_{X(0)} < \infty$. We have $X(R + s) = w_{X(R)}(s)$ for all small $s$, from which it follows that $X(R) \neq X(R-)$ over $\{R < \infty\}$, and $R$ is the first jump time. Clearly this implies that $R$ is an $\mathscr{F}_t^0$-stopping time, and $R = t + R \circ \theta_t$ over $\{t < R\}$. Also, since all points of $E$ are permanent for $R$, $R$ is an exact terminal time ([1] III, (4.10)). This completes the proof.

COROLLARY 1.2. *There exists a canonical path system which is $B^+ \times \mathscr{E}$-measurable in $(t, x)$.*

PROOF. Let $\Delta$ be adjoined to $E$ as a discrete point, and let

$$\begin{aligned} X_\Delta(t) &= X(t); && 0 \leqq t < R \\ &= \Delta; && \text{otherwise}. \end{aligned}$$

Then $X_\Delta(t)$ defines a Hunt process over $E \cup \{\Delta\}$ by [1] III, (3.16) and (3.7). We set $L(x) = \sup\{t: P^x\{R > t\} > 0\}$. Then the function

$$h(t, x) = w_x(t)\,; \qquad 0 \leqq t < L(x)$$
$$= \Delta\,; \qquad \text{otherwise}$$

satisfies, for $A \in \mathscr{E}$ and $0 \leqq t$, $\{x: h(t, x) \in A\} = \{x: P^x\{X_\Delta(t) \in A\} > 0\} \in \mathscr{E}$, and by right-continuity we have that $h(t, x)$ is $B^+ \times \mathscr{E}_\Delta$ measurable. Then it is easy to see that the function $h(t - [t/L(x)]L(x), x)$ satisfies the requirements.

We can now give the proof of Theorem 0. For this purpose, we let $f(t, x)$ be any $B^+ \times \mathscr{E}$-measurable canonical path system and, denoting the condition of Definition 0 by (0), we prove that (0) $\rightarrow$ (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (ii) $\rightarrow$ (0). The implication (0) $\rightarrow$ (i) $\rightarrow$ (ii) is already clear. Assuming (ii) it follows that for each $x$, small $\varepsilon$, and $0 \leqq \varepsilon' < \varepsilon$, we have $f(\varepsilon' + s, x) = f(s, f(\varepsilon', x))$ for $0 \leqq s < \varepsilon''$ sufficiently small. Indeed, this follows from the Markov property at time $t = \varepsilon'$ whenever $P^x\{X(t) = f(t, x), 0 \leqq t < \varepsilon\} > 0$. On the other hand, the strong Markov property shows that for each stopping time $T$, $P^x\{X(T + s) = f(s, X(T)), 0 \leqq s < \varepsilon$ for some $\varepsilon > 0\} = 1$. In particular, let $T_0 = \inf\{t:$ for no $\varepsilon > 0$ does it hold that $X(t + s) = f(s, X(t)), 0 \leqq s < \varepsilon\}$. Since $X(t + s)$ and $f(s, X(t))$ are both jointly measurable in $(s, t, w)$ and, for $c \geqq 0$,

$$\{w: T_0 \geqq c\} = \{w: \lim_{n\to\infty} I_{S(n)}(t, w) \geqq I_{[0,c)}(t) \text{ for all } t\} \in \mathscr{F}_c^0\,,$$

where $$S(n) = \{X(t + s) = f(s, X(t)), 0 \leqq s < 1/n\}\,,$$

one notes that $T_0$ is a stopping time. If we show that for all $x$, $P^x\{T_0 = \infty\} = 1$, then (iii) follows by restricting $\Omega$ first to $\{w_0: X(s, w_0) = f(s, X(0, w_0)), 0 \leqq s < \varepsilon$, for some $\varepsilon > 0\}$. Now over $\{T_0 < \infty\}$ we have arbitrarily small $\varepsilon' > 0$ such that for arbitrarily small $s$, $X(T_0 + \varepsilon' + s) \neq X(s, X(T_0 + \varepsilon'))$. But except on a negligible set we have for all small $s$ and $\varepsilon'$ that $X(T_0 + \varepsilon' + s) = f(\varepsilon' + s, X(T_0))$ and $X(T_0 + \varepsilon') = f(\varepsilon', X(T_0))$, which contradicts the properties of $f$ since $X(s, f(\varepsilon', X(T_0))) = f(s, f(\varepsilon', X(T_0)))$ for small $s$.

