The Annals of Applied Probability
2001, Vol. 11, No. 1, 261–284

ASYMPTOTIC BEHAVIOR OF ABSORBING MARKOV CHAINS CONDITIONAL ON NONABSORPTION FOR APPLICATIONS IN CONSERVATION BIOLOGY

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We find a Lyapunov-type sufficient condition for discrete-time Markov chains on a countable state space including an absorbing set to almost surely reach this absorbing set and to asymptotically stabilize conditional on nonabsorption. This result is applied to Bienaymé–Galton–Watson-like branching processes in which the offspring distribution depends on the current population size. This yields a generalization of the Yaglom limit. The techniques used mainly rely on the spectral theory of linear operators on Banach spaces and especially on the notion of quasi-compact linear operator.

1. Introduction. Let $\mathbf{N}$ and $\mathbf{N}^*$ denote the sets of nonnegative integers and positive integers, respectively. The "classical" Bienaymé–Galton–Watson (BGW) branching process has been the most studied discrete-time branching process so far [see Athreya and Ney (1972), Jagers (1975) and Asmussen and Hering (1983)]. Denoting by $(p_k)_{k\in\mathbf{N}}$ the probability distribution of the offspring (integer) number at the next time step $n+1$ of any individual part of the population at any time n, and assuming as classically that $p_0+p_1<1$ and $p_0>0$, one property of BGW branching processes $(Z_n)_{n\in\mathbf{N}}$ is that provided the average offspring number per individual is less than 1 (i.e., $m=\sum_{k=1}^{\infty}kp_k<1$), extinction is certain (i.e., $\lim_{n\to\infty}P_\pi(Z_n=0)=1$) and there is a *unique* probability distribution $(b_i)_{i\in\mathbf{N}^*}$ limit in distribution of the process conditioned on current nonextinction:

$$\lim_{n\to\infty}P_\pi\bigl(Z_n=i\big|Z_n>0\bigr)=b_i,\qquad i\in\mathbf{N}^*,$$

provided $(\pi_i)_{i\in\mathbf{N}}$ satisfies certain conditions stated by Seneta and Vere-Jones (1966) or Asmussen and Hering (1983). Then $(b_i)_{i\in\mathbf{N}^*}$ is called the *Yaglom limit* of (Z_n).

Some authors investigated similar properties for generalizations of BGW branching processes by introducing population-size-dependence [see Fujimagari (1976), Lebreton (1981), Klebaner (1984, 1985) and Höpfner (1985, 1986)], different types of individuals [see Joffe and Spitzer (1967),

Received June 1998; revised February 2000.
AMS 2000 *subject classifications*. Primary 60J10, 60J80; secondary 47B07, 47B37, 47B65, 60J20, 60J85, 92D25.
Key words and phrases. Homogeneous Markov chain, extinction, absorbing set, quasi-stationary distribution, Yaglom limit, density-dependence, population-size-dependent Bienaymé–Galton–Watson branching processes, quasi-compact linear operator, nonnegative operator, irreducible matrix, infinite dimensional matrix, spectral theory, conservation biology, population viability analysis, demography, population dynamics, Lyapunov-type condition.

Buiculescu (1975) and Hoppe and Seneta (1978), pages 224–228] or a random environment [see Athreya and Karlin (1971) and Tanny (1981)]. This paper aims to generalize the existence of a Yaglom limit to discrete-time Markov chains on a countable state space including an absorbing set, under biologically realistic and practical conditions. Indeed, such properties have already been found by, for example, Seneta and Vere-Jones (1966), Kesten (1995) and Ferrari, Kesten and Martínez (1996), but under conditions that will be seen impractical for models of biological population extinction, the applied field we are interested in (Section 7).

This paper is organized as follows: we first introduce the notation (Section 2), state the two main results of the paper (Section 3), and apply them to population-size-dependent BGW branching processes (Section 4). We then propose in Section 5 some basic spectral results, used in Section 6 to prove the results in Section 3. We conclude this paper with important final remarks (Section 7).

The characteristics of this work are (1) a heavy reliance on the linear operator theory, and especially on the notion of quasi-compact operator. This imposes a rather stringent assumption (3.I), which is the main shortcoming of this paper; on the other hand, (2) a large generality as far as the irreducible classes of the Markov chain are concerned and (3) the formulation of sufficient conditions that appear to be biologically realistic and verifiable [see Gosselin (1997 and 1998b)].

2. Notation and setting. We consider a countable state space $\mathbf{G}$ and a discrete-time homogeneous Markov chain $(Z_n)_{n\in\mathbf{N}}$ on $\mathbf{G}$. P_π represents the probability of an event when the distribution of Z_0 is $\pi = (\pi_i)_{i\in\mathbf{G}}$, and $P = (p_{i,j})_{(i,j)\in\mathbf{G}^2}$ is the transition matrix of (Z_n). We first assume the following:

(2.I) There is a partition $\{\mathbf{G}_0, \mathbf{G}^*\}$ of $\mathbf{G}$ such that $\mathbf{G}_0$ is an *absorbing set* [i.e., for every i in $\mathbf{G}_0$, $P_i(Z_1 \in \mathbf{G}_0) = 1$] and $\mathbf{G}^*$ is *nonabsorbing* [i.e., here, for every i in $\mathbf{G}^*$, $P_i(Z_n \in \mathbf{G}_0) > 0$ for some n in $\mathbf{N}^*$].

For every subset $\mathbf{G}'$ of $\mathbf{G}$ we denote by $Q_{\mathbf{G}'}$ the substochastic matrix, restriction of P on $\mathbf{G}'$: $Q_{\mathbf{G}'} = (p_{i,j})_{(i,j)\in\mathbf{G}'^2}$. Under (2.I), we particularly denote $Q = Q_{\mathbf{G}^*}$. There is then a unique partition $\{\mathbf{G}_h\}_{h\in\mathbf{I}}$ of $\mathbf{G}^*$, constituted by the equivalence classes of communicating states (i.e., for every j in any $\mathbf{G}_h$, $\mathbf{G}_h$ is the union of $\{j\}$ with the set of states i in $\mathbf{G}$ that lead to j and that j leads to). By construction, when the Markov chain leaves $\mathbf{G}_h$, with probability 1 it can no longer return $\mathbf{G}_h$. Therefore, $p_{i,j}^{(n)} = (Q_{\mathbf{G}_h}^n)_{i,j}$ for every n in $\mathbf{N}^*$ and i and j in $\mathbf{G}_h$. Besides, provided $\mathbf{G}_h$ is not restricted to one element that does not lead to itself, the matrix $Q_{\mathbf{G}_h}$ is irreducible, that is, for every i and j in $\mathbf{G}_h$, $p_{i,j}^{(n)} > 0$ for some n in $\mathbf{N}^*$.

Under (2.I), considering a realization ϖ of the Markov chain, we say that $\{Z_n(\varpi)\}_{n\in\mathbf{N}}$ goes *extinct* if there is n in $\mathbf{N}$, such that $Z_n(\varpi) \in \mathbf{G}_0$.

We next consider a map t from $\mathbf{G}$ to $\mathbf{R}^+$ such that we have the following:

(2.II) $t(\mathbf{G}_0) = \{0\}$, and t has positive values on $\mathbf{G}^*$ and for any $c > 0$, there is at most a finite number of j in $\mathbf{G}^*$ such that $t(j) < c$.

(2.I) and (2.II) are supposed to be met throughout the paper. We will also often refer to the following conditions.

(2.III) $\exists\ i \in \mathbf{G}^*, \exists n \in \mathbf{N}^*,\ p_{i,i}^{(n)} > 0$;
(2.IV) $\sup_{i \in \mathbf{G}^*} {}_{(i)}m_t < \infty$, where ${}_{(i)}m_t = \sum_{j \in \mathbf{G}} (t(j)/t(i)) p_{i,j}$.

3. Statement of the main results.

3.1. *Certain extinction.* We first propose a sufficient condition for the certain (ultimate) extinction of (Z_n). Under assumptions (2.I) and (2.II), each state i in $\mathbf{G}^*$ is transient and, whatever the initial distribution π,

$$P_\pi\left\{\lim_{n\to\infty} t(Z_n) = 0 \text{ or } \lim_{n\to\infty} t(Z_n) = \infty\right\} = 1.$$

PROPOSITION 3.1. *Assume* (2.I), (2.II) *and* (2.IV). *If* ${}_{(i)}m_t \le 1$ *for all but a finite number of i in $\mathbf{G}^*$, then, whatever the initial distribution π,*

$$P_\pi\left\{\lim_{n\to\infty} t(Z_n) = 0\right\} = \lim_{n\to\infty} P_\pi(Z_n \in \mathbf{G}_0) = 1.$$

PROOF. If $\pi = \delta_i$, where i is in $\mathbf{G}^*$, the proof is similar to that of Propositions 4 and 5, pages 55 and 56, in Lebreton (1981). The general case readily follows. This result also stems from Theorem 5 in Foster (1953) if we have ${}_{(i)}m_t \le 1$ for any i in $\mathbf{G}^*$. $\square$

3.2. *Asymptotic results conditional on nonextinction.* Extinction is the fate of any model fulfilling the conditions of Proposition 3.1. The aim of this paper is to prove that, under certain additional conditions, some stochastic equilibrium is then asymptotically reached *conditional on nonextinction*. We therefore study the convergence of the following quantities, where states i and j are in $\mathbf{G}^*$ and times n in $\mathbf{N}$ and n' in $\mathbf{Z}$:

$$\alpha_{\pi,j}(n,n') = P_\pi\big(Z_n = j \big| Z_{n+n'} \in \mathbf{G}^*\big),$$

$$\delta_{\pi,j}(n,n') = P_\pi\big(Z_n = j \big| Z_n \in \mathbf{G}^*, Z_{n+n'} \in \mathbf{G}_0\big),$$

$$b_j(n) = \frac{1}{n}\sum_{\nu=1}^{n} 1_{\{Z_\nu = j\}}, \tag{3.1}$$

$$\tau_{\pi,j}(n) = \frac{1}{n}\sum_{\nu=1}^{n} P_\pi\big(Z_\nu = j \big| Z_n \in \mathbf{G}^*\big) = E_\pi\big(b_j(n) \big| Z_n \in \mathbf{G}^*\big).$$

Denoting by $r(Q)$ the spectral radius (cf. Definition 5.1) of the bounded linear operator Q on the Banach space l_t^∞ of all the sequences $(x_j)_{j \in \mathbf{G}^*}$ with

$(t(j)^{-1}|x_j|)$ bounded, we will prove the convergence of the quantities in (3.1) when the following two assumptions are added:

(3.I) If $\mathbf{G}^*$ is infinite, $\limsup_{t(i)\to\infty} {}_{(i)}m_t < r(Q)$.

