The Annals of Mathematical Statistics
1972, Vol. 43, No. 6, 1944-1952

GAUSSIAN PROCESSES AND GAUSSIAN MEASURES[1]

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The subject of this paper is the study of the correspondence between Gaussian processes with paths in linear function spaces and Gaussian measures on function spaces. For the function spaces $C(I)$, $C^n[a, b]$, $AC[a, b]$ and $L_2(T, \mathscr{A}, \nu)$ it is shown that if a Gaussian process has paths in these spaces then it induces a Gaussian measure on them, and, conversely, that every Gaussian measure on these spaces is induced by a Gaussian process with paths in these spaces.

1. Introduction. Gaussian processes are used in connection with problems such as estimation, detection, mutual information, etc. These problems are often effectively formulated in terms of Gaussian measures on appropriate linear spaces of functions. Even though both concepts, the Gaussian process and the Gaussian measure, have been extensively studied, it seems that the connection between them has not been adequately explained. Two important questions arising in this context are the following:

(Q_1) Given a Gaussian process with paths in a linear function space, is there a Gaussian measure on the function space which is induced by the given process?

(Q_2) Given a Gaussian measure on a linear function space, is there a Gaussian process with paths in the function space which induces the given measure?

In this paper questions Q_1 and Q_2 are answered in the affirmative for the following commonly encountered function spaces: $C(I)$, I an arbitrary interval; $C^n(I)$ and $AC(I)$, I a compact interval; and $L_2(T, \mathscr{A}, \nu)$, where $(T, \mathscr{A}, \nu)$ is an arbitrary σ-finite measure space. It is clear from the analysis in this paper that it should be possible to answer these questions for other function spaces for which satisfactory representations for the continuous linear functionals are known.

It should be mentioned that, throughout the literature, whenever the need arises to have a Gaussian measure induced on an appropriate function space by a Gaussian process, then, either (i) it is assumed that the Gaussian process is mean square continuous and that the index set is a compact interval [10], [12], [15], and thus unnecessary assumptions are made on the process, or (ii) the term "Gaussian process" is used to mean a generalized Gaussian process, i.e., a Gaussian measure, or a measurable map which induces a Gaussian measure [2], [7] and thus the problem of inducing the Gaussian measure from a Gaussian process is not considered.

Received October 21, 1970; revised March 31, 1972.

[1] This research was supported by the National Science Foundation under Grant GU-2059.

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AMS 1970 *subject classifications.* Primary 60B05, 60G15.

Key words and phrases. Gaussian processes, Gaussian measures on linear function spaces.

Throughout this paper real linear topological spaces, and real stochastic processes are considered. The basic notation, definitions and properties that are consistently used in subsequent sections are given in the following.

Let $\mathscr{X}$ be a linear topological space, $\mathscr{X}^*$ its topological dual, $\mathscr{B}(\mathscr{X})$ the smallest σ-algebra generated by all open subsets of $\mathscr{X}$, and $\mathscr{B}_c(\mathscr{X})$ the smallest σ-algebra with respect to which all elements of $\mathscr{X}^*$ are measurable. A probability measure μ on $(\mathscr{X}, \mathscr{B}_c(\mathscr{X}))$ is said to be Gaussian if every $F \in \mathscr{X}^*$ is a Gaussian random variable (rv) on $(\mathscr{X}, \mathscr{B}_c(\mathscr{X}), \mu)$. All spaces considered in this paper are separable Fréchet spaces for which it is known that $\mathscr{B}_c(\mathscr{X}) = \mathscr{B}(\mathscr{X})$ ([1] page 100). If in particular $\mathscr{X}$ is a separable Hilbert space H, with inner product and norm denoted respectively by $\langle \cdot, \cdot \rangle$ and $||\cdot||$, and if μ is a Gaussian measure on $(H, \mathscr{B}(H))$, its mean and its covariance operator are uniquely defined [13] as the element $u_0 \in H$ and the bounded, linear, nonnegative, self-adjoint and trace class operator S on H, which satisfy for all $v, w \in H$

$$\int_H \langle u, v\rangle \, d\mu(u) = \langle u_0, v\rangle \tag{1}$$

$$\int_H \langle u - u_0, v\rangle \langle u - u_0, w\rangle \, d\mu(u) = \langle Sv, w\rangle \, . \tag{2}$$

Also, if $\mathrm{tr}\,(S)$ denotes the trace of S, then [13]

$$\int_H ||u||^2 \, d\mu(u) = ||u_0||^2 + \mathrm{tr}\,(S) < +\infty \, . \tag{3}$$

A stochastic process $(\Omega, \mathscr{F}, P; X(t, \omega), t \in T)$ is said to be Gaussian if for every finite n and $t_1, \cdots, t_n \in T$ the random variables $X(t_1, \omega), \cdots, X(t_n, \omega)$ are jointly Gaussian.