To show (iii) $\rightarrow$ (ii) we have only to use (iii) in the case $t = 0$. It is easily checked that the only place where (i) was used in defining $f(t, x)$ for (ii) was in the choice of $w_x'$ such that $P^x\{X(s) = X(s, w_x'), 0 \leqq s < \varepsilon$ for some $\varepsilon > 0\} = 1$ in the proof of Theorem 1.1. It is clear that (iii) with $t = 0$ will suffice for this purpose. Finally, to show (ii) $\rightarrow$ (0) we again discard the set where $\{T_0 < \infty\}$ whence, for each stopping time $T$, $X(T + s) = f(s, X(T))$ for $0 \leqq s < \varepsilon$ sufficiently small. Then for $S \in \mathscr{G}^+(T)$ we have $S = \lim_{n\to\infty} S \cap \{X(T + s) = f(s, X(T)), 0 \leqq s < 1/n\}$ where each set on the right has the form $S_n \cap \{X(T + s) = f(s, X(T)), 0 \leqq s < 1/n\}$ for some $S_n \in \sigma(X(T))$ since its indicator involves only times $T + s$ with $s < 1/n$. Then $S = (\limsup S_n) \in \sigma(X(T))$ and the proof of Theorem 0 is complete.

We return to the reconstruction of $X(t)$ in terms of the canonical path system $\{w_x(t), x \in E\}$. It is convenient to introduce a kind of partial preordering in $E$ involving the times $L(x) = \sup\{t: P^x\{R > t\} > 0\}$.

DEFINITION 1.2. We will write $x \leqq y$ if for some $t < L(x)$, $y = w_x(t)$.

It is clear that $x \leqq y$ and $y \leqq z$ implies $x \leqq z$, although one may have $x \leqq y$ and $y \leqq x$ without $x = y$ (as in the case of uniform motion on a circle). We let $(x)$ denote the equivalence class of $x$ under the relation $x \equiv y$ if and only if $x \leqq y$ and $y \leqq x$, and write $(x) \leqq (y)$ for the induced ordering of equivalence classes. By Zorn's lemma each $(x)$ is contained in some maximal totally ordered subset $(A)$ of classes, and we proceed to analyze the structure of such a subset. A class $(x)$ will be called a "holding class" if it contains only the single point $x$ and $x = w_x(t)$ for all $t < L(x)$ (in this case $x$ is either a holding point or a trap of $X(t)$, and $L(x) = \infty$). Let us observe that $(A)$ can contain at most one class which either consists of more than one element or is a holding class, and such a class must be the maximum element of $(A)$. Indeed, no class can exceed a holding class, and the same is true of a class having more than one element for if $x = w_y(t_1)$, $0 < t_1 < L(y)$, and $y = w_x(t_2)$, $0 < t_2 < L(x)$, then $x = w_x(t_1 + t_2)$, $t_1 + t_2 < L(x) = \infty$, and $\{w_x(t),\ 0 \leqq t\} \subset (x)$.

THEOREM 1.3. *The totality of elements of the members of $(A)$ may be expressed as the range of a continuous function $w(t)$ defined on a right-open, left-open-or-closed interval (which may be unbounded in either direction) such that, for all $s$ in the domain, $w(s + t) = w_{w(s)}(t)$, $0 \leqq t < L(w(s))$. Such a $w$ is unique up to translations of the domain.*

PROOF. *Case* 1. $(A)$ has only a single member $(z)$. Then either $(z)$ is a holding class or $(z)$ has more than one element. In the former case we set $w(t) = z$ for all $t$, while in the latter if $z \in (z)$ then let $c = \inf\{s > 0 : w_z(s) = z\}$ and define $w(t) = w_z(t - [t/c]c)$, $-\infty < t < \infty$. The properties of $w(t)$ and the uniqueness assertion are clear.

