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AN EQUIVALENT TO THE MARTINGALE SQUARE FUNCTION INEQUALITY[1]

BY LOUIS GORDON

Stanford University

A direct proof is given for an inequality relating the expected absolute value of stopped Brownian motion to the expected time to stopping. This inequality was originally proved by means of the martingale square function inequality. The latter is then derived from the former through use of a Skorokhod embedding. The first inequality is also applied to prove a martingale strong law of large numbers.

1. Introduction and summary. The square function inequality for independent random variables with zero means,

$$m(p)E(\sum_1^n X_j^2)^p \leqq E(\sum_1^n X_j)^{2p} \leqq M(p)E(\sum_1^n X_j^2)^p , \qquad \text{for} \quad p > \tfrac{1}{2}$$

is proved in Marcinkiewicz and Zygmund (1938). The result is extended to martingale difference sequences in Burkholder (1966). Burkholder's result is used in Millar (1968) to study the sample function behavior of continuous-time martingales. A corollary to Millar's work is found in Burkholder and Gundy (1970), which extends the results of Burkholder. There, it is shown that for $W(t)$ a standard Wiener process and T a bounded stopping time,

$$k(p)ET^p \leqq E|W(T)|^{2p} \leqq K(p)ET^p \tag{1}$$

for $p > \frac{1}{2}$ and constants $k(p)$ and $K(p)$.

In Section 2, we provide a conceptually simple direct proof of (1) by means of the Itô calculus for Brownian motion. The left-hand inequality is somewhat improved in that we only require that a moment of T of sufficiently large order be finite.

Section 3 is devoted to a proof of the martingale square function inequality using the Skorokhod embedding for martingales and equation (1). It follows that (1) and the square function inequality are equivalent.

Chow (1967) employs the square function inequality to prove a strong law of large numbers for martingales. An alternative proof using (1) is found in Section 4. The similar use of the two inequalities here and in Chow (1960) first suggested the possibility of the equivalence shown in Section 3. We also provide a partial generalization to martingales of a strong law due to Chung (1947).

An esthetic justification for new proofs of old results is perhaps in order. Inequality (1) may be of sufficient interest in itself to deserve a direct proof. A

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debit and credit for the above program is its use of purely probabilistic methods; the origins of and insights into the square function inequality gained via orthogonal series and analysis are thereby obscured. Neither does the proof yield maximal inequalities as in Burkholder and Gundy (1970). Hopefully, the combination of the Skorokhod embedding and Itô calculus to obtain information about functions of martingales may also be of interest.

2. Inequalities for Brownian motion. We here present sufficient conditions on a stopping time T for the standard Wiener process W so that there exist constants $k(p)$ and $K(p)$ independent of the choice of T for which $k(p)ET^p \leqq E|W(T)|^{2p} \leqq K(p)ET^p$, for $p > \frac{1}{2}$. The proof involves use of a result from the Itô calculus (e.g., see McKean (1969)).

We first pause to give a heuristic discussion of the necessary results. We may write $W(t + dt) = W(t) + dW(t)$ where the Brownian differential sticks out into the future and so is independent of the past. The differential has mean zero and we approximate $(dW(t))^2$ by dt because of the law of the iterated logarithm and Lévy's theorem on the almost sure behavior of sums of the squared differences of the Wiener process for increasingly fine partitions of time (e.g., see Freedman (1971), page 42). Hence, if $f(t, w)$ is sufficiently smooth,

$$E\,df(t, W(t)) = E\{f_1(t, W(t)) + \tfrac{1}{2}f_{22}(t, W(t))\}\,dt\,,$$

where $f_1 = \partial f/\partial t$ and $f_{22} = \partial^2 f/\partial w^2$. If we can add the differentials we obtain $Ef(t, W(t)) = E\int_0^t f_1(s, W(s)) + \frac{1}{2}f_{22}(s, W(s))\,ds$. Further, the same argument applies to differentials $dW(T + t)$ for T a stopping time by the strong Markov property. We therefore expect that

$$Ef(T, W(T)) = E\int_0^T f_1(t, W(t)) + \tfrac{1}{2}f_{22}(t, W(t))\,dt.$$

This argument is made rigorous by means of the Itô integral discussed in McKean (1969). In particular, for T a stopping time, $f(0, 0) = 0$, and f well-behaved,

$$f(T, W(T)) = \int_0^T f_2(t, W(t))\,dW(t) + \int_0^T f_1(t, W(t)) + \tfrac{1}{2}f_{22}(t, W(t))\,dt$$

where $f_2 = \partial f/\partial w$ and the right integral is an Itô integral. Further, if $E\int_0^T f_2^2(t, W(t))\,dt < \infty$ then $E\int_0^T f_2(t, W(t))\,dW(t) = 0$. We refer to these two identities as Itô's formula.