(3.II) There is only one k in $\mathbf{I}$ such that $r(Q_{\mathbf{G}_k}) = r(Q)$, where $r(Q_{\mathbf{G}_k})$ is the spectral radius of the operator $Q_{\mathbf{G}_k}$ on $l^{\infty}_{\mathbf{G}_k, t} = \{x \in \mathbf{C}^{\mathbf{G}_k} / \sup_{j\in\mathbf{G}_k} t(j)^{-1} |x_j| < \infty\}$. The matrix $Q_{\mathbf{G}_k}$ such that $r(Q_{\mathbf{G}_k}) = r(Q)$ is aperiodic.

We will see in Section 5 that, when $\mathbf{G}^*$ is infinite and conditions (2.I) to (2.IV) are satisfied, (3.I) is a sufficient condition for the quasi compactness of the linear operator Q on $n(Q_{G_k}) = n(Q)l^{\infty}_t$. Under conditions (2.I) to (3.I), $r(Q)$ is positive and there is at least one, and at most a finite number of, k in $\mathbf{I}$ such that (Proposition 5.4). Assumption (3.II) then stipulates that the spectral radius $r(Q)$ is reached in *only* one irreducible class, $\mathbf{G}_k$, assumed to be aperiodic.

We here introduce for convenience the set $\mathbf{C}_k$ of the vectors $(x_i)_{i\in\mathbf{G}^*}$ such that $x_i \geq 0$ for every i on $\mathbf{G}^*$, $\sum_{i\in\mathbf{G}^*} t(i)x_i < \infty$ and there is i with $x_i > 0$ that leads to $\mathbf{G}_k$ [i.e., $P_i(Z_n \in \mathbf{G}_k) > 0$ for some n in $\mathbf{N}^*$].

THEOREM 3.1. *Assume* (2.I)*,* (2.II)*,* (2.III)*,* (2.IV)*,* (3.I) *and* (3.II)*. Then, $r(Q)$ is in $(0, 1)$, the matrix Q has unique right and left nonnegative eigenvectors, $(u_i)_{i\in\mathbf{G}^*}$ and $(v_i)_{i\in\mathbf{G}^*}$, associated with the eigenvalue $r(Q)$, such that*

$$\sup_{i\in\mathbf{G}^*} u_i t(i)^{-1} < \infty, \quad \sum_{i\in\mathbf{G}^*} t(i)v_i < \infty \quad \textit{and} \quad \sum_{i\in\mathbf{G}^*} v_i = \sum_{i\in\mathbf{G}^*} u_i v_i = 1$$

and there is $0 < \varepsilon < 1$ such that, as n and n' tend to infinity:

(a) *If $\pi = (\pi_i)_{i\in\mathbf{G}^*}$ is a probability distribution such that $\sum_{i\in\mathbf{G}^*} t(i)\pi_i < \infty$,*

$$1 - P_{\pi}(Z_n \in \mathbf{G}_0) = r(Q)^n(u, \pi) + o(r(Q)^n(1-\varepsilon)^n),$$

in particular,

$$\lim_{n\to\infty} \frac{P_j(Z_n \in \mathbf{G}^*)}{r(Q)^n} = u_j, \qquad j \in \mathbf{G}^*.$$

(b) *If $\pi = (\pi_i)_{i\in\mathbf{G}^*}$ is a probability distribution that belongs to $\mathbf{C}_k$, and if we denote for every n'' in $\mathbf{N}$, $p^{(n'')}_{i,0} = \sum_{j\in\mathbf{G}_0} p^{(n'')}_{i,j}$, we have*

$$\sum_{j\in\mathbf{G}^*} t(j)\left\{\left|\alpha_{\pi,j}(n, n'') - \frac{1 - p^{(n'')}_{j,0} v_j}{r(Q)^{n''}}\right| + \left|\delta_{\pi,j}(n, n'') - \frac{p^{(n'')}_{j,0} v_j}{1 - r(Q)^{n''}}\right| 1_{\{n''>0\}}\right\} = o\big((1-\varepsilon)^n\big)$$

and

$$\sum_{j\in\mathbf{G}^*} t(j)\left|\alpha_{\pi,j}(n, -n'') - r(Q)^{n''} v_j\right| = o\big((1-\varepsilon)^n\big);$$

with $n'' = 0$. These results imply that the Markov chain has a Yaglom limit, which is $(v_j)_{j\in\mathbf{G}^*}$,

$$\lim_{n\to\infty} P_\pi\big(Z_n = j \big| Z_n \in \mathbf{G}^*\big) = v_j, \qquad j \in \mathbf{G}^*$$

and

(c) *If $\pi = (\pi_i)_{i\in\mathbf{G}^*}$ is a probability distribution that belongs to $\mathbf{C}_k$, for any $\alpha > 0$,*

$$\lim_{n\to\infty} P_\pi\left(\sum_{j\in\mathbf{G}^*} \big|b_j(n) - u_j v_j\big| > \alpha \middle| Z_n \in \mathbf{G}^*\right) = 0$$

and

$$\sum_{j\in\mathbf{G}^*} |\tau_{\pi,j}(n) - u_j v_j| = O\Big(\frac{1}{n}\Big) \text{ and } \sum_{j\in\mathbf{G}^*} |\alpha_{\pi,j}(n,n') - u_j v_j| = o\big((1-\varepsilon)^{\inf(n,n')}\big).$$

4. Application to population-size-dependent BGW branching processes. A population-size-dependent BGW branching process $(Z_n)_{n\in\mathbf{N}}$ is a Markov chain with state space $\mathbf{N}$ such that for every i in $\mathbf{N}^*$ its one-time step transition probability, $(p_{i,j})_{j\in\mathbf{N}}$, is the ith fold convolution of a probability distribution $({}_{(i)}p_k)_{k\in\mathbf{N}}$. As a result, denoting by ${}_{(i)}W_{r,n}$ the random variable representing the offspring number at the next time step $n+1$ of the rth individual in a population of size i at time n,

$$Z_{n+1} = \sum_{r=1}^{Z_n} {}_{(Z_n)}W_{r,n},$$

with the convention $\sum_{r=1}^{0} = 0$, where ${}_{(i)}W_{r,n}$ has the probability distribution $({}_{(i)}p_k)_{k\in\mathbf{N}}$, and the random variables $({}_{(i)}W_{r,n})_{n\in\mathbf{N},\, i\in\mathbf{N}^*,\, 1\le r\le i}$ are independent one from the other and independent of Z_0.

We here consider $\mathbf{G}^* = \mathbf{N}^* = \mathbf{N} - \{0\}$ and assume

(4.I) For every i in $\mathbf{N}^*$, ${}_{(i)}p_0 + {}_{(i)}p_1 < 1$, ${}_{(i)}p_0 > 0$ and ${}_{(i)}p_1 > 0$.

Under (4.I), $\mathbf{G}^*$ is an irreducible, aperiodic class of (Z_n) and assumptions (2.I) and (2.III) are met. Under (4.I), (3.II) is also met as soon as we consider a map t such that (2.IV) and (2.II) are met. We also denote by t_0 the map such that $t_0(n) \equiv n$, by ${}_{(i)}f(s) = \sum_{k=0}^{\infty} {}_{(i)}p_k s^k$, $s \in [0,1]$, the probability generating function associated with $({}_{(i)}p_k)_{k\in\mathbf{N}}$ and by ${}_{(i)}g_t(s) = \sum_{j\in\mathbf{N}} p_{i,j} s^{t(j)}$, $s \in [0,1]$, the function associated with the probability distribution $(p_{i,j})_{j\in\mathbf{N}}$ and the map t. Then, for every i in $\mathbf{N}^*$,

$${}_{(i)}g_{t_0}(s) = {}_{(i)}f(s)^i \quad \text{and} \quad {}_{(i)}m_{t_0} = E\big({}_{(i)}W_{1,1}\big) = {}_{(i)}f'(1) = \sum_{k=1}^{\infty} k\ {}_{(i)}p_k.$$

4.1. *The case when* $t = t_0^\nu$. We first look for more practical formulations of assumptions (2.IV), (3.I) and (3.II) when we consider the map t_0^ν such that $t_0^\nu(n) \equiv n^\nu$, $n \in \mathbf{N}$. We find a relationship between $\limsup_{i\to\infty} {}_{(i)}m_{t_0^\nu}$ and $\limsup_{i\to\infty} {}_{(i)}m_{t_0}$, quantities involved in (3.I) applied, respectively, to $t = t_0^\nu$ and $t = t_0$.