Let $(\Omega, \mathscr{F}, P; X(t, \omega), t \in T)$ be a stochastic process such that $X(\cdot, \omega) \in \mathscr{X}$ almost surely (a.s.) $[P]$, where $\mathscr{X}$ is a separable Fréchet space of real functions on T. Without loss of generality it will then be assumed that $X(\cdot, \omega) \in \mathscr{X}$ for all $\omega \in \Omega$. If the map $\Phi: (\Omega, \mathscr{F}) \to (\mathscr{X}, \mathscr{B}(\mathscr{X}))$ defined by

$$\Phi(\omega) = X(\cdot, \omega) \tag{4}$$

is measurable, then the probability measure μ_X induced by $X(t, \omega)$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is defined for all $B \in \mathscr{B}(\mathscr{X})$ by

$$\mu_X(B) = P\{\Phi^{-1}(B)\} = P\{\omega \in \Omega : X(\cdot, \omega) \in B\} \, . \tag{5}$$

Thus question Q_1 is whether the property of being Gaussian carries over from the process $\{X(t, \omega), t \in T\}$ to the measure μ_X. Question Q_2 consists in (i) the existence of a stochastic process $\{X(t, \omega), t \in T\}$ which induces the given measure μ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$, and (ii) whether the property of being Gaussian carries over from μ to $\{X(t, \omega), t \in T\}$.

Related to the measurability of Φ the following should be mentioned. Since for separable Fréchet spaces $\mathscr{B}_c(\mathscr{X}) = \mathscr{B}(\mathscr{X})$, it is clear that the map Φ defined by (4) is measurable if and only if for every $F \in \mathscr{X}^*$, $F \circ \Phi$ is $\mathscr{F}$-measurable (for $\mathscr{X}$ a Banach space see [9]). Thus the measurability of Φ depends on the properties of the process $\{X(t, \omega), t \in T\}$ and of the space $\mathscr{X}$. For the spaces

$C(I)$, $C^n(I)$ and $AC(I)$ the measurability of Φ is proven without additional assumptions on the process, and for the space $L_2(T, \mathscr{A}, \nu)$ the product measurability of $\{X(t, \omega), t \in T\}$ is required. The problem of the product measurability of a stochastic process has been widely studied (see for instance [3] and [4]).

2. Gaussian processes and Gaussian measures on spaces of continuous functions. In this section it is shown that both questions Q_1 and Q_2 have affirmative answers for the following linear spaces of continuous functions: (i) $C(I)$, the separable Fréchet space of real continuous functions on the arbitrary interval I, endowed with the topology of uniform convergence on compact subsets of I ([6] page 205); (ii) $C^n(I)$, the separable Banach space of n-times continuously differentiable real functions on $I = [a, b]$, $-\infty < a < b < +\infty$, with the norm

$$||x|| = \sum_{k=0}^{n} \sup_{a \leq t \leq b} |x^{(k)}(t)| ,$$

where $x^{(0)} = x$ and $x^{(k)}$, $k = 1, 2, \cdots, n$, denotes the kth derivative of x ([5] page 242); (iii) $AC(I)$, the separable Banach space of absolutely continuous real functions on $I = [a, b]$, $-\infty < a < b < +\infty$, with the norm $||x|| = |x(a)| + \int_a^b |x'(t)| \, dt$, where x' is the derivative of x ([5] page 242).

THEOREM 1. *The following are true for $\mathscr{X} = C(I)$ or $C^n(I)$ or $AC(I)$.*

(a) *If $(\Omega, \mathscr{F}, P; X(t, \omega), t \in I)$ is a Gaussian process with paths in $\mathscr{X}$, then the map Φ defined by* (4) *is measurable and the probability $\mu_X = P \circ \Phi^{-1}$ induced on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is Gaussian.*

(b) *If μ is a Gaussian measure on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$, there exists a Gaussian process $(\Omega, \mathscr{F}, P; X(t, \omega), t \in I)$ with paths in $\mathscr{X}$ which induces μ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$.*

PROOF OF (a). Since $\mathscr{B}(\mathscr{X}) = \mathscr{B}_c(\mathscr{X})$, for the measurability of Φ it suffices to show that $F \circ \Phi$ is $\mathscr{F}$-measurable for all $F \in \mathscr{X}^*$. Also, in order to show that $\mu_X = P \circ \Phi^{-1}$ is Gaussian, it suffices to show that for all $F \in \mathscr{X}^*$, F is a Gaussian rv on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu_X)$, or equivalently, because of (5), that $F \circ \Phi$ is a Gaussian rv on $(\Omega, \mathscr{F}, P)$.