*Case* 2. $(A)$ has only single-element classes. Let $(x)$ be any such class in $(A)$, and set $w(0) = x$ and $w(t) = w_x(t)$ for $0 \leqq t < L(x)$. Since $P^x\{X(t) = w_x(t),\ 0 \leqq t \leqq L(x)\} \leqq P^x\{R = L(x)\} = 0$ in case $L(x) < \infty$ by quasi-left-continuity, we see that every element $y$ with $(x) < (y)$ is represented as $w_x(t)$ for some $0 < t < L(x)$. Next, every $y$ with $(y) \in (A)$ and $(y) < (x)$ satisfies $w_y(t) = x$, $0 < t < L(y)$. If $(A)$ has a least element $(y_0) < (x)$ with $w_{y_0}(t_0) = x$, then we define $w(-t) = w_{y_0}(t_0 - t)$ for $0 \leqq t \leqq t_0$. Otherwise, let $(t_n)$ increase to $\sup(t : x = w_y(t),\ (y) \in (A),\ t < L(y)\}$, and $(y_n) \in (A)$ be such that $x = w_{y_n}(t_n)$. Then define $w(-t) = w_{y_n}(t_n - t)$ for all $0 \leqq t \leqq t_n$, $n \geqq 1$. The consistency of these definitions and the properties of $w(t)$ are again clear.

*Case* 3. $(A)$ has both a single-element class and a several-element class. Let $(x) \in (A)$ have only the element $x$, and let $(z) \in (A)$ be the maximum of $(A)$. We set $x = w(0)$ and for $0 \leqq t < \infty$, $w(t) = w_x(t)$. Next, for $(y) \in (A)$, $(y) < (x)$, we proceed just as in Case 2. The continuity requirement on $w(t)$ dictates the choice of $w(\min\{t : w(t) \in (z)\})$, and from this both the properties of $w$ and the uniqueness are evident. The proof is complete.

To each such set we can assign a continuous function $E(t)$ such that, if $w(t)$, $t < b$, defines the set, then for each $x = w(t)$ and $t < t + s < b$, $P^x\{R > s\} =$

$E(t+s)/E(t)$. Here $E(t)$ is determined uniquely up to a constant factor and is continuous on the domain of $w(t)$. To define $E(t)$ we set $E(t_0) = 1$ for any $t_0$ in the domain and then

$$\begin{aligned} E(t) &= P^{w(t_0)}\{R > t_0 - t\}\,; \qquad & t_0 < t \\ &= (P^{w(t)}\{R > t_0 - t\})^{-1}\,; \qquad & t < t_0\,. \end{aligned}$$

The continuity of $E(t)$ follows from the quasi-left continuity of $X(t)$. We note the further properties that if $w(t_1) = w(t_2)$ then $E(t_1 + k(t_2 - t_1))E^{-1}(t_1) = (E(t_2)/E(t_1))^k$, $1 \leqq k$, while if $w(t) = w(t_1)$ for all $t \geqq t_1$ then $E(t_1 + t_2) = E(t_1)\exp(-\varphi t_2)$ for $0 \leqq t_2$ where $0 \leqq \varphi$ is a constant.

We can treat the distribution of $X(R)$ similarly. For each $x$ there exists a wide-sense conditional distribution $q(t', A)$ of $X(R)$ given $R = t'$ relative to $P^x$, since $E$ is compact and metrizable. This is defined uniquely up to sets of $t'$ which are of measure 0 for the distribution $1 - E(t+s)/E(t)$ where $x = w(t)$. Now if $s_x = \sup\{s \geqq 0;\ w(t+s') \text{ is one-to-one in } 0 \leqq s' \leqq s\}$ then either $s_x = 0$ and $w(t+s) = w(t)$ for all $s \geqq 0$, or else $w(t+s)$ is one-to-one in $[0, s_x)$ and then either repeats itself cyclically over a cycle of length at most $s_x$, or remains constant after time $t + s_x$. Where $w$ is constant we may assume that $q(t', A)$ is likewise constant since then $w(t')$ is a trap or holding point, while in the cyclic case the Markov property justifies using a cyclic repetition of $q(t', A)$ over the same cycles. Thus we can introduce a function $p(x', A)$ satisfying $p(w(t+t'), A) = q(t', A)$, and for any $t_1 \geqq t$ with $x_1 = w(t_1)$, it is clear that $p(w(t_1 + t'), A)$ defines a wide-sense conditional distribution of $X(R)$ given $X(R-) = w(t_1 + t')$ relative to $P^{x_1}$. Recalling the sequence $(y_n)$ such that $x = w_{y_n}(t_n)$, and assuming that $(x)$ consists of a single element as in Theorem 1.3, Case 2, we can extend the definition of $p(x', A)$ to the entire range of $w(t)$ by defining recursively $p_n(x', A) = p_{n-1}(x', A)$ for $x' = w_{y_{n-1}}(s)$, $0 \leqq s < L(y_{n-1})$, and extending the definition to $x' = w_{y_n}(s)$, $0 \leqq s < t_n - t_{n-1}$, for each $n$. In the limit $n \to \infty$ we obtain a function $p(x', A)$ providing a wide sense conditional distribution of $X(R)$ given $X(R-) = x'$ for any $P^x$ with $x$ in the range of $w(t)$.