PROPOSITION 4.1. *Let* (Z_n) *be a population-size-dependent BGW branching process. If* $\nu \in \mathbf{N}^*$ *and* $\sup_{i\in\mathbf{N}^*} \mathbf{E}({}_{(i)}W_{1,1}^\nu) < \infty$, *then* (2.IV) *is met with* $t = t_0^\nu$ *and*

$$\limsup_{i\to\infty} {}_{(i)}m_{t_0^\nu} = \Bigl(\limsup_{i\to\infty} {}_{(i)}m_{t_0}\Bigr)^\nu.$$

PROOF. First,

$$(4.1)\qquad {}_{(i)}m_{t_0^\nu} = \sum_{j\in\mathbf{N}^*} \frac{j^\nu}{i^\nu} p_{i,j} = \frac{\mathbf{E}\Bigl\{\bigl({}_{(i)}W_{1,1} + {}_{(i)}W_{2,1} + \cdots + {}_{(i)}W_{i,1}\bigr)^\nu\Bigr\}}{i^\nu},$$

and since, by Hölder's inequality,

$$\Bigl({}_{(i)}W_{1,1} + {}_{(i)}W_{2,1} + \cdots + {}_{(i)}W_{i,1}\Bigr)^\nu \le i^{\nu-1} \sum_{j=1}^{i} {}_{(i)}W_{j,1}^\nu,$$

we get $\sup_{i\in\mathbf{N}^*} {}_{(i)}m_{t_0^\nu} \le \sup_{i\in\mathbf{N}^*} \mathbf{E}({}_{(i)}W_{1,1}^\nu)$, which proves (2.IV) with $t = t_0^\nu$.

Second, let us define for every i and ρ in $\mathbf{N}^*$, ${}_{(i)}m(\rho) = \mathbf{E}({}_{(i)}W_{1,1}^\rho)$. Of course, ${}_{(i)}m(1) = {}_{(i)}m_{t_0}$. Since $M := \sup_{i\in\mathbf{N}^*} \mathbf{E}({}_{(i)}W_{1,1}^\nu) < \infty$, we get $\sup_{i\in\mathbf{N}^*} {}_{(i)}m(\rho) \le M$ for $\rho \le v$. We then recall the result of Theorem 2.5 in Klebaner (1984),

$$(4.2)\qquad \begin{aligned} &\mathbf{E}\{({}_{(i)}W_{1,1} + {}_{(i)}W_{2,1} + \cdots + {}_{(i)}W_{i,1})^\nu\} \\ &\qquad = {}_{(i)}m_{t_0}^\nu + f_\nu(i, {}_{(i)}m(1), {}_{(i)}m(2), \dots, {}_{(i)}m(\nu)), \end{aligned}$$

where $f_\nu(i, {}_{(i)}m(1), {}_{(i)}m(2), \dots, {}_{(i)}m(\nu))$ is a polynomial of order $\nu - 1$ in i and a nonnegative, nondecreasing function with respect to ${}_{(i)}m(\rho)$, $1 \le \rho \le \nu$. It follows from (4.1) and (4.2) that

$${}_{(i)}m_{t_0}^\nu \le \sum_{j\in\mathbf{N}^*} \frac{j^\nu}{i^\nu} p_{i,j} \le \frac{{}_{(i)}m_{t_0}^\nu i^\nu + f_\nu(i, M, M, \dots, M)}{i^\nu}, \qquad i \in \mathbf{N}^*,$$

which proves $\limsup_{i\to\infty} {}_{(i)}m_{t_0^\nu} = (\limsup_{i\to\infty} {}_{(i)}m_{t_0})^\nu$. $\square$

Proposition 4.1 does not give any insight about the other quantity involved in condition (3.I), that is, $r(Q)$, whence Lemma 4.1, whose proof is obvious, based, for example, on Gel'fand's theorem in Section 5.1.

LEMMA 4.1. *Under assumptions* (2.I), (2.II) *and* (2.IV), *we have* $r(Q) \ge \lim_{n\to\infty} r({}_{(n)}Q)$, *where* ${}_{(n)}Q$ *is the finite matrix* $(p_{i,j})_{(i,j)\in\mathbf{G}^{*2},\, t(i),\, t(j)\le n}$.

This allows us to propose a sufficient condition for the results of Theorem 3.1 to hold for $t = t_0^\nu$.

THEOREM 4.1. *Let* (Z_n) *be a population-size-dependent BGW branching process. Assume* (4.I). *If* $\limsup_{i\to\infty} {}_{(i)}m_{t_0} < 1$, *and for every* ν *in* $\mathbf{N}^*$, $\sup_{i\in\mathbf{N}^*} \mathbf{E}({}_{(i)}W_{1,1}^\nu) < \infty$, *then there is* $N > 0$ *such that* $\nu \in \mathbf{N}^*$ *and* $\nu > N$ *imply that the results of Theorem* 3.1 *hold with* $t = t_0^\nu$.

PROOF. Let $r_{\infty, t_0^\nu}(Q)$ denote the spectral radius of the bounded linear operator Q on the Banach space $l_{t_0^\nu}^\infty$ (cf. definitions in Sections 3.2 and 5.1). First, using the same notation as in Lemma 4.1, assumption (2.III) yields $\lim_{n\to\infty} r({}_{(n)}Q) > 0$, which proves $\inf_{\nu\in\mathbf{R}^{+*}} r_{\infty,t_0^\nu}(Q) > 0$. Now, since for every ν in $\mathbf{N}^*$, $\sup_{i\in\mathbf{N}^*} \mathbf{E}({}_{(i)}W_{1,1}^\nu) < \infty$, and $\limsup_{i\to\infty} {}_{(i)}m_{t_0} < 1$, we get from Proposition 4.1 that condition (2.IV) is met for $t = t_0^\nu$ for every ν and

$$\lim_{\nu\to\infty}\left[\limsup_{i\to\infty} {}_{(i)}m_{t_0^\nu}\right] = 0.$$

This proves that condition (3.I) is met for $t = t_0^\nu$ for every ν sufficiently large. $\square$

4.2. *The case when* $t = A^{t_0}1_{\{t_0>0\}}$. Let $A > 1$ and $t_{0,A} = A^{t_0}1_{\{t_0>0\}}$. We have

$$\begin{aligned} {}_{(i)}m_{t_{0,A}} &= \sum_{j\in\mathbf{N}^*} \frac{A^j}{A_i} p_{i,j} = \frac{{}_{(i)}g_{t_0}(A) - {}_{(i)}g_{t_0}(0)}{A^i} \\ &= \frac{{}_{(i)}f(A)^i - {}_{(i)}f(0)^i}{A^i}, \qquad i \in \mathbf{N}^*. \end{aligned}$$

It is then easily proved that for some $A > 1$, assumption (2.IV) is met for $t = t_{0,A}$ if and only if $\sup_{i\in\mathbf{N}^*} {}_{(i)}f(A)^i A^{-i} < \infty$.

PROPOSITION 4.2. *Let* (Z_n) *be a population-size-dependent BGW branching process such that* $\limsup_{i\to\infty} {}_{(i)}m_{t_0} < 1$. *If* $\sup_{i\in\mathbf{N}^*} {}_{(i)}f(A') < \infty$ *for some* $A' > 1$, *then for every* $A > 1$ *sufficiently small,* $\sup_{i\in\mathbf{N}^*} {}_{(i)}f(A) < \infty$ *and* $\lim_{i\to\infty} {}_{(i)}f(A)^i\ A^{-i} = 0$.

PROOF. Each ${}_{(i)}f$ is then indefinitely differentiable on $[0, A')$. Then, for every s in $[0, A')$, by Taylor's theorem, there is $\theta_{s,i} \in (1, s)$ such that

$${}_{(i)}f(s) = 1 + {}_{(i)}m_{t_0}(s-1) + \frac{(s-1)^2}{2}\,{}_{(i)}f''(\theta_{s,i}).$$

However, denoting $M = \sup_{i\in\mathbf{N}^*} {}_{(i)}f(A')$, we have ${}_{(i)}p_k \le MA'^{-k}$, $k \in \mathbf{N}$, $i \in \mathbf{N}^*$ and

$${}_{(i)}f''(s) \le \sum_{k=2}^{\infty} k(k-1)MA'^{-k}s^{k-2} := g(s), \qquad i \in \mathbf{N}^*,\ s \in (1, A'),$$

where g is indefinitely differentiable on $[0, A')$. Then, using

$$\limsup_{i\to\infty} {}_{(i)}m_{t_0} < 1,$$

there are $\varepsilon > 0$ and $I > 0$ such that

$$i \geq I \Rightarrow {}_{(i)}f(s) \leq 1 + (1-\varepsilon)(s-1) + g(s)\frac{(s-1)^2}{2} := h(s), \qquad s \in (1, A').$$

By the mean-value theorem, there are $a < 1$ and $1 < A \leq A'$ such that $h(A) \leq aA$. Then, $i \geq I$ implies ${}_{(i)}f(A) \leq aA$, which proves Proposition 4.2. $\square$

REMARKS. First, the last conditions of Proposition 4.2 and Theorem 4.1 are met if there is a maximum offspring number per individual, that is, there is $K > 0$ such that for every i, $k \geq K$ implies ${}_{(i)}p_k = 0$. Second, if $\limsup_{i\to\infty} {}_{(i)}m_{t_0} < 1$ and $\sup_{i\in\mathbf{N}^*} {}_{(i)}f(A') < \infty$ for some $A' > 1$, we also have for every ν in $\mathbf{N}^*$, $\sup_{i\in\mathbf{N}^*} E({}_{(i)}W_{1,1}^{\nu}) < \infty$. Then, assumption (3.I) is met both for $t = t_{0,A}$ for some $A > 1$ (Proposition 4.2) and for $t = t_0^{\nu}$ for any sufficiently big ν (Theorem 4.1).