It should be noted that if a stochastic process $(\Omega, \mathscr{F}, P; X(t, \omega), t \in I)$ has continuous paths, $X(\cdot, \omega) \in C(I)$ for all $\omega \in \Omega$, then it is product measurable, i.e., $X(t, \omega): (I \times \Omega, \mathscr{B}(I) \times \mathscr{F}) \rightarrow (R, \mathscr{B}(R))$ is measurable (R is the real line). For $I = [a, b]$ this is seen as follows, and the extension to arbitrary intervals I is clear. For every $n = 1, 2, \cdots$ define $t_{k,n} = a + (b - a)(k/n)$, $k = 0, 1, \cdots, n$, and

$$X_n(t, \omega) = \sum_{k=0}^{n-2} \chi_{[t_{k,n}, t_{k+1,n}]}(t) X(t_{k,n}, \omega) + \chi_{[t_{n-1,n}, t_{n,n}]}(t) X(t_{n-1,n}, \omega)$$

for all $\omega \in \Omega$, where χ is the indicator function. It is clear that for all n, $X_n(t, \omega)$ is product measurable, and that $\lim_n X_n(t, \omega) = X(t, \omega)$ for all $t \in I$, $\omega \in \Omega$. It follows that $X(t, \omega)$ is product measurable.

PROOF OF (a) FOR $\mathscr{X} = C(I)$. Let $F \in \mathscr{X}^*$. Then there exists a regular Borel measure λ on I with compact support such that $F(x) = \int_I x(t) \, d\lambda(t)$ for all $x \in \mathscr{X}$

([6] page 203). Let $a, b \in I$ be such that λ assigns zero measure to all Borel subsets of I that are disjoint from $[a, b]$. Then $F(x) = \int_a^b x(t)\, d\lambda(t)$ for all $x \in \mathscr{X}$, and thus $F \in C^*[a, b]$. It follows that there exists a real function g of bounded variation on $[a, b]$ such that $F(x) = \int_a^b x(t)\, dg(t)$ for all $x \in \mathscr{X}$. Since the x's are continuous, we can write

$$F(x) = \lim_{n\to\infty} \sum_{k=1}^n x(t_{k-1})[g(t_{k,n}) - g(t_{k-1,n})] \tag{6}$$

for all $x \in \mathscr{X}$, where $t_{k,n} = a + (b-a)(k/n)$, $k = 0, 1, \cdots, n$. It follows from (4) and (6) that

$$(F \circ \Phi)(\omega) = \lim_{n\to\infty} \sum_{k=1}^n X(t_{k-1,n}, \omega)[g(t_{k,n}) - g(t_{k-1,n})] \tag{7}$$

for all $\omega \in \Omega$. Hence $F \circ \Phi$ is $\mathscr{F}$-measurable and also Gaussian, since the a.s. limit of a sequence of Gaussian rv's is a Gaussian rv.

Proof of (a) for $\mathscr{X} = C^n(I)$. Let $F \in \mathscr{X}^*$. Then there exists a regular Borel measure λ on $[a, b]$ and $\alpha_k \in R$, $k = 0, 1, \cdots, n-1$ such that

$$F(x) = \sum_{k=0}^{n-1} \alpha_k x^{(k)}(a) + \int_a^b x^{(n)}(t)\, d\lambda(t) \tag{8}$$

for all $x \in \mathscr{X}$ ([5] page 344). The proof is completed as for $\mathscr{X} = C(I)$, by noting that the processes $\{X^{(k)}(t, \omega), t \in I\}_{k=0}^n$ are jointly Gaussian, where $X^{(k)}(t, \omega)$ is the kth derivative of the ω-path of $X(t, \omega)$.