It is quite trivial if not very elegant, to extend the definitions of $E(t)$ and $p(x', A)$ to other functions $w(t)$ in a consistent manner. The ranges of such $w$ clearly may overlap, but if they contain a point $x$ in common then they also contain all $y$ with $(x) \leqq (y)$. Proceeding by transfinite induction we choose a point $x$ at each step for which $p(x, A)$ is undefined, and choose a $w$ such that $x = w(0)$. If $p_w(x', A)$ is undefined on the entire range of $w$ then we define both it and $E_w(t)$ as above. Otherwise we set $t = \inf\{t' : p_w(w(t'), A) \text{ is already defined}\}$, and extend the definitions of $p_w(x', A)$ and $E_w(t')$ consistently to the remainder of the range $w(t')$, $t' \leqq t$. Repeating this procedure as often as necessary we finally exhaust $E$ and complete the definitions.

Remark. In terms of the fields $\mathscr{F}(R-)$ of [2] this means that $P^x(X(R) \in A \mid \mathscr{F}(R-)) = p(X(R-), A)$. Indeed, $\mathscr{F}(R-)$ is generated by the pair $(X(0), R)$

by Corollary 1.2, and $P^x\{X(0) = x\} = 1$. Such a stopping time $R$ is called in [5] a Markov time for the strict past, where it is shown (Theorem 2 of [5]) that under hypothesis (L) of Meyer every totally inaccessible exact terminal time has this property.

It is now a simple matter to reconstruct a process equivalent to $X(t)$, $X(0) = x_0$, from the data $(\{w(t), E_w(t)\}, p(x', A))$ obtained above. To review briefly, for each $x \in E$ there are a $w$ and $t_0$ with $x = w(t_0)$ and

$$P^x\{R \leqq t\} = 1 - \frac{E_w(t_0 + t)}{E_w(t_0)} \qquad \text{for } t \geqq 0 .$$

We denote this by $F_x(t)$, since for each $x$ it does not depend on the choice of $w$ and $t_0$. Next, we have $P^x\{X(R) \in A \mid R\} = P^x\{X(R) \in A \mid X(R-)\} = p(X(R-), A)$ in the usual sense of equality of conditional probabilities. In view of Theorem 0 (iii), the jumps of $X(t)$ are well ordered, and since $P^{x_0}$ a.e. path has only a countable number of jumps one deduces easily the existence of a countable ordinal $\alpha_\infty$ such that $P^{x_0}\{X(t)$ has more than $\alpha_\infty$ jumps$\} = 0$. We shall construct a $P^{x_0}$-equivalent process $X_0(t)$ on the sample space $\times_{\alpha=1}^{\alpha_\infty} (R_\alpha{}^+ x E_\alpha)$ consisting of a product of compact half-lines $R_\alpha{}^+ = [0, \infty) \cup \{\infty\}$ and spaces $E_\alpha \cup \{\Delta_\alpha\}$ where $\Delta$ is some fixed object, a copy of which is adjoined to each $E$ as a discrete point. We begin by defining on $R_1{}^+$ a measure with distribution $F_{x_0}(t)$, where $1 - F_{x_0}(\infty)$ is the measure of $\{\infty\}_1$. Next we set $X_{x_0}(t) = w(t_0 + t)$, $0 \leqq t < t_1 \wedge L(x_0)$ for $t_1 \in R_1{}^+$ (using the notation $a \wedge b = \min(a, b)$). Now letting $p(\Delta, \Delta) = 1$ and $w(\infty) = \Delta$ for each of the functions $w(t)$, we define $P_1(B_1 \times A_1) = \int_{B_1} p(w(t_0 + t_1), A_1) F_{x_0}(dt_1)$ for each pair $(B_1, A_1)$ of Borel sets in $R_1{}^+$ and $E_1$. Clearly $P_1$ extends to a unique measure on the product $\sigma$-field of $R_1{}^+ \times E_1$, and we define