Proposition 4.2 allows us to propose a more convenient formulation of Theorem 3.1 when considering $t = t_{0,A}$.

THEOREM 4.2. *Let* (Z_n) *be a population-size-dependent BGW branching process. Assume* (4.I), $\limsup_{i\to\infty} {}_{(i)}m < 1$ *and* $\sup_{i\in\mathbf{N}^*} {}_{(i)}f(A') < \infty$ *for some* $A' > 1$. *Then, for all* $A > 1$ *small enough, the results of Theorem* 3.1 *hold with* $t = t_{0,A} = A^{t_0}1_{\{t_0>0\}}$.

4.3. *Applications.* We now consider the more specific process (Z_n) such that, for every i in $\mathbf{N}^*$, the offspring number probability distribution $({}_{(i)}p_k)_{k\in\mathbf{N}}$ is a power series distribution associated with a power series that does not depend on i [see, e.g., Johnson and Kotz (1969), Chapter 2, Section 3]; that is, the probability generating functions ${}_{(i)}f$ have the shape ${}_{(i)}f(s) = g(s_{(i)}\lambda)/g({}_{(i)}\lambda)$, $s \in [0,1]$, where $0 < {}_{(i)}\lambda \leq 1$ and $g(s) = \sum_{k=0}^{\infty} c_k s^k$, $s \in [0,1]$, is a probability generating function, which we assume such that $c_1 > 0$, $c_0 > 0$, $c_0 + c_1 < 1$ and $\sum_{k=2}^{\infty} k(k-1)c_k < \infty$. Due to these last assumptions, (Z_n) satisfies (4.I). Furthermore,

$${}_{(i)}f'(1) = {}_{(i)}\lambda g'({}_{(i)}\lambda)/g({}_{(i)}\lambda) := h({}_{(i)}\lambda), \qquad i \in \mathbf{N}^*.$$

We can prove that $h({}_{(i)}\lambda)$ is a nondecreasing function of ${}_{(i)}\lambda$ and, since g and g' are continuous on $[0,1]$,

$$\limsup_{i\to\infty} {}_{(i)}f'(1) = \limsup_{i\to\infty} {}_{(i)}\lambda \; g'\Big(\limsup_{i\to\infty} {}_{(i)}\lambda\Big)\Big[g\Big(\limsup_{i\to\infty} {}_{(i)}\lambda\Big)\Big]^{-1}.$$

Set $\lambda = \sup\{x \in [0,1];\; h(x) \leq 1\}$. If $\limsup_{i\to\infty} {}_{(i)}\lambda < \lambda$ and $\sup_{i\in\mathbf{N}^*} {}_{(i)}\lambda < 1$, Theorems 4.1 and 4.2 prove that the outcomes of Theorem 3.1 apply with $t = t_0^{\nu}$ or $t = t_{0,A}$, for some values of ν and $A > 1$, respectively. If $\lim_{i\to\infty} {}_{(i)}\lambda = 0$ and $\sup_{i\in\mathbf{N}^*} {}_{(i)}\lambda \leq 1$, we can prove with similar methods to the above that Theorem 3.1 applies with $t = t_0^2$.

Finally, for population-size-independent BGW branching processes, the results of Theorem 3.1 hold with $t = t_0^2$ when $p_0 > 0$, $p_1 > 0$, $p_0 + p_1 < 1$, $m < 1$ and $f''(1) < \infty$ (cf. notation in Section 1). Indeed, (2.I), (2.III), (2.IV) and (3.II) are then satisfied and, introducing the vector β in $l^\infty_{t_0^2}$ such that $\beta_i = i$, $i \in \mathbf{N}^*$, we have $Q\beta = m\beta$, which insures us that $r(Q) \geq m$. But, simultaneously, Proposition 4.1 implies $\limsup_{i\to\infty\,(i)} m_{t_0^2} = m^2 < m$, which proves that (3.I) is also satisfied.

5. Basic spectral results. The most classical tool in the study of branching processes is the iteration of probability generating functions. Under our conditions, the use of such techniques appears limited [see Gosselin (1993, 1998a)]. We therefore prefer to use a linear operator approach. We present in this section some reminders and results from this theory.

5.1. *Reminders from the linear operator theory.* In this subsection we recall for the sake of convenience some notions and results of the spectral theory of linear operators on a Banach space. Basic references are Section 6 in Krein and Rutman (1950), Dunford and Schwartz (1958), Kato (1966), Dieudonné (1972) and Istratescu (1981).

Let E denote a complex Banach space and $\mathscr{L}(E)$ the set of bounded linear operators from E onto E. In this section, U is in $\mathscr{L}(E)$ and $\|U\|_E$ denotes its norm, $\sigma(U)$ its spectrum and $\rho(U) = \mathbf{C} - \sigma(U)$ its resolvant set. We denote by $r_E(U) = \sup_{\lambda\in\sigma(U)} |\lambda|$ the *spectral radius* of U, which is, from Gel'fand's theorem, the limit of the nonincreasing sequence $(\|U^k\|_E^{k^{-1}})_{k\in\mathbf{N}^*}$. We call *resolvant* the map from $\rho(U)$ to $\mathscr{L}(E)$ that associates to each ζ in $\rho(U)$ the bounded operator $R(\zeta, U) = (\zeta I - U)^{-1}$. For $|\zeta| > r_E(U)$, $R(\zeta, U) = \sum_{k=0}^{\infty}(U^k/\zeta^{k+1})$.

An *eigenvalue* λ of U is a complex number such that there is a nonnull vector x in E such that $Ux = \lambda x$. Then x is called an *eigenvector* of U associated with the eigenvalue λ. Every eigenvalue of U is in its spectrum $\sigma(U)$. A spectral value (resp., eigenvalue) λ of U is called a *peripheral spectral value* (resp., *eigenvalue*) if $|\lambda| = r_E(U)$.

If λ is an isolated spectral value and if γ denotes the oriented boundary of a closed neighborhood V of λ whose intersection with $\sigma(U) - \{\lambda\}$ is void and whose boundary is regular enough, the operator $\Pi(\lambda, U) = (2i\pi)^{-1}\times \int_\gamma R(\zeta, U)d\zeta$ is a projection called the *spectral projection* associated with U and λ. This projection does not depend on γ and commutes with U. The *Laurent expansion* of the resolvent around λ is $R(\zeta, U) = \sum_{k\in\mathbf{Z}} a_k(\zeta - \lambda)^k$, with, for k positive, $a_{-k} = (U - \lambda I)^{k-1}\Pi(\lambda, U)$. In particular, λ is called a *pole of the resolvant* if $(U - \lambda I)\Pi(\lambda, U)$ is nilpotent. The *order* (or index) ω of λ is then the largest positive integer ω such that $a_{-\omega} = (U - \lambda I)^{\omega-1}\Pi(\lambda, U) \neq 0$. Every pole of the resolvant of U is an eigenvalue of U. The eigenvalue λ is called *simple* if it is isolated in the spectrum and if $\mathrm{Im}(\Pi(\lambda, U))$ is one-dimensional. Its order is then one.

We next recall the key notions of compactness and quasi compactness of linear operators.

DEFINITION 5.1. U in $\mathscr{L}(E)$ is said to be compact if the image by U of every bounded subset of E is relatively compact.

If E has finite dimension, every U in $\mathscr{L}(E)$ is compact. The spectrum of a compact operator on E consists of the set of its eigenvalues plus, if E is infinite dimensional, zero. These and other spectral properties make compact operators (spectrally) closer to operators in finite-dimensional Banach spaces than noncompact operators, whose spectrum can be continuous.

The transposed operator of U, denoted by U', acts on the dual space E'. It has the same spectrum as U, and U is compact on E if and only if its transpose U' is compact on E'. For every linear form y in E' and x in E, we set $(y, x) = y(x)$.

DEFINITION 5.2 [see Sasser (1964)]. U is said to be quasi-compact on E if there exist a positive integer n and V in $\mathscr{L}(E)$ such that $U^n - V$ is a compact operator in $\mathscr{L}(E)$ and $r_E(V) < r_E(U)^n$.

Obviously, U is quasi-compact on E if and only if its transpose U' is quasi-compact on E', and a compact operator is quasi-compact if and only if its spectral radius is positive. Furthermore, a quasi-compact operator has a nonvoid peripheral spectrum [see Brunel and Revuz (1974)] and every peripheral spectral value of a quasi-compact operator is an isolated eigenvalue and a pole of the resolvant [see Sasser (1964)].