Proof of (a) for $\mathscr{X} = AC(I)$. Let $F \in \mathscr{X}^*$. Then there exists a $f \in L_\infty(I)$ and $\alpha \in R$ such that

$$F(x) = \alpha x(a) + \int_a^b x'(t) f(t)\, dt \tag{9}$$

for all $x \in \mathscr{X}$ ([5] page 343). It follows from (4) and (9) that

$$(F \circ \Phi)(\omega) = \alpha X(a, \omega) + \int_a^b X'(t, \omega) f(t)\, dt\,. \tag{10}$$

Since $\{X(t, \omega), t \in [a, b]\}$ is a product measurable Gaussian process with paths in $\mathscr{X}$, it is easily seen that $\{X'(t, \omega), t \in [a, b]\}$ is a product measurable Gaussian process and X, X' are jointly Gaussian. It follows by (10) that $F \circ \Phi$ is $\mathscr{F}$-measurable, and by (10) and Theorem 2.8 of [3] (page 64) that $F \circ \Phi$ is Gaussian.

Proof of (b). For $\mathscr{X} = C(I)$ and $\mathscr{X} = C^n(I)$, take $(\Omega, \mathscr{F}, P) \equiv (\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu)$ and $X(t, \omega) = \omega(t)$. The result is obvious for $\mathscr{X} = C(I)$, and also for $\mathscr{X} = C^n(I)$ if we note that $\delta_t(x) = x(t)$ belongs to $\mathscr{X}^*$ for all $t \in I$.

For $\mathscr{X} = AC(I)$, take $(\Omega, \mathscr{F}, P) \equiv (\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu)$ and

$$X(t, \omega) = \omega(a) + \int_a^t \omega'(s)\, ds \tag{11}$$

for all $t \in I = [a, b]$ and $\omega \in \Omega = \mathscr{X}$. Then clearly $X(\cdot, \omega) \in \mathscr{X}$ for all $\omega \in \Omega$ and by (a) it induces a Gaussian measure $\mu_X = P \circ \Phi^{-1}$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$. Since $\mathscr{B}(\mathscr{X}) = \mathscr{B}_c(\mathscr{X})$, in order to show that $\mu_X = \mu$, it suffices to prove $\mu_X(B) = \mu(B)$ for all cylinder sets B, i.e., for all sets of the form

$$B = \{x \in \mathscr{X} : (F_1(x), \cdots, F_n(x)) \in B_n\} \tag{12}$$

where $F_1, \cdots, F_n \in \mathscr{X}^*$ and B_n is an n-dimensional Borel set. By (5), $\mu_X(B) = \mu\{\Phi^{-1}(B)\}$ where

$$\Phi^{-1}(B) = \{x \in \mathscr{X}: ((F_1 \circ \Phi)(x), \cdots, (F_n \circ \Phi)(x)) \in B_n\} . \tag{13}$$

For $k = 1, \cdots, n$, let $F_k(x) = \alpha_k x(a) + \int_a^b x'(t) f_k(t)\, dt$ for all $x \in \mathscr{X}$. Then it follows by (11) that $(F_k \circ \Phi)(x) = \alpha_k x(a) + \int_a^b x'(t) f_k(t)\, dt = F_k(x)$. Hence, by (12) and (13), $\Phi^{-1}(B) = B$ and thus $\mu_X(B) = \mu\{\Phi^{-1}(B)\} = \mu(B)$, which completes the proof.

REMARK 1. Let T be any index set and $\mathscr{X}$ a linear topological space of real functions on T. Then a careful inspection of the proof of Theorem 1 for $C(I)$ and $C^n(I)$ reveals that questions Q_1 and Q_2 have affirmative answers and for Q_2 one can take $(\Omega, \mathscr{F}, P) \equiv (\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu)$ and $X(t, \omega) = \omega(t)$, if the following sufficient condition is met:

For every $t \in T$ the evaluation map $\delta_t \in \mathscr{X}^*$ and the linear span of $\{\delta_t, t \in T\}$ is weak* sequentially dense in $\mathscr{X}^*$.

3. Gaussian processes and Gaussian measures on L_2. In this section we consider questions Q_1 and Q_2 for Hilbert spaces of real valued square integrable functions. Let T be an arbitrary index set, $\mathscr{A}$ a σ-algebra of subsets of T and ν a non-negative, σ-finite measure on $\mathscr{A}$. It will be assumed throughout this section that $\mathscr{X} = L_2(T, \mathscr{A}, \nu)$ is separable. A sufficient condition for this is that $\mathscr{A}$ has a countable set of generators ([8] pages 168, 177). This formulation includes the following cases: (i) T is a measurable subset of the real line, $\mathscr{A}$ the Borel subsets of T and ν the Lebesgue measure; (ii) T is the set of integers, $\mathscr{A}$ the set of all subsets of T and ν the counting measure.