THEOREM 1. *For $p > \frac{1}{2}$ there exist constants $k(p)$ and $K(p)$ for which $E|W(T)|^{2p} \leqq K(p)ET^p$, for T a stopping time. If, in addition, $ET^p < \infty$ then $k(p)ET^p \leqq E|W(T)|^{2p}$.*

PROOF. We write W^r instead of $|W|^r$ and apply Itô's formula to functions of the form $t^\gamma W^{2p-2\gamma}$. The reader should not worry that the functions be non-differentiable at 0. We may approximate by $t^\gamma(\varepsilon + W^2)^{p-\gamma}$ or observe that W spends 0 time at 0 on a closed subset of time. We further note that

$$E\int_0^T (t^\gamma W^{2p-2\gamma})^2\,dt < \infty$$

when $T < B < \infty$, for B a constant. To obtain the bounds, we may truncate the time T and prove the result. The upper bound then follows immediately from Fatou's lemma. We therefore make free use of Itô's formula.

First, we prove the inequalities in the following table and then apply Hölder's inequality. We write $p = 1 + \alpha$.

To prove:

TABLE 1

	$k(p)ET^{1+\alpha} \leqq EW^{2+2\alpha}(T)$	$K(p)ET^{1+\alpha} \geqq EW^{2+2\alpha}(T)$
$\alpha > \frac{1}{2}$	$(1+\alpha)ET^{\alpha}W^2(T) \geqq ET^{1+\alpha}$	$\binom{2+2\alpha}{2}ETW^{2\alpha}(T) \geqq EW^{2+2\alpha}(T)$
$\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$	$(1+\alpha)ET^{\alpha}W^2(T) \geqq ET^{1+\alpha}$	$\binom{2+2\alpha}{2}ET^{1-n\alpha}W^{2(n+1)\alpha}(T) \geqq EW^{2+2\alpha}(T)$
$\alpha \in (-\frac{1}{2}, 0)$	$ET^{\gamma}W^{2+2\alpha-2\gamma}(T) \geqq \frac{1}{2}\gamma ET^{1+\alpha}$, $\gamma = (1+2\alpha)/4$	$(1+\alpha)ET^{\alpha}W^2(T) \leqq ET^{1+\alpha}$

The inequalities are proved in order. Note that

$$EW^{2+2\alpha}(T) = \binom{2+2\alpha}{2}E\int_0^T W^{2\alpha}(t)\,dt$$

by Itô's formula where $\binom{a}{2} = \frac{1}{2}a(a-1)$. For example, $EW^2(T) = ET$.

(1) $(1+\alpha)ET^{\alpha}W^2(T) = (1+\alpha)E\int_0^T t^{\alpha} + \alpha t^{\alpha-1}W^2(t)\,dt \geqq (1+\alpha)E\int_0^T t^{\alpha}\,dt$ for $\alpha > 0$.

(2) $\binom{2+2\alpha}{2}ETW^{2\alpha}(T) = \binom{2+2\alpha}{2}E\int_0^T W^{2\alpha}(t) + \binom{2\alpha}{2}tW^{2\alpha-2}(t)\,dt \geqq \binom{2+2\alpha}{2}E\int_0^T W^{2\alpha}(t)\,dt$ for $\alpha > \frac{1}{2}$.

(3) $\binom{2+2\alpha}{2}ET^{1-n\alpha}W^{2(n+1)\alpha}(T) = \binom{2+2\alpha}{2}E\int_0^T W^{2\alpha}[(1-n\alpha)(W^2/t)^{n\alpha} + (W^2/t)^{n\alpha-1}]\,dt$ for $\alpha \in [1/(n+1), 1/n)$. But $(1-n\alpha)x^{n\alpha} + x^{n\alpha-1} \geqq 1$ for $x \in (0, \infty)$.