5.2. *Criteria for compactness of linear operators on l^∞, l_t^∞, l^1 and l_t^1.* We associate with the set $\mathbf{G}^*$ and the map t introduced in Section 2, the following norms: for every sequence x indexed by $\mathbf{G}^*$, let

$$\|x\|_\infty := \sup_{j\in\mathbf{G}^*} |x_j|, \quad \|x\|_{\infty, t} := \sup_{j\in\mathbf{G}^*} t(j)^{-1}|x_j|,$$

$$\|x\|_1 := \sum_{j\in\mathbf{G}^*} |x_j| \quad \text{and} \quad \|x\|_{1, t} := \sum_{j\in\mathbf{G}^*} t(j)|x_j|.$$

We then define the four associated Banach spaces $l^\infty = \{x \in \mathbf{C}^{\mathbf{G}^*}/\|x\|_\infty < \infty\}$, $l_t^\infty = \{x \in \mathbf{C}^{\mathbf{G}^*}/\|x\|_{\infty, t} < \infty\}$, $l^1 = \{x \in \mathbf{C}^{\mathbf{G}^*}/\|x\|_1 < \infty\}$ and $l_t^1 = \{x \in \mathbf{C}^{\mathbf{G}^*}/\|x\|_{1, t} < \infty\}$. In this paper, we say that an operator U on one of these Banach spaces is *linked to the matrix* $(u_{i, j})_{(i, j)\in\mathbf{G}^{*2}}$, and we denote shortly $U = (u_{i, j})$, if for any x in this Banach space, we have $(Ux)_i = \sum_{j\in\mathbf{G}^*} u_{i,j}\, x_j$, $i \in \mathbf{G}^*$.

It is well known that $(l^1)' = l^\infty$ and $(l_t^1)' = l_t^\infty$. Moreover, any bounded linear operator U on l^1 is linked to an infinite-dimensional matrix $U = (u_{i, j})_{(i, j)\in\mathbf{G}^{*2}}$ [see for instance Vere-Jones (1968)]. Its transpose U' on l^∞ is then linked to the transposed matrix ${}^tU = (u_{j, i})_{(i, j)\in\mathbf{G}^{*2}}$. The same holds for a bounded operator on l_t^1 and its transpose on l_t^∞. We denote the norm and spectral radius of a bounded operator U on l^1 (resp., l^∞, l_t^1, l_t^∞) by $\|U\|_1$ and $r_1(U)$ [resp., $\|U\|_\infty$ and $r_\infty(U)$, $\|U\|_{1,t}$ and $r_{1,t}(U)$, $\|U\|_{\infty,t}$ and $r_{\infty,t}(U)$]. We now give two results about the compactness of linear operators on these Banach spaces.

PROPOSITION 5.1. *Assume* (2.II). *A bounded linear operator $U = (u_{i,j})$ on l^1 is compact if and only if*

$$\lim_{n\to\infty}\left(\sup_{j\in\mathbf{G}^*}\sum_{i\in\mathbf{G}^*/t(i)>n}|u_{i,j}|\right) = 0.$$

For the proof: In the previous formula, the limit does not depend on the map t, provided this map fulfills (2.II). This result therefore directly stems from page 278 in Taylor and Lay (1958).

PROPOSITION 5.2. *Let $U = (u_{i,j})$ be a linear operator on l^1 and U_t be the operator on l_t^1 linked to the matrix $\{(t(j)/t(i))u_{i,j}\}$. Then, U is bounded (resp., compact) on l^1 if and only if U_t is bounded (resp., compact) on l_t^1. In particular, a bounded linear operator $U = (u_{i,j})$ on l_t^1 is compact if and only if*

$$\lim_{n\to\infty}\left[\sup_{j\in\mathbf{G}^*}\sum_{i\in\mathbf{G}^*/t(i)>n}\frac{t(i)}{t(j)}|u_{i,j}|\right] = 0.$$

The proof is straightforward and left to the reader.

5.3. *Quasi compactness and spectral properties of the operators ${}^tQ$ on l_t^1 and Q on l_t^∞.* We now return to the substochastic matrix Q, restriction of the transition matrix P to $\mathbf{G}^*$ (see Section 2). We have chosen to work on the Banach space l_t^∞ instead of l^∞ because sufficient conditions for the compactness of the operator Q on l^∞ found in pages 24 and 25 of Gosselin (1993), are more restrictive than the results to come and those in Gosselin (1998a). First, we obtain the following.

PROPOSITION 5.3. *Let $\mathbf{G}^*$ be infinite. Under assumptions* (2.I), (2.II), (2.III), (2.IV) *and* (3.I)*, the operators Q on l_t^∞ and ${}^tQ$ on l_t^1 are quasi-compact.*

PROOF. First, the operator Q is then bounded on l_t^∞ (Proposition 5.2). Second, denote by Q_n the finite rank operator on l_t^∞ linked to the matrix on $\mathbf{G}^{*2}$ having its "first" (according to t) rows defined by $(p_{i,j})_{(i,j)\in\mathbf{G}^{*2},\, t(i)\le n}$, its other rows being null. We get

$$\|Q - Q_n\|_{\infty,t} = \sup_{i\in\mathbf{G}^*/t(i)>n}\sum_{j\in\mathbf{G}^*}\frac{t(j)}{t(i)}p_{i,j} = \sup_{i\in\mathbf{G}^*/t(i)>n}{}_{(i)}m_t.$$

From $\limsup_{t(i)\to\infty\,(i)} m_t < r(Q)$, it results that for n sufficiently large,

$$\|Q - Q_n\|_{\infty,\,t} < r(Q),$$

which, together with the compactness of the operator Q_n on l_t^∞ (Propositions 5.1 and 5.2), proves the quasi compactness of Q on l_t^∞ and that of ${}^tQ$ on l_t^1. $\square$

We next prove in Proposition 5.4 a first series of spectral properties of the operators ${}^tQ$ on l_t^1 and Q on l_t^∞, when they are quasi-compact.

PROPOSITION 5.4. *Assume* (2.I), (2.II), (2.III), (2.IV), (3.I). *For every k in $\mathbf{I}$, denote by r_k the spectral radius of the operator $Q_{\mathbf{G}_k}$ on $l^\infty_{\mathbf{G}_k,\,t} = \{x \in \mathbf{C}^{\mathbf{G}_k}/ \sup_{j\in\mathbf{G}_k} t(j)^{-1}|x_j| < \infty\}$. Then:*

(a) *$r(Q)$ is in $(0,1)$ and there is at least one, and at most a finite number of, k in $\mathbf{I}^*$ such that $r_k = \sup_{h\in\mathbf{I}^*} r_h = r(Q)$.*

(b) *There are r in $\mathbf{N}^*$ and a partition $(\mathbf{G}'_s)_{s\in\{1,2,\ldots,2r+1\}}$ of $\mathbf{G}^*$ such that*

$$Q = \begin{pmatrix} Q_{\mathbf{G}'_1} & 0 & \cdots & 0 & 0 \\ Q_{\mathbf{G}'_2\to\mathbf{G}'_1} & Q_{\mathbf{G}'_2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ Q_{\mathbf{G}'_{2r}\to\mathbf{G}'_1} & Q_{\mathbf{G}'_{2r}\to\mathbf{G}'_2} & \cdots & Q_{\mathbf{G}'_{2r}} & 0 \\ Q_{\mathbf{G}'_{2r+1}\to\mathbf{G}'_1} & Q_{\mathbf{G}'_{2r+1}\to\mathbf{G}'_2} & \cdots & Q_{\mathbf{G}'_{2r+1}\to\mathbf{G}'_{2r}} & Q_{\mathbf{G}'_{2r+1}} \end{pmatrix},$$

where (i) *for every s in $\{0,1,\ldots,r\}$, $\mathbf{G}'_{2s+1}$ is the union of the classes $\mathbf{G}_h$ in $\mathbf{G}^*$ with spectral radius strictly less than $r(Q)$, such that $\mathbf{G}_{h'}$ leads to $\mathbf{G}'_{2s}$ (only if $s > 0$) but not to $\mathbf{G}'_{2s+2}$ (only if $s < r$). $\mathbf{G}'_{2s+1}$ may be empty, in which case the corresponding matrices are not present in the above matrix, and, if not, $r(Q_{\mathbf{G}'_{2s+1}}) < r(Q)$; and* (ii) *for every s in $\{1,2,\ldots,r\}$, $\mathbf{G}'_{2s}$ is in $\{\mathbf{G}_h\}_{h\in\mathbf{I}}$, $r(Q_{\mathbf{G}'_{2s}}) = r(Q)$ and $\mathbf{G}'_{2s}$ does not lead to $\bigcup_{s'=s+1}^{r}\mathbf{G}'_{2s'}$;*

(c) *The operator ${}^tQ$ on l_t^1 (resp., Q on l_t^∞) has at least a nonnegative eigenvector associated with the eigenvalue $r(Q)$. For every such eigenvector v (resp., u), there is s (resp., s'') in $\{1,2,\ldots,r\}$ such that $v_i = 0$ for every i in $\bigcup_{s'=2s+1}^{2r+1}\mathbf{G}'_{s'}$ (resp., $u_i = 0$ for every i in $\bigcup_{s'=1}^{2s''-1}\mathbf{G}'_{s'}$) and $(v_i)_{i\in\mathbf{G}'_{2s}}$ (resp., $(u_i)_{i\in\mathbf{G}'_{2s''}}$) is an eigenvector of the operator ${}^tQ_{\mathbf{G}'_{2s}}$ (resp., $Q_{\mathbf{G}'_{2s''}}$) on $l^1_{\mathbf{G}'_{2s},\,t}$ (resp., $l^\infty_{\mathbf{G}'_{2s''},\,t}$) associated with the eigenvalue $r(Q)$.*

PROOF. First, from (2.II) and Gel'fand's theorem, $r(Q) > 0$. We now suppose that the index set $\mathbf{I}$ is finite. We can then write Q the following way, possibly by reordering $\mathbf{I}$:

$$Q = \begin{pmatrix} Q_{\mathbf{G}_1} & 0 & \dots & 0 \\ Q_{\mathbf{G}_2\to\mathbf{G}_1} & Q_{\mathbf{G}_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ Q_{\mathbf{G}_\kappa\to\mathbf{G}_1} & Q_{\mathbf{G}_\kappa\to\mathbf{G}_2} & \cdots & Q_{\mathbf{G}_\kappa} \end{pmatrix}.$$