THEOREM 2. (a) *Let $(\Omega, \mathscr{F}, P; X(t, \omega), t \in T)$ be a $\mathscr{A} \times \mathscr{F}$-measurable Gaussian process with mean $m(t)$, autocorrelation $r(t, s)$, covariance $R(t, s)$, and with paths in $\mathscr{X} = L_2(T, \mathscr{A}, \nu)$. Then the map Φ defined by* (4) *is measurable and the probability $\mu_X = P \circ \Phi^{-1}$ induced on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is Gaussian with mean m and covariance the integral type operator with kernel $R(t, s)$.*

(b) *Let μ be a Gaussian measure on $(\mathscr{X} = L_2(T, \mathscr{A}, \nu), \mathscr{B}(\mathscr{X}))$. Then there exists a $\mathscr{A} \times \mathscr{F}$-measurable Gaussian process $(\Omega, \mathscr{F}, P; X(t, \omega), t \in T)$ with paths in $\mathscr{X}$ which induces μ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$.*

COROLLARY 1. *Let $(\Omega, \mathscr{F}, P; X(t, \omega), t \in T)$ be a $\mathscr{A} \times \mathscr{F}$-measurable Gaussian process. Then*

$$\int_T X^2(t, \omega)\, d\nu(t) < +\infty \quad \text{a.s. } [P] \tag{14}$$

if and only if

$$\int_T r(t, t)\, d\nu(t) < +\infty . \tag{15}$$

The "only if" part is shown in the proof of Theorem 2 (a) and the "if" part is an application of Fubini's theorem.

Thus, for a product measurable Gaussian process condition (15) determines whether or not $X(t, \omega)$ induces a Gaussian measure on $L_2(T, \mathscr{A}, \nu)$. For instance, if $T = R$, the real line, $\mathscr{A} = \mathscr{B}(R)$, and if $\{X(t, \omega), t \in R\}$ is stationary, then $X(t, \omega)$ does not induce a measure on $L_2(R, \mathscr{B}(R), \text{Leb})$, but it induces a Gaussian measure on $L_2(R, \mathscr{B}(R), \nu)$ for every finite measure ν. Also, if $T = [0, +\infty) = R^+$, $\mathscr{A} = \mathscr{B}(R^+)$, and if $\{X(t, \omega), t \in R^+\}$ is the Wiener process, then $r(t, t) = t$ and even though $X(t, \omega)$ does not induce a measure on $L_2(R^+, \mathscr{B}(R^+), \text{Leb})$, it induces a Gaussian measure for example on $L_2(R^+, \mathscr{B}(R^+), \nu)$, where ν is defined by $[d\nu/d\,\text{Leb}](t) = e^{-t}$. It is easily seen that if T is a Borel subset of the real line and $\mathscr{A}$ the σ-algebra of Borel subsets of T, then there always exist measures ν on $(T, \mathscr{A})$ such that (15) is satisfied. One such measure is obtained as follows. Define ν_0 on $(T, \mathscr{A})$ by $[d\nu_0/d\,\text{Leb}](t) = f(t)g(t)$, where $g(t) > 0$, $g \in L_1(T, \mathscr{A}, \text{Leb})$, $f(t) = r(t, t)$ for $0 \leqq r(t, t) < 1$, and $f(t) = r^{-1}(t, t)$ for $1 \leqq r(t, t)$. Clearly ν_0 satisfies (10) and is also a finite measure.

It follows from Corollary 1 that if $\{X(\text{t}, \omega), t \in T\}$ is $\mathscr{A} \times \mathscr{F}$-measurable and Gaussian and if $f\colon (T, \mathscr{A}) \to (R, \mathscr{B}(R))$ is measurable, then $f(t)X^2(t, \omega) \in L_1(T, \mathscr{A}, \nu)$ a.s. $[P]$ if and only if $f(t)r(t, t) \in L_1(T, \mathscr{A}, \nu)$ (apply Corollary 1 to the process $Y(t, \omega) = |f(t)|^{\frac{1}{2}}X(t, \omega))$. This result is known in the particular cases where T is a compact interval of the real line, ν the Lebesgue measure, and $X(t, \omega)$ the Wiener process [16] or a mean square continuous Gaussian process [17]. In fact in [16] and [17] a zero-one law is proven for this property; a similar zero-one law in our more general setting is proven in [14].

PROOF OF THEOREM 2 (a). It should be noted at the outset that since $X(t, \omega)$ is $\mathscr{A} \times \mathscr{F}$-measurable, it follows that $m(t)$ is $\mathscr{A}$-measurable and $r(t, s)$, $R(t, s)$ are $\mathscr{A} \times \mathscr{A}$-measurable.