(4) Choose $\alpha \in (-\frac{1}{2}, 0)$ and consider $\gamma = (1+2\alpha)/4$. Then

$$ET^{\gamma}W^{2+2\alpha-2\gamma}(T) \geqq E\int_0^T t^{\alpha}[\gamma(t/W^2)^{\gamma-\alpha-1} + \tfrac{1}{2}(1+2\alpha-2\gamma)(t/W^2)^{\gamma-\alpha}]\,dt\,.$$

Also, $\gamma x^{\gamma-\alpha-1} + \frac{1}{2}(1+2\alpha-2\gamma)x^{\gamma-\alpha} \geqq \gamma$. So

$$ET^{\gamma}W^{2+2\alpha-2\gamma}(T) \geqq \gamma E\int_0^T t^{\alpha}\,dt\,.$$

(5) $(1+\alpha)ET^{\alpha}W^2(T) = (1+\alpha)E\int_0^T \alpha t^{\alpha-1}W^2(t) + t^{\alpha}\,dt \leqq (1+\alpha)E\int_0^T t^{\alpha}\,dt$ for $\alpha \in (-\frac{1}{2}, 0)$.

We now apply Hölder's inequality to the previous table. For example, if $\alpha > \frac{1}{2}$, $EW^{2+2\alpha}(T) \leqq \binom{2+2\alpha}{2}ETW^{2\alpha}(T)$ so that

$$\binom{2+2\alpha}{2}(ET^{1+\alpha})^{1/(1+\alpha)} \geqq (EW^{2+2\alpha}(T))^{1/(1+\alpha)}\,.$$

We exhibit the following table of constants:

TABLE 2

	$\alpha \geqq \frac{1}{2}$	$\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$	$\alpha \in (-\frac{1}{2}, 0)$
$k(1+\alpha)$	$(1+\alpha)^{-(1+\alpha)}$	$(1+\alpha)^{1-1/\alpha}$	$(\frac{1}{2}\gamma)^{(1+\alpha)/\gamma}$, $\gamma = (1+2\alpha)/4$
$K(1+\alpha)$	$\binom{2+2\alpha}{2}^{1+\alpha}$	$\binom{2+2\alpha}{2}^{(n+1)\alpha/(1+\alpha)}$	$(1+\alpha)^{-(1+\alpha)}$

for which $k(p)ET^p \leqq EW^{2p}(T) \leqq K(p)ET^p$ when T is bounded.

We now have that the right inequality holds for all stopping times T while the left holds for all truncated times $T \wedge B$ for B a finite constant. Let

$$W^*(T \wedge B) = \sup_t |W(T \wedge B \wedge t)|$$

and assume $ET^p < \infty$. Then $W(T \wedge B \wedge t)$ is a martingale with continuous sample paths so that by the martingale maximum inequality and Doob (1953), page 317,

$$E[W^*(T \wedge B)]^{2p} \leqq C_p' EW^{2p}(T \wedge B).$$

We now apply the right-hand inequality of (1) to obtain

$$E[W^*(T \wedge B)]^{2p} \leqq C_p E(T \wedge B)^p .$$

By monotone convergence we obtain

$$E[W^*(T)]^{2p} \leqq C_p ET^p < \infty .$$

Since $W(T \wedge B) \to W(T)$ and $|W(T \wedge B)| \leqq W^*(T)$ it follows from the dominated convergence theorem that

$$k(p)ET^p \leqq EW^{2p}(T) .$$

The reader may also note that the same proof we use for the right-hand inequality in (1) yields

$$EW^{2p}(T) \leqq p^{-p}ET^p$$

for $0 < p < 1$.

3. The square function inequality. We next show that the preceding inequality implies the square function inequality. The reader should observe that Khintchine's inequality is used here and in Burkholder (1966). We use Theorem 1 to give a partial proof of this inequality.

LEMMA 2. (Khinchine's Inequality). *Let Q_j be* i.i.d. *random variables taking values* 1 *or* -1 *with probabilities* $\frac{1}{2}$ *each. For* $p > \frac{1}{2}$, *there exist positive constants* $b(p)$ *and* $B(p)$ *for which*

$$b(p)(\sum x_j^2)^p \leqq E(\sum Q_j x_j)^{2p} \leqq B(p)(\sum x_j^2)^p ,$$

for arbitrary constants x_j.