Then, for every n in $\mathbf{N}$,

$$Q^n = \begin{pmatrix} Q^n_{\mathbf{G}_1} & 0 & \dots & 0 \\ \vdots & Q^n_{\mathbf{G}_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \dots & \dots & Q^n_{\mathbf{G}_\kappa} \end{pmatrix},$$

which, together with the fact that the entries of Q are nonnegative, implies $r(Q) \geq \max_{h\in\mathbf{I}} r_h$. Furthermore, due to $r(Q) > 0$ and the quasi compactness of ${}^tQ$ on l^1_t (Proposition 5.3), from Theorem 3 in Sasser (1964), there exists a nonnegative eigenvector v (resp., u) of ${}^tQ$ (resp., Q) in l^1_t (resp., l^∞_t) associated with the eigenvalue $r(Q)$. Then, let us denote by k the smallest element in $\mathbf{I}$ such that there is j in $\mathbf{G}_k$ with $u_j > 0$. Due to the shape of Q, $(u_j)_{j\in\mathbf{G}_k}$ is a right eigenvector of $Q_{\mathbf{G}_k}$ associated with the eigenvalue $r(Q)$. Acting similarly with v, we find (c) and, from the above, $r_k = r(Q)$. By regrouping the sets $\mathbf{G}_h$ such that $r_h < r(Q)$, and by keeping alone the sets $\mathbf{G}_h$ with $r_h = r(Q)$, we easily find the shape of Q asserted in (b).

Now suppose that $\mathbf{I}$ is infinite, for example $I = N^*$, in which case Q has the following shape:

$$Q = \begin{pmatrix} Q_{\mathbf{G}_1} & Q_{\mathbf{G}_1\to\mathbf{G}_2} & \cdots & Q_{\mathbf{G}_1\to\mathbf{G}_h} & \cdots \\ Q_{\mathbf{G}_2\to\mathbf{G}_1} & Q_{\mathbf{G}_2} & \ddots & \vdots & \cdots \\ \vdots & \vdots & \ddots & Q_{\mathbf{G}_{h-1}\to\mathbf{G}_h} & \ddots \\ Q_{\mathbf{G}_h\to\mathbf{G}_1} & Q_{\mathbf{G}_h\to\mathbf{G}_2} & \cdots & Q_{\mathbf{G}_h} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us denote for every h in $\mathbf{I} = \mathbf{N}^*$, $\mathbf{F}_h = \bigcup_{h'=1}^h \mathbf{G}_{h'}$ and $\mathbf{H}_h = \mathbf{G}^* - \mathbf{F}_h$. Given the properties of the sets $\{\mathbf{G}_h\}_{h\in\mathbf{I}}$, we cannot with a positive probability leave and then return to $\mathbf{F}_h$ more than $h-1$ times. Then, denoting for $n \geq 2$ $Q_{\mathbf{F}_h,\mathbf{H}_h}(n) = \sum_{n'=0}^{n-2} Q_{\mathbf{F}_h\to\mathbf{H}_h} Q^{n'}_{\mathbf{H}_h} Q_{\mathbf{H}_h\to\mathbf{F}_h} Q^{n-2-n'}_{\mathbf{F}_h}$, we get for every $n \geq 2$,

$$(5.1)\quad (Q^n)_{\mathbf{F}_h} = \sum_{r=0}^{h-1} \sum_{\substack{n_0,\dots,n_r \\ n_0\geq 0,\, n_r=n, \\ n_s+2\leq n_{s+1},\, s\in\{0,\dots,r-1\}}} Q^{n_0}_{\mathbf{F}_h} Q_{\mathbf{F}_h,\mathbf{H}_h}(n_1-n_0)\cdots Q_{\mathbf{F}_h,\mathbf{H}_h}(n_r-n_{r-1}),$$

with, for every $n \geq 2$,

$$\|Q_{\mathbf{F}_h,\mathbf{H}_h}(n)\|_{\infty,t} \leq \|Q_{\mathbf{F}_h\to\mathbf{H}_h}\|_{\infty,t}\|Q_{\mathbf{H}_h\to\mathbf{F}_h}\|_{\infty,t} \times \sum_{\nu=0}^{n-2}\|Q^{\nu}_{\mathbf{F}_h}\|_{\infty,t}\|Q_{\mathbf{H}_h}\|^{n-2-\nu}_{\infty,t}. \tag{5.2}$$

Let us first prove that for every h in $\mathbf{I}$ either $r(Q_{\mathbf{F}_h}) \geq r(Q)$ or $r(Q_{\mathbf{H}_h}) \geq r(Q)$. Indeed, from Gel'fand's theorem, for every $\varepsilon' > 0$, there is $K_{h,\varepsilon'} > 0$ such that

$$\|Q^n_{\mathbf{F}_h}\|_{\infty,t} \leq K_{h,\varepsilon'}(r(Q_{\mathbf{F}_h}) + \varepsilon')^n$$

and

$$\|Q^n_{\mathbf{H}_h}\|_{\infty,t} \leq K_{h,\varepsilon'}(r(Q_{\mathbf{H}_h}) + \varepsilon')^n, \qquad n \in \mathbf{N},$$

which involves due to (5.2),

$$\|Q_{\mathbf{F}_h,\mathbf{H}_h}(n)\|_{\infty,t} \leq \|Q_{\mathbf{F}_h,\mathbf{H}_h}\|_{\infty,t}\|Q_{\mathbf{H}_h,\mathbf{F}_h}\|_{\infty,t} \times K^2_{h,\varepsilon'}\big[\sup\{r(Q_{\mathbf{F}_h}); r(Q_{\mathbf{H}_h})\} + \varepsilon'\big]^{n-2}.$$

Then due to (5.1) and to similar results concerning $(Q^n)_{\mathbf{H}_h\to\mathbf{F}_h}$, $(Q^n)_{\mathbf{H}_h}$ and $(Q^n)_{\mathbf{F}_h\to\mathbf{H}_h}$, and since $\{(n_0,\dots,n_r); n_0 \geq 0, n_r = n, n_s + 2 \leq n_{s+1}, s \in \{0,\dots, r-1\}\}$ has $(n-r)!/[r!(n-2r)!]$ elements, we get that for every h in $\mathbf{I}$ and $\varepsilon' > 0$, there is $K'_{h,\varepsilon'} > 0$ such that

$$\|Q^n\|_{\infty,t} \leq K'_{h,\varepsilon'} n^h\big[\sup\{r(Q_{\mathbf{F}_h}); r(Q_{\mathbf{H}_h})\} + \varepsilon'\big]^n, \qquad n \in \mathbf{N}.$$

We deduce from this last inequality and from Gel'fand's theorem that either $r(Q_{\mathbf{F}_h}) \geq r(Q)$ or $r(Q_{\mathbf{H}_h}) \geq r(Q)$.

We then choose h such that $\|Q_{\mathbf{H}_h}\|_{\infty,t} < r(Q_{\mathbf{F}_h})$. Such an h exists due to the previous result and since (1) there is h' with $r(Q_{\mathbf{F}_{h'}}) > 0$ due to (2.III), (2) $(r(Q_{\mathbf{F}_{h'}}))_{h'\in\mathbf{N}^*}$ is nondecreasing, and (3) from (3.I), $\lim_{h\to\infty}\|Q_{\mathbf{H}_h}\|_{\infty,t} < r(Q)$. For such a value of h, the above results clearly imply $r(Q_{\mathbf{F}_h}) \geq r(Q)$. Since, from the part of this proof devoted to the case where $\mathbf{I}$ is finite, $r(Q_{\mathbf{F}_h}) = \max_{h'\in\{1,2,\dots,h\}} r_{h'} \leq r(Q)$, we get $r(Q) = r(Q_{\mathbf{F}_h}) > 0$, for every h such that $\|Q_{\mathbf{H}_h}\|_{\infty,t} < r(Q_{\mathbf{F}_h})$. Now, by (3.I), we get first, $\limsup_{h\to\infty} r_h \leq \limsup_{h\to\infty}\|Q_{\mathbf{G}_h}\|_{\infty,t} < r(Q)$ and second, there is K in $\mathbf{N}^*$ such that $h \geq K$ implies $r(Q_{\mathbf{F}_h}) = \sup_{h'\in\mathbf{N}^*} r_{h'} = \max_{h'\in\mathbf{N}^*} r_{h'}$. Consequently, $r(Q) = \max_{h\in\mathbf{N}^*} r_h > 0$ and $\max_{h\in\mathbf{N}^*} r_h$ is reached for at most a finite number of h, which we denote by $k_1, k_2, \dots, k_r$, ordered so that for every i in $\{1, 2, \dots, r-1\}$, $\mathbf{G}_{k_i}$ does not lead to $\cup_{j=i+1}\mathbf{G}_{k_j}$. Let us denote for every s in $\{1, 2, \dots, r\}$ $\mathbf{G}'_{2s} = \mathbf{G}_{k_s}$. Defining $\mathbf{G}'_{2s+1}$ as in the statement of Proposition 5.4(b), we can then write the matrix Q as in (b).