$$\begin{aligned} X_{x_0}(t_1 \wedge L(x_0)) &= e_1 \qquad \text{if } \ t_1 < L(x_0) \\ &= \Delta \qquad \text{if } \ t_1 \geqq L(x_0) , \end{aligned}$$

where $e_1$ is the coordinate in $E_1$. It is not hard to see that $X_{x_0}(t \wedge t_1 \wedge L(x_0))$ is $P_1 \leftrightarrow P^{x_0}$ equivalent to $X(t \wedge R_1)$, where $R_1$ is the first jump time of $X(t)$.

We now proceed by transfinite induction.

*Case* 1. $\alpha$ is not a limit ordinal. We may assume as induction hypothesis that a measure $P_\alpha$ is given on the product $\sigma$-field generated at coordinates of index at most $\alpha$, and that a process $X_{x_0}(t \wedge \sum_{i=1}^{\alpha} t_i)$ has been defined over $\{t_j < L(X_{x_0}(\sum_{i<j} t_i))$ for all $j \leqq \alpha$ such that $\sum_{i<j} t_i < \infty\}$ in such a way that the complement of this set has $P_\alpha$-measure 0 and $X_{x_0}(t \wedge \sum_{i=1}^{\alpha} t_i)$ is $P_\alpha \leftrightarrow P^{x_0}$ equivalent to $X(t \wedge R_\alpha)$, $0 \leqq t < \infty$, where $R_\alpha$ is the $\alpha$th jump time of $X(t)$, or $\infty$ if this is undefined. We shall extend the measure to the $\sigma$-field generated at the coordinates of index at most $\alpha + 1$, and extend the definition of $X_{x_0}$ to obtain an equivalent of $X(t \wedge R_{\alpha+1})$. Over $\{\sum_{i=1}^{\alpha} t_i = \infty\}$ we already have $X_{x_0}(t)$ defined for all $t$, except on the $P_\alpha$-null set where $t_j \geqq L(X_{x_0}(\sum_{i<j} t_i))$ holds for some $j \leqq \alpha$. Denoting the right side by $L_j$, we define $X_{x_0}(t) = \Delta$ for $t \geqq \sum_{i<j} t_i + L_j$ and we may assume it defined for $t$ smaller than the first such expression for

which $t_j$ satisfies the above inequality. Over $\{\sum_{i=1}^{\alpha} t_i < \infty\}$ the point $X_{x_0}(\sum_{i=1}^{\alpha} t_i)$ is thus defined by hypothesis, and using this value in place of $x_0$ we define a conditional distribution

$$\begin{aligned} P_{\alpha+1}(B_{\alpha+1} \times A_{\alpha+1} \,|\, y = X_{x_0}(\textstyle\sum_{i=1}^{\alpha} t_i)) \\ &= \textstyle\int_{B_{\alpha+1}} p(w(t_\alpha + t_{\alpha+1}), A_{\alpha+1}) F_y(dt_{\alpha+1}) , \end{aligned}$$

where $w$ and $t_\alpha$ satisfy $y = w(t_\alpha)$. The extension of this to the product $\sigma$-field on $R^+_{\alpha+1} \times E_{\alpha+1}$ provides a conditional distribution given the $\sigma$-field on the coordinates up to $\alpha$, and hence a measure $P_{\alpha+1}$ on the product $\sigma$-field on $(\alpha + 1)$-coordinates (in the case $y = \Delta$, we simply choose $t_\alpha = \infty$, and define $F_\Delta(\{\infty\}) = 1$, so that the measure is concentrated at $(\infty_{\alpha+1}, \Delta_{\alpha+1})$). Over $\{X_{x_0}(\sum_{i=1}^{\alpha} (t_i)) \neq \Delta\}$ we extend the definition of $X_{x_0}$ up to $t = \sum_{i=1}^{\alpha+1} t_i$ by setting $X_{x_0}(\sum_{i=1}^{\alpha} t_i + t) = w(t_\alpha + t)$, $0 \leqq t < t_{\alpha+1} \wedge L(w(t_\alpha))$ where $t_{\alpha+1}$ is the coordinate in $R^+_{\alpha+1}$, and finally