PROOF. Let $W(t)$ be a standard Wiener process and $R_n = \sum_0^n T_j$ where $T_0 = 0$,

$$T_n = \inf\{t \,|\, |W(t + R_{n-1}) - W(R_{n-1})| = |x_n|\} .$$

By the usual scale change transformation, $T_n = x_n^2 T_n^*$ where the T_n^* are i.i.d. as the hitting time of $|W|$ to 1. We use Jensen's inequality and Theorem 1 to conclude that

$$K_1(\sum x_j^2 ET_1^*)^p \leqq E(\sum x_j Q_j)^{2p} \leqq K_2(\sum x_j^2)^p E(T_1^*)^p .$$

Where $K_1 = k(p)$, $K_2 = K(p)$ if $p \geqq 1$ and $K_1 = K(p)$, $K_2 = k(p)$ if $p < 1$.

The Skorokhod representation for sums of independent variates with zero mean is well known. The representation can be easily generalized by a conditional argument to martingale difference sequences (e.g., see Freedman (1971), page 90). For our purposes the approach in Breiman (1967) is most conveniently generalized.

In particular, we choose to represent every martingale $\hat{S}_n$ as a Wiener process stopped at times R_n, with $S_n = W(R_n)$, where $R_0 = 0$, $R_n = T_n + R_{n-1}$ and $T_n = T_n(U, V)$, the first hitting time of $W(t + R_{n-1}) - W(R_{n-1})$ to U or V. Here (U, V) is chosen independent of the post-R_{n-1} process according to

$$P\{U \in du,\ V \in dv\} = \alpha|v| I_{\{uv<0\} \cup \{u=0, v<0\}} F(du \mid S_1, \cdots, S_{n-1}) F(dv \mid S_1, \cdots, S_{n-1})$$

where F is the distribution of $\hat{S}_n - \hat{S}_{n-1} = \hat{X}_n$, conditional on $\hat{S}_1, \cdots, \hat{S}_{n-1}$ and $\alpha^{-1} = E\{|\hat{X}_n| \mid S_1, \cdots, S_{n-1}\}$ whenever the latter is non-zero; we concentrate (U, V) on (0, 0) otherwise. The X_j constitute a martingale difference sequence.

We now give a proof of the square function inequality using the Skorokhod representation and Theorem 1. The crucial observation is that if R_n is an increasing sequence of Brownian stopping times, then for $X_n = W(R_n) - W(R_{n-1})$, $k(p)ER_n^{\,p} \leqq E(\sum_1^n d_j X_j)^{2p}$ for any sequence d_j of 1's and -1's. This enables us to combine Theorem 1 with Khintchine's inequality to obtain a proof of the square function inequality.

THEOREM 3. (Square function inequality). *Let X_j be a martingale difference sequence. Then if $p > \frac{1}{2}$ there exist constants $m(p)$ and $M(p)$ for which*

$$m(p)E(\textstyle\sum_1^n X_j^2)^p \leqq E(\sum_1^n X_j)^{2p} \leqq M(p)E(\sum_1^n X_j^2)^p .$$

PROOF. We assume in our proof that all stopping times have finite moments of all orders. The general theorem follows from this specialization by truncation, and a limiting argument. Either the X_j themselves may be truncated or one may truncate the stopping times in the Skorokhod representation at the first exit time from $[-N, N]$ for large N.

Let d_k be an arbitrary sequence of 1's and -1's, and R_n an increasing sequence of stopping times with finite moments of large order. From Theorem 1, $k(p)ER_n^{\,p} \leqq E(\sum_1^n d_j X_j)^{2p} \leqq K(p)ER_n^{\,p}$ for $X_j = W(R_j) - W(R_{j-1})$, $R_0 = 0$, because a sign change of the post-R_k process alters neither its Brownian character nor its relation to the R_{k+j} as stopping times. In particular, for Q_j independent of everything else, as described in Lemma 2, $k(p)ER_n^{\,p} \leqq E(\sum_1^n Q_j X_j)^{2p} \leqq K(p)ER_n^{\,p}$. Hence, for Skorokhod representation times R_j, we have from Khintchine's inequality and Theorem 1, that

$$k(p)b(p)E(\textstyle\sum X_j^2)^p \leqq k(p)E(\sum X_j Q_j)^{2p} \leqq k(p)K(p)ER_n^{\,p} \leqq K(p)E(\sum X_j)^{2p} .$$