Applying the above arguments to the space $\mathbf{G}'_{2s+1}$, it follows that $r(Q_{\mathbf{G}'_{2s+1}}) = \max_{h\in\mathbf{N}^*/\mathbf{G}_h\in\mathbf{G}'_{2s+1}} r_h < r(Q)$. Relying on this shape of the matrix Q and on $r(Q) > 0$, (c) is a consequence of Theorem 3 in Sasser (1964).

We finally prove $r(Q) < 1$. Let us consider the vector e whose entries all equal one, and a nonnegative eigenvector v of ${}^tQ$ in l^1_t associated with the eigenvalue $r(Q)$. From ${}^tQv = r(Q)v$ and $e \in l_t^\infty$, which results from (2.II), we have

$$P_b(Z_n \in \mathbf{G}^*) = {}^te\,{}^tQ^n b = r(Q)^n,$$

where $b = v/(e, v)$ is a probability distribution on $\mathbf{G}^*$. Whence $r(Q) \le 1$, which implies, due to (3.I), $\limsup_{t(i)\to\infty\ (i)} m_t < 1$. But this last inequality itself involves, by Proposition 3.1, $\lim_{n\to\infty} P_b(Z_n \in \mathbf{G}^*) = 0$ and, thus, $r(Q) < 1$.

REMARK. Proposition 5.4 in particular implies that, under (2.I) to (2.IV), $r(Q) \ge 1$ or $\limsup_{t(i)\to\infty\ (i)} m_t \ge 1$ entails that, if $\mathbf{G}^*$ is infinite, $\limsup_{t(i)\to\infty\ (i)} m_t \ge r(Q)$.

6. Proof of Theorem 3.1. Under (3.II), we denote by k the unique element of $\mathbf{I}$ such that $r(Q_{\mathbf{G}_k}) = r(Q)$. We also denote $e = (1)_{i\in\mathbf{G}^*}$, for every j in $\mathbf{G}^*$ e_j the probability distribution in both l_t^∞ and l_t^1, such that $(e_j)_j = 1$, and $M = 1/\inf_{j\in\mathbf{G}^*} t(j)$, where t is the map introduced in (2.II). Thanks to (2.II), M is finite and e is in l_t^∞. Adding assumption (3.II), the results in Proposition 5.4 are supplemented as follows.

LEMMA 6.1. *Assume* (2.I)*,* (2.II)*,* (2.III)*,* (2.IV)*,* (3.I) *and* (3.II)*. Then*:

(a) *$r(Q)$ is the only peripheral eigenvalue of the operator ${}^tQ$ on l_t^1. We can write in a unique way*

$$ {}^tQ^n = r(Q)^n v\,{}^tu + {}^tS^n, \qquad n \ge 1, \tag{6.1}$$

where $r_{1,t}({}^tS) < r(Q)$, u and v are nonnegative vectors such that $v \in l_t^1$, $u \in l_t^\infty$, $\sum_{j\in\mathbf{G}^*} v_j = 1$, ${}^tSv = 0$, and $Su = 0$.

(b) *For every h in $\mathbf{I}$, either $\mathbf{G}_h$ does not lead to $\mathbf{G}_k$ (resp., $\mathbf{G}_k$ does not lead to $\mathbf{G}_h$) and $(u_j)_{j\in\mathbf{G}_h} \equiv 0$ (resp., $(v_j)_{j\in\mathbf{G}_h} \equiv 0$), or $\mathbf{G}_h$ leads to $\mathbf{G}_k$ (resp., $\mathbf{G}_k$ leads to $\mathbf{G}_h$) and $u_j > 0$ for any j in $\mathbf{G}_h$ (resp., $v_j > 0$ for any j in $\mathbf{G}_h$).*

PROOF. Under the conditions of Lemma 6.1, Proposition 5.4 applies. Therefore, $r(Q) \in (0, 1)$, the operators ${}^tQ$ on l_t^1 and Q on l_t^∞ are quasi-compact and have nonnegative eigenvectors, v and u, respectively, associated with the eigenvalue $r(Q)$. Moreover, from (3.II), Q has the following shape:

$$Q = \begin{pmatrix} Q_{\mathbf{G}'_1} & 0 & 0 \\ Q_{\mathbf{G}'_2\to\mathbf{G}'_1} & Q_{\mathbf{G}'_2} & 0 \\ Q_{\mathbf{G}'_3\to\mathbf{G}'_1} & Q_{\mathbf{G}'_3\to\mathbf{G}'_2} & Q_{\mathbf{G}'_3} \end{pmatrix},$$

where $\mathbf{G}'_2 = \mathbf{G}_k$ and $r(Q_{\mathbf{G}'_2}) = r(Q)$, and $\mathbf{G}'_1$ (resp., $\mathbf{G}'_3$), if not empty, satisfies $r(Q_{\mathbf{G}'_1}) < r(Q)$ [resp., $r(Q_{\mathbf{G}'_3}) < r(Q)$].

We next prove the uniqueness of the eigenvectors u and v associated with the eigenvalue $r(Q)$, as well as (b). Given the shape of the matrix Q and $r(Q_{\mathbf{G}'_1}) < r(Q)$, for u to be an eigenvector of Q on l_t^∞ associated with the eigenvalue $r(Q)$, $(u_j)_{j\in\mathbf{G}'_1}$ must be null and $(u_j)_{j\in\mathbf{G}'_2}$ a right eigenvector of $Q_{\mathbf{G}'_2}$

associated with the eigenvalue $r(Q)$. Since $Q_{\mathbf{G}'_2}$ is irreducible and the operator $Q_{\mathbf{G}'_2}$ on $l^{\infty}_{\mathbf{G}'_2, t}$ is quasi-compact, $(u_j)_{j \in \mathbf{G}'_2}$ is unique up to a constant factor and can be chosen such that for every i in $\mathbf{G}'_2$, $u_i > 0$ [see Theorem 3 in Sasser (1964) and Theorem 5.2, page 329, in Schaefer (1974)]. Additionally, since $r_{\infty, t}(Q_{\mathbf{G}'_3}) < r(Q)$,

$$(u_j)_{j \in \mathbf{G}'_3} = \sum_{n=0}^{\infty} \frac{Q^n_{\mathbf{G}'_3}}{r(Q)^{n+1}} Q_{\mathbf{G}'_3 \to \mathbf{G}'_2} (u_j)_{j \in \mathbf{G}'_2} \geq \frac{1}{r(Q)} Q_{\mathbf{G}'_3 \to \mathbf{G}'_2} (u_j)_{j \in \mathbf{G}'_2}.$$

Then, since $u_j > 0$ for every j in $\mathbf{G}'_2$, and since all the states in $\mathbf{G}'_3$ lead to $\mathbf{G}'_2$, we get $u_j > 0$, for any j in $\mathbf{G}'_3$. We hence reach (b), together with the uniqueness of the eigenvector u associated with the eigenvalue $r(Q)$. We similarly prove equivalent results for the eigenvector v of ${}^tQ$. Of course, since $u_j > 0$ and $v_j > 0$ for every j in $\mathbf{G}'_2$, we get $(u, v) > 0$.

Now, the peripheral spectrum of ${}^tQ$ on l^1_t is the same as the one of ${}^tQ_{\mathbf{G}'_2}$ on $l^1_{\mathbf{G}'_2, t}$. From, for example, Vere-Jones (1967), we can easily prove that the aperiodicity of the matrix ${}^tQ_{\mathbf{G}'_2}$ implies that $r(Q)$ is the only peripheral eigenvalue of both ${}^tQ_{\mathbf{G}'_2}$ and ${}^tQ$. Let us then show that $r(Q)$ is a simple eigenvalue of ${}^tQ$ on l^1_t and Q on l^{∞}_t. First, due to the properties of quasi-compact operators, $r(Q)$ is an isolated eigenvalue and a pole of the resolvant. Let ω be its index and ${}^t\Pi$ the associated spectral projection (see Section 5.1). As in the proof of Theorem 3 in Sasser (1964), ${}^t\Gamma = \{r(Q)I - {}^tQ\}^{\omega-1}\, {}^t\Pi$ is nonnegative and nonnull. Furthermore, by ${}^tQ\; {}^t\Gamma = r(Q)\; {}^t\Gamma$ and the uniqueness of v, there is x in l^1_t such that ${}^t\Gamma x = \gamma v$ with $\gamma > 0$. Then, due to ${}^t\Gamma = \lim_{\rho \downarrow r(Q)} \{\rho - r(Q)\}^{\omega} R(\rho, {}^tQ)$ and the shape of $R(\rho, {}^tQ)$ for $\rho > r(Q)$, we get as in the proof of Theorem 4 in Sasser (1964),

$$(u, {}^t\Gamma x) = (u, x) \lim_{\rho \downarrow r(Q)} \{\rho - r(Q)\}^{\omega - 1},$$

which, if $\omega > 1$, is equal to zero and contradicts $(u, {}^t\Gamma x) = \gamma(u, v) > 0$. This proves $\omega = 1$. Since $r(Q)$ is an isolated eigenvalue and a pole of index one of the operator ${}^tQ$ on l^1_t to which corresponds a unique (up to a factor) eigenvector v in l^1_t, it is a simple eigenvalue of the operators ${}^tQ$ on l^1_t and Q on l^{∞}_t. The rest of the proof of Theorem 4 in Sasser (1964) also holds, and we get (a). $\square$

PROOF OF THEOREM 3.1. Under the conditions of Theorem 3.1, Lemma 6.1 yields

$${}^tQ^n = r(Q)^n v\, {}^tu + {}^tS^n, \qquad n \in \mathbf{N},$$

where, since $r({}^tS) < r(Q)$, there is $\varepsilon \in (0, 1)$ such that $\lim_{n \to \infty} r(Q)^{-n} \times (1 - \varepsilon)^{-n} \|{}^tS^n\|_{1, t} = 0$. Moreover, $(u, \pi) > 0$ since π is in C_k. Assertion (a) in Theorem 3.1 is then easily proved based on the above spectral decomposition and on

$$1 - P_{\pi}(Z_n \in \mathbf{G}_0) = {}^te\, {}^tQ^n \pi.$$

Recall $|(y, x)| \leq \|y\|_{\infty, t}\|x\|_{1, t}$. We additionally set $\varphi_j = u_j$, $\varphi'_j(n) = r(Q)^{-n} \times {}^t e\, {}^t S^n e_j$, $\mu_j = (u, \pi) v_j$, $\mu'_j(n) = r(Q)^{-n}\, {}^t e_j\, {}^t S^n \pi$, and $\chi = (u, \pi)$. Then, of course, $\varphi = \{\varphi_j\}_{j \in \mathbf{G}^*} = u$ and $\varphi'(n) = r(Q)^{-n} S^n e$ are in $l^\infty_{\mathbf{G}^*, t}$, while $\mu = (u, \pi) v$ and $\mu'(n) = r(Q)^{-n}\, {}^t S^n \pi$ are in l^1_t.