Because of the separability of $\mathscr{X}$, an application of Fubini's theorem shows that the map Φ defined by (4) is measurable and thus it induces the probability $\mu_X = P \circ \Phi^{-1}$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$.

In order to prove that μ_X is Gaussian it suffices to show that for every fixed $f \in \mathscr{X}$, $F(u) = \langle u, f\rangle$, $u \in \mathscr{X}$, is a Gaussian rv on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu_X)$, or equivalently, because of (5), that $\xi(\omega) = \int_T X(t, \omega)f(t)\,d\nu(t)$ is a Gaussian rv on $(\Omega, \mathscr{F}, P)$.

It also suffices to prove that $\xi(\omega)$ is Gaussian for $\nu(T) < +\infty$. This is seen as follows. By the σ-finiteness of ν there exists an increasing sequence $\{T^{(n)}\}_{n=1}^{\infty}$ of sets in $\mathscr{A}$ such that $\nu(T^{(n)}) < +\infty$, all n, and $\bigcup_{n=1}^{\infty} T^{(n)} = T$. Then

$$\xi^{(n)}(\omega) = \int_{T^{(n)}} X(t, \omega)f(t)\,d\nu(t) \to_{n\to\infty} \xi(\omega)$$

on Ω and if $\xi^{(n)}$ is Gaussian for all n, ξ is also Gaussian.

Now for $k = 1, 2, \cdots$, define $T_k = \{t \in T\colon r(t, t) \leqq k\}$, $X_k(t, \omega) = \chi_{T_k}(t)X(t, \omega)$ and $\xi_k(\omega) = \int_T X_k(t, \omega)f(t)\,d\nu(t)$. This idea of truncating the process X was suggested to us by Dr. L. A. Shepp. Clearly $T_k \uparrow T$ as $k \to \infty$ and this implies that $\xi_k(\omega) \to_{k\to\infty} \xi(\omega)$ on Ω. Thus it suffices to prove that ξ_k is Gaussian for all k.

We now fix k, assume $\nu(T) < +\infty$, and will show that ξ_k is Gaussian. Clearly

$\{X_k(t, \omega), t \in T\}$ is a Gaussian process and its autocorrelation $r_k(t, s)$ satisfies

$$\int_T r_k(t, t)\, d\nu(t) < +\infty . \tag{16}$$

Then $r_k \in L_2(T \times T, \mathscr{A} \times \mathscr{A}, \nu \times \nu)$ and by Fubini's theorem

$$E[\xi_k^2] = \int_T \int_T r_k(t, s) f(t) f(s)\, d\nu(t)\, d\nu(s) < +\infty .$$

Let H_k be the closure in $L_2(\Omega, \mathscr{F}, P)$ of the linear manifold generated by the random variables $X_k(t, \omega)$, $t \in T$. Since $\xi_k \in L_2(\Omega, \mathscr{F}, P)$ we can write $\xi_k = \zeta + \eta$ where $\zeta \in H_k$ and $\eta \perp H_k$. Then ζ is of the form $\zeta = \lim_n \zeta_n$ in $L_2(\Omega, \mathscr{F}, P)$, where $\zeta_n(\omega) = \sum_{i=1}^{N_n} a_{n,i} X(t_{n,i}, \omega)$, $t_{n,i} \in T$, $a_{n,i} \in R$. For all $t \in T$ we have $E[X_k(t)\xi_k] = E[X_k(t)\zeta] \equiv g(t)$ and also, by the definition of ξ_k, $g(t) = \int_T r_k(t, s) f(s)\, d\nu(s)$ and $g(t) = \lim_n E[X_k(t)\zeta_n] = \lim_n g_n(t)$, where $g_n(t) = \sum_{i=1}^{N_n} a_{n,i} r_k(t, t_{n,i})$. Now $|g(t)|^2 \leqq E[\zeta^2] r_k(t, t)$, $t \in T$, and (16) imply that $g \in \mathscr{H}$, and similarly $g_n \in \mathscr{H}$ for all n. Also $|g_n(t) - g(t)|^2 \leqq E[(\zeta_n - \zeta)^2] r_k(t, t)$, $t \in T$, along with $\lim_n E[(\zeta_n - \zeta)^2] = 0$, (16) and the bounded convergence theorem imply that $\lim_n g_n = g$ in $\mathscr{H}$. It follows from these results that