Hence, for $m(p) = k(p)b(p)/K(p)$ and, similarly, for $M(p) = K(p)B(p)/k(p)$ we have

$$m(p)E(\textstyle\sum_1^n X_j^2)^p \leqq E(\sum_1^n X_j)^{2p} \leqq M(p)(\sum_1^n X_j^2)^p .$$

4. A martingale strong law. We employ the upper inequality of Theorem 1 to give a proof of a strong law for martingales found in Chow (1967). The use of Theorem 1 in this proof parallels the use of the square function inequality in

Chow's (1960) proof. This similarity first suggested that Theorem 3 might be proved in the manner above.

THEOREM 4. *Let* $R_n = \sum_1^n T_j$ *be an increasing family of stopping times. Then, for* $p > 1$, *on the set* $\{\sum_1^\infty T_j^{2p}/j^{1+p} < \infty\}$, $n^{-1}W(R_n) \to 0$ a.s.

PROOF. We may truncate the stopping times by examining $R_n \wedge R^*$, where R^* is the first time $(\sum T_j^{2p}/j^{1+p}) + (t - R_n)^{2p}/(n+1)^{p+1} = D$ for $R_n < t \leqq R_{n+1}$ and D a large constant. We write $R_0 = 0$. Hence we may as well assume the R_n have moments of all positive orders and that $E \sum_1^\infty T_j^p/j^{1+p} < \infty$.

From the Jensen inequality and Kronecker Lemma, $\sum_1^n T_j/nj$ and $n^{-2}R^n \to 0$ in L^p. It follows from the usual scale-change argument that $W(R_n)/n \to_p 0$. Also, for $\mathscr{F}_n$ the σ-algebra generated by the process up to time R_n,

$$[W(R_n)/n]^{2p} - \sum_1^n [E\{W^{2p}(R_j) \mid \mathscr{F}_{j-1}\} - W^{2p}(R_{j-1})]/j^{2p}$$

is a supermartingale. Therefore, it converges almost surely if

$$E \sum_1^n [E\{W^{2p}(R_j) \mid \mathscr{F}_{j-1}\} - EW^{2p}(R_{j-1})]/j^{2p}$$

is bounded. The sum whose expectation is taken is a sum of positive terms so that if its expectation is bounded, $W(R_n)/n$ converges almost surely.

However,

$$\begin{aligned} E \sum_1^\infty [W^{2p}(R_j) - W^{2p}(R_{j-1})]/j^{2p} &\leqq c_1 E \sum_1^\infty R_n^p/n^{1+2p} \\ &\leqq c\, E \sum_1^\infty \sum_1^n T_j^p/n^{2+p} \end{aligned}$$

by Theorem 1 and Jensen's inequality. The latter is just $cE \sum_1^\infty T_j^p/j^{1+p} < \infty$.

We obtain as a corollary the martingale strong law of Chow (1967).

COROLLARY 1. *If* X_j *is a martingale difference sequence and if* $\sum_1^\infty EX_j^{2p}/j^{1+p} < \infty$ *for* $p \geqq 1$ *then* $\sum_1^n X_j/n \to 0$ a.s.

PROOF. Use a Skorokhod representation, R_n, of the partial sums for which $cE[W(R_n) - W(R_{n-1})]^{2p} > E(R_n - R_{n-1})^p$. (See Theorem 5, below.) Then for $T_j = R_n - R_{n-1}$, $\sum_1^\infty T_j^p/j^{1+p} < \infty$ a.s.

We also partially generalize the following strong law of Chung (1947) to martingale difference sequences.

COROLLARY 2. *Let* φ *be a positive function on the positive reals for which* $\varphi(t)/t^2$ *is decreasing and* $\varphi(t)/t^r$ *is increasing for some* $r > 1$. *Then* $E \sum_1^\infty \varphi(|X_j|)/\varphi(j) < \infty$ *implies* $\sum_1^n X_j/n \to 0$ *almost surely.*

PROOF. We choose a Skrokhod representation with $cE\varphi(|X_j|) \geqq E\varphi(T_j^{\frac{1}{2}})$ (see Theorem 5). Then $\sum \varphi(T_j^{\frac{1}{2}})/\varphi(j) < \infty$ implies $\varphi(T_j^{\frac{1}{2}})/\varphi(j) < T_j/j^2$ only finitely often. Hence $\sum_1^\infty T_j/j^2 < \infty$ and Theorem 4 applies.