$$\begin{aligned} X_{x_0}((\textstyle\sum_{i=1}^{\alpha} t_i) + (t_{\alpha+1} \wedge L(w(t_\alpha)))) &= e_{\alpha+1} ; \qquad \text{if} \quad t_{\alpha+1} < L(w(t_\alpha)) \\ &= \Delta ; \qquad \text{if} \quad t_{\alpha+1} \geqq L(w(t_\alpha)) , \end{aligned}$$

where $e_{\alpha+1}$ is the coordinate in $E_{\alpha+1}$. On the complementary set, $X(t) = \Delta$ for all $t \geqq \sum_{i=1}^{\alpha} (t_i)$. It is evident from the strong Markov property of $X(t)$ at the $\alpha$th jump time $R_\alpha$ that the process $X_{x_0}(t \wedge \{\sum_{i=1}^{\alpha} t_i + (t_{\alpha+1} \wedge L(w(t_\alpha)))\})$ is equivalent to $X(t \wedge R_{\alpha+1})$, and equals $X_{x_0}(t \wedge \sum_{i=1}^{\alpha+1} t_i)$ except on a subset of the corresponding null set for the case $\alpha + 1$.

*Case* 2. $\alpha$ is a limit ordinal. In this case the jump time $R_\alpha$ of $X(t)$ is "accessible" since if $\alpha_n \uparrow \alpha$ then $R_{\alpha_n} \to R_\alpha$ where defined, and $\alpha_n < \alpha$ implies $R_{\alpha_n} < R_\alpha$ on $\{R_\alpha > \infty\}$. By a basic result of Meyer ([4] XIV, T37), $X(t)$ is continuous at $R_\alpha$, $P^{x_0}$-a.s. over $\{R_\alpha < \infty\}$. As induction hypothesis we may assume that consistent measures $P_\beta$ have been defined for every $\beta < \alpha$ and consistent definitions of processes $X_{x_0}(t \wedge \sum_{i=1}^{\beta} t_i)$ have been made such that each is $P_\beta \leftrightarrow P^{x_0}$ equivalent to $X(t \wedge R_\beta)$, $0 \leqq t < \infty$. In this case, we may pass to the limit $\beta \to \alpha$ to define a process $X_{x_0}(t)$ for $0 \leqq t < \sum_{i<\alpha} t_i$, and extend the consistent measures $P_\beta$ uniquely to a measure $P_{\alpha-}$ over the $\sigma$-field generated at all coordinates $\beta < \alpha$. Indeed, the existence of such an extension follows by comparison with the measure $P^{x_0}$ on the $\sigma$-field generated by $X(t \wedge R_\alpha)$. It is similarly clear that $\lim_{t \uparrow \sum_{i<\alpha} t_i} X_{x_0}(t) = X_{x_0}(\sum_{i<\alpha} t_i)$ exists almost everywhere over $\{\sum_{i<\alpha} t_i < \infty\}$, and with this definition the process $X_{x_0}(t \wedge \sum_{i<\alpha} t_i)$ is equivalent to $X(t \wedge R_\alpha)$. Over the subset where this limit does not exist we set $X_{x_0}(\sum_{i<\alpha} t_i) = \Delta$. We now continue exactly as in Case 1 from the point $X_{x_0}(\sum_{i<\alpha} t_i)$, over $\{\sum_{i<\alpha} t_i < \infty\}$. The strong Markov property of $X(t)$ at $R_\alpha$ shows that the extension continues to define a process equivalent to $X(t)$, and the induction terminates at $\alpha = \alpha_\infty$ in an obvious way (replacing $\alpha_\infty$ by $\alpha_\infty + 1$, we can simply set $X_{x_0}(t) = \Delta$ for $t \geqq \sum_{i=1}^{\alpha_\infty+1} t_i$). This completes the construction.

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