$$\begin{aligned} E[\zeta^2] &= \lim_n \lim_m E[\zeta_n \zeta_m] \\ &= \lim_n \lim_m \textstyle\sum_{i=1}^{N_n} \sum_{j=1}^{N_m} a_{n,i} a_{m,j} r_k(t_{n,i}, t_{m,j}) \\ &= \lim_n \textstyle\sum_{i=1}^{N_n} a_{n,i} \lim_m g_m(t_{n,i}) = \lim_n \sum_{i=1}^{N_n} a_{n,i} g(t_{n,i}) \\ &= \lim_n \textstyle\int_T g_n(s) f(s)\, d\nu(s) = \int_T g(s) f(s)\, d\nu(s) \\ &= \textstyle\int_T \int_T r_k(t, s) f(t) f(s)\, d\nu(t)\, d\nu(s) = E[\xi_k^2] . \end{aligned}$$

Hence $\xi_k = \zeta \in H_k$ and thus ξ_k is Gaussian, since all random variables in H_k are Gaussian.

Hence μ_X is Gaussian and by (3), (5) and Fubini's theorem we have

$$+\infty > \int_{\mathscr{H}} \|u\|^2\, d\mu_X(u) = \int_\Omega \left(\int_T X^2(t, \omega)\, d\nu(t)\right) dP(\omega) = \int_T r(t, t)\, d\nu(t) . \tag{17}$$

Since $r(t, t) = m^2(t) + R(t, t)$, it follows that $m \in \mathscr{H}$ and that the kernel $R(t, s)$ defines a trace class operator on $\mathscr{H}$. By using (1), (2) and (5) it is easily seen that the mean element of μ_X is m and that its covariance operator is the integral type operator with kernel $R(t, s)$.

REMARK 2. With respect to the proof of Theorem 2(a) it should be noted that an alternative way of showing that $\xi_k \in H_k$ is by using the quadratic mean integral introduced in [11]. Also in the particular case where T is an interval on the real line, $\mathscr{A}$ is the Borel subset of T and ν is the Lebesgue measure, that ξ is Gaussian follows from Theorem 2.8 of [3] (page 64). However, this theorem of Doob is clearly not applicable in our general setting. A careful inspection of the proof of Theorem 2(a) reveals that an appropriate generalization of Theorem 2.8 of [3] to the present more general set up is provided therein.

PROOF OF THEOREM 2(b). Let u_0 and S be the mean and the covariance operator of μ, and let $\{\phi_n\}_{n=1}^\infty$ be a complete set of eigenfunctions of S with corresponding eigenvalues $\{\lambda_n\}_{n=1}^\infty$ (some of which may be equal to zero). There exists a set

$T_1 \in \mathscr{A}$ with $\nu(T_1) = 0$ such that on T_1^c, $\phi_n(t)$ is defined and finite valued for all n. Define for all n

$$\begin{aligned} \psi_n(t, u) &= \langle \phi_n, u \rangle \phi_n(t) &\quad \text{on} \quad T_1^c \times \mathscr{X} \\ &= 0 &\quad \text{on} \quad T_1 \times \mathscr{X} . \end{aligned}$$

Then clearly $\{\psi_n(t, u), t \in T\}_{n=1}^\infty$ are jointly Gaussian processes on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu)$. Also it is easily seen from (2) that for all n, $\psi_n \in L_2(T \times \mathscr{X}, \mathscr{A} \times \mathscr{B}(\mathscr{X}), \nu \times \mu) = L_2(T \times \mathscr{X})$ and that for all n and m

$$\int_T \int_{\mathscr{X}} \psi_n(t, u)\psi_m(t, u)\, d\mu(u)\, d\nu(t) = \delta_{n,m}(\lambda_n + \langle u_0, \phi_n \rangle^2) .$$

Hence $\{\psi_n\}_{n=1}^\infty$ are orthogonal in $L_2(T \times \mathscr{X})$ and by (3) and $\operatorname{tr}(S) = \sum_{n=1}^\infty \lambda_n$ we have

$$\sum_{n=1}^\infty \int_T \int_{\mathscr{X}} \psi_n^2(t, u)\, d\mu(u)\, d\nu(t) = \|u_0\|^2 + \operatorname{tr}(S) < +\infty .$$