Both corollaries depend strongly on the construction of a Skorokhod representation as in Breiman (1967). It is described in Section 3 above. We here prove a generalization of Breiman's Proposition 1. The proof given makes use of Theorem 1 rather than use of integration by parts.

THEOREM 5. *Let ϕ be positive function on the positive reals for which $\phi(t)/t^p$ is decreasing and $\phi(t)/t^r$ is increasing for some $p > r > 1$.*

Let X have distribution F with mean 0. Let (U, V) be distributed independent of W as $\alpha I_{\{uv<0\}\cup\{u=0,v<0\}}|v|F(du)F(dv)$ where $\alpha^{-1} = \int_0^\infty vF(dv)$. Denote by $T(U, V)$ the first hitting time of W to U or V. Then there exists c depending only on ϕ for which $cE\phi(|X|) \geqq E\phi(T^{\frac{1}{2}}(U, V))$ and $W(T(U, V))$ is distributed as F.

PROOF. Let $0 \leqq a \leqq 1$ be a constant and T_a be the first hitting time of W to a or $a - 1$. Then T_a has moments of all orders and

$$P\{W(T_a) = a\} = 1 - a .$$

Hence for $q > 1$ there exist constants d_q and D_q for which

$$ET_a^{q/2} \leqq d_q E|W(T_a)|^q$$

which equals

$$d_q[a^q(1 - a) + (1 - a)^q a] \leqq D_q a .$$

We then have that $E\phi(T^{\frac{1}{2}}(U, V)) = E\phi((|U| + |V|)T_A^{\frac{1}{2}})$ where

$$A = (|U| \wedge |V|)/(|U| + |V|)$$

by the scale change formula, and $T_A^{\frac{1}{2}}$ depends on (U, V) only through the value of A. Hence, under the hypotheses on ϕ, $E\phi(T(U, V)) \leqq E\{\phi(|U| + |V|)(T_A^{r/2} + T_A^{p/2}) \leqq DE\{A\phi(|U| + |V|)\}$, for some constant D, the latter inequality by conditioning on (U, V). The remainder of the proof follows Breiman (1967).

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Another proof of (1) for p an integer may be found in Rosenkrantz and Sawyer (1972); their work and mine were done independently. Their constants are significantly superior to mine. They employ polynomial martingales homogeneous in t and $W^2(t)$ whose leading terms are $W^n(t) - \binom{n}{2}tW^{n-2}(t)$. The expected magnitudes of these terms are compared in the first table of Section 2.

REFERENCES

[1] BREIMAN, L. (1967). On the tail behavior of sums of independent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **9** 20–25.

[2] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[3] BURKHOLDER, D. L. and GUNDY, R. F. (1970). Extrapolation and interpolation and quasi-linear operators on martingales. *Acta Math.* **124** 249–304.

[4] CHOW, Y. S. (1960). A martingale inequality and the law of large numbers. *Proc. Amer. Math. Soc.* **11** 107–111.

[5] CHOW, Y. S. (1967). On a strong law of large numbers for martingales. *Ann. Math. Statist.* **38** 610.

[6] CHUNG, K. L. (1947). Note on some strong laws of large numbers. *Amer. J. Math.* **69** 189–192.

[7] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[8] FREEDMAN, D. (1971). *Brownian Motion and Diffusion.* Holden-Day, San Francisco.

[9] MARCINKIEWICZ, J. and ZYGMUND, A. (1938). Quelques theorems sur les fonctions independantes. *Studia Math.* **7** 104–120.

[10] MCKEAN, H. P. (1969). *Stochastic Integrals.* Academic Press, New York.

[11] MILLAR, P. W. (1968). Martingale integrals. *Trans. Amer. Math. Soc.* **133** 145–166.

[12] ROSENKRANTZ, W. and SAWYER, S. (1972). An elementary derivation of the moment inequalities for Skorokhod stopping times. *Notices Amer. Math. Soc.* **19** A226.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305