It follows that the series $\sum_{n=1}^\infty \psi_n(t, u)$ converges in $L_2(T \times \mathscr{X})$ to say $Y(t, u) \in L_2(T \times \mathscr{X})$. Hence for a subsequence $\lim_k \sum_{n=1}^{N_k} \psi_n(t, u) = Y(t, u)$ a.e. $[\nu \times \mu]$. Let $E = \{(t, u) \in T \times \mathscr{X}: \sum_{n=1}^{N_k} \psi_n(t, u)$ does not converge to $Y(t, u)\}$. Then $(\nu \times \mu)(E) = 0$ and it follows by Fubini's theorem that there exist sets (i) $T_2 \in \mathscr{A}$ with $\nu(T_2) = 0$ such that for all $t \in T_2^c$ and $\omega \in E_t^c \in \mathscr{B}(\mathscr{X})$, where $\mu(E_t) = 0$, we have $\lim_k \sum_{n=1}^{N_k} \psi_n(t, u) = Y(t, u)$, and (ii) $N_0 \in \mathscr{B}(\mathscr{X})$ with $\mu(N_0) = 0$ such that for all $u \in N_0^c$ and $t \in E_u^c \in \mathscr{A}$, where $\nu(E_u) = 0$, we have $\lim_k \sum_{n=1}^{N_k} \psi_n(t, u) = Y(t, u)$. If we let $T_0 = T_1 \cup T_2$, then $T_0 \in \mathscr{A}$ and $\nu(T_0) = 0$. Define $X(t, u)$ on $T \times \mathscr{X}$ by

$$\begin{aligned} X(t, u) &= Y(t, u) &\quad \text{on} \quad \{T_0^c \times \mathscr{X}) \cup (T \times N_0^c)\} \cap E^c \\ &= 0 &\quad \text{elsewhere} . \end{aligned} \tag{18}$$

Then clearly $\{X(t, u), t \in T\}$ is a $\mathscr{A} \times \mathscr{F}$-measurable process on the probability space $(\Omega, \mathscr{F}, P) \equiv (\mathscr{X}, \mathscr{B}(\mathscr{X}), \mu)$. Since by definition for $t \in T_0^c$, $X(t, u) = \lim_k \sum_{n=1}^{N_k} \psi_n(t, u)$ a.e. $[\mu]$, and since $\{\psi_n(t, u), t \in T\}_{n=1}^\infty$ are jointly Gaussian, it follows that $\{X(t, u), t \in T\}$ is Gaussian. Since by (18), $X(t, u) = Y(t, u)$ a.e. $[\nu \times \mu]$ and $Y \in L_2(T \times \mathscr{X})$, it follows that $X \in L_2(T \times \mathscr{X})$ and thus

$$+\infty > \int_T \int_{\mathscr{X}} X^2(t, u)\, d\mu(u)\, d\nu(t) = \int_T r(t, t)\, d\nu(t) ,$$

i.e., (15) is satisfied. Hence $\{X(t, u), t \in T\}$ satisfies the assumptions of Theorem 2(a) and thus it induces a Gaussian measure μ_X on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$. It will be shown that $\mu_X = \mu$. For this it suffices to prove that $\mu_X(B) = \mu(B)$ for all cylinder sets B. Let

$$B = \{u \in \mathscr{X}: (\langle u, f_1 \rangle, \cdots, \langle u, f_n \rangle) \in B_n\}$$

where $f_1, \cdots, f_n \in \mathscr{X}$ and B_n is an n-dimensional Borel set. Then by (5), $\mu_X(B) = \mu\{\Phi^{-1}(B)\}$, where

$$\Phi^{-1}(B) = \{u \in \mathscr{X}: (\int_T X(t, u) f_1(t)\, d\nu(t), \cdots, \int_T X(t, u) f_n(t)\, d\nu(t)) \in B_n\} .$$

Note that for $u \in N_0^c$, $\mu(N_0) = 0$, $X(t, u) = \lim_k \sum_{n=1}^{N_k} \psi_n(t, u)$ a.e. $[\nu]$ and thus $X(t, u) = \lim_k \sum_{n=1}^{N_k} \langle \phi_n, u \rangle \phi_n(t)$ a.e. $[\nu]$. Also, since $\{\phi_n\}_{n=1}^\infty$ is a complete orthonormal set in $\mathscr{X}$, we have that for all $u \in \mathscr{X}$, $\sum_{n=1}^{\infty} \langle \phi_n, u \rangle \phi_n = u$ in $\mathscr{X}$. It follows that for $u \in N_0^c$, $\mu(N_0) = 0$, we have $X(t, u) = u(t)$ a.e. $[\nu]$. Hence $\int_T X(t, u) f_k(t)\, d\nu(t) = \int_T u(t) f_k(t)\, d\nu(t)$ a.e. $[\mu]$ for $k = 1, \cdots, n$ and thus $\mu_X(B) = \mu\{\Phi^{-1}(B)\} = \mu(B)$, which completes the proof.

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