(b) For $n'' \geq 0$, we now study the l^1_t - difference between the vectors $\alpha_\pi(n, n'')$ with components

$$\alpha_{\pi, j}(n, n'') = \frac{{}^t e\, {}^t Q^{n''} e_j\, {}^t e_j\, {}^t Q^n \pi}{{}^t e\, {}^t Q^{n+n''} \pi} = \frac{\left(1 - p^{(n'')}_{j, 0}\right) r(Q)^n \left[\mu_j + \mu'_j(n)\right]}{r(Q)^{n+n''} \chi + {}^t e\, {}^t S^{n+n''} \pi}$$

and

$$\alpha'_\pi(n'') = \left\{ \frac{(1 - p^{(n'')}_{j, 0}) \mu_j}{r(Q)^{n''} \chi} \right\}_{j \in \mathbf{G}^*},$$

apparently the limit of $\alpha_\pi(n, n'')$ when n tends to infinity. Since $0 \leq p^{(n'')}_{i, 0} = \sum_{j \in \mathbf{G}_0} p^{(n'')}_{i, j} \leq 1$, $i \in \mathbf{G}^*$, we get

$$\begin{aligned}
\|\alpha_\pi(n, n'') - \alpha'_\pi(n'')\|_{1, t} &\leq \left\| r(Q)^n \mu \right\|_{1, t} \left| \frac{1}{r(Q)^{n+n''} \chi} - \frac{1}{{}^t e\, {}^t Q^{n+n''} \pi} \right| \\
&\quad + \frac{\left\| {}^t S^n \right\|_{1, t} \|\pi\|_{1, t}}{{}^t e\, {}^t Q^{n+n''} \pi} \\
&\leq \frac{\left\| {}^t S^n \right\|_{1, t} \|\pi\|_{1, t}}{r(Q)^{n+n''}} \\
&\quad \times \frac{1 + \frac{\|\mu\|_{1, t}}{r(Q)^{n''} \chi} \|e\|_{\infty, t} \left\| {}^t S^{n''} \right\|_{1, t}}{\chi - r(Q)^{-n-n''} \|e\|_{\infty, t} \left\| {}^t S^{n+n''} \right\|_{1, t} \|\pi\|_{1, t}}.
\end{aligned}$$

Given $\|e\|_{\infty, t} = M < \infty$ and $\lim_{n \to \infty} r(Q)^{-n} (1 - \varepsilon)^{-n} \|{}^t S^n\|_{1, t} = 0$, we get

$$\|\alpha_\pi(n, n'') - \alpha'_\pi(n'')\|_{1, t} = o[(1 - \varepsilon)^n].$$

We prove the result about $\{\alpha_{\pi, j}(n, -n'')\}$ with the same methods.

Now considering n'' only in $\mathbf{N}^*$,

$$\delta_{\pi, j}(n, n'') = \frac{p^{(n'')}_{j, 0}\, {}^t e_j\, {}^t Q^n \pi}{\sum_{i \in \mathbf{G}^*} p^{(n'')}_{i, 0}\, {}^t e_j\, {}^t Q^n \pi} = \frac{p^{(n'')}_{j, 0} \mu_j + p^{(n'')}_{j, 0} \mu'_j(n)}{\sum_{i \in \mathbf{G}^*} p^{(n'')}_{i, 0} \mu_i + \sum_{i \in \mathbf{G}^*} p^{(n'')}_{i, 0} \mu'_i(n)}.$$

Since ${}^t Q \mu = r(Q) \mu$ (Lemma 6.1), we get

$$\sum_{i \in \mathbf{G}^*} p^{(n'')}_{i, 0} \mu_i = \sum_{i \in \mathbf{G}^*} \left(1 - \sum_{i' \in \mathbf{G}^*} p^{(n'')}_{i, i'}\right) \mu_i = \chi - r(Q)^{n''} \chi > 0.$$

Relying on $\left\| \sum_{i \in \mathbf{G}^*} p^{(n'')}_{i, 0} e_i \right\|_{\infty, t} \leq M < \infty$ and $0 \leq p^{(n'')}_{i, 0} \leq 1$, $i \in \mathbf{G}^*$, we obtain the convergence of $\delta_\pi(n, n'')$ as that of $\alpha_\pi(n, n'')$.

(c) From

$$\tau_{\pi,\,j}(n) = \frac{r(Q)^n \sum_{v=1}^{n}\{\mu_j + \mu'_j(v)\}\{\varphi_j + \varphi'_j(n-v)\}}{n(r(Q)^n\chi + {}^te^tS^n\pi)},$$

we infer that $\tau_{\pi,\,j}(n)$ will tend to $(\varphi_j\mu_j/\chi) = u_jv_j$ when n tends to infinity. Actually first, since $u \in l_t^\infty$ and $v \in l_t^1$, $(u_jv_j)_{j\in\mathbf{G}^*}$ is in l^1. Second, denoting

$$\Xi_n = \sum_{\nu=1}^{n}\sum_{j\in\mathbf{G}^*}\{|\varphi_j\mu'_j(\nu)| + |\mu_j\varphi'_j(n-\nu)| + |\mu'_j(\nu)\varphi'_j(n-\nu)|\},$$

we get

$$\sum_{j\in\mathbf{G}^*}|\tau_{\pi,\,j}(n) - u_jv_j| \le \frac{n^{-1}\Xi_n + \chi^{-1}r(Q)^{-n}|{}^te\,{}^tS^n\pi|}{\chi - r(Q)^{-n}|{}^te\,{}^tS^n\pi|}.$$

But, $r(Q)^{-n}|{}^te\,{}^tS^{-n}\pi| \le r(Q)^{-n}\|e\|_{\infty,\,t}\|{}^tS^{-n}\|_{1,\,t}\|\pi\|_{1,\,t}$ and

$$\Xi_n \le \sum_{\nu=1}^{n}\big\{\|\varphi\|_{\infty,\,t}\|\mu'(\nu)\|_{1,\,t} + \|\varphi'(n-\nu)\|_{\infty,\,t}\|\mu\|_{1,\,t} + \|\varphi'(n-\nu)\|_{\infty,\,t}\|\mu'(\nu)\|_{1,\,t}\big\},$$

where

$$\|\mu'(\nu)\|_{1,\,t} \le r(Q)^{-\nu}\|{}^tS^\nu\|_{1,\,t}\|\pi\|_{1,\,t}$$

and

$$\|\varphi'(\nu)\|_{\infty,\,t} \le r(Q)^{-\nu}\|{}^tS^\nu\|_{1,\,t}\|e\|_{\infty,\,t}.$$

From

$$\lim_{n\to\infty} r(Q)^{-n}(1-\varepsilon)^{-n}\|{}^tS^n\|_{1,\,t} = 0,$$

it then follows that $(\Xi_n)_{n\in\mathbf{N}}$ is a bounded sequence. Together with $\chi = (u, \pi) > 0$, this shows $\sum_{j\in\mathbf{G}^*}|\tau_{\pi,\,j}(n) - u_jv_j| = O(1/n)$.

We obtain the last part of (c) as the previous result, by writing

$$\alpha_{\pi,\,j}(n, n') = \frac{\mu_j\varphi_j + \mu'_j(n)\varphi_j + \mu_j\varphi'_j(n') + \mu'_j(n)\varphi'_j(n')}{\chi + r(Q)^{-n-n'}\,{}^te\,{}^tS^{n+n'}\pi}.$$

We now show the first part of (c). From the convergence of $\{\tau_{\pi,\,j}(n)\}$, we get,

$$\lim_{n\to\infty}\big|\mathbf{E}_\pi[b_j(n)\big|Z_n \in \mathbf{G}^*] - u_jv_j\big| = 0, \qquad j \in \mathbf{G}^*,$$

where $b_j(n) = \sum_{\nu=1}^{n}\gamma_j(\nu)/n$ with $\gamma_j(\nu) = 1_{\{Z_\nu = j\}}$. As Reddingius (1971), we then study $\mathrm{Var}_\pi[b_j(n)|Z_n \in \mathbf{G}^*]$, which can be written as

$$\mathrm{Var}_\pi[b_j(n)|Z_n \in \mathbf{G}^*] = \frac{1}{n^2}\sum_{\nu=1}^{n}\mathrm{Var}_\pi[\gamma_j(\nu)|Z_n \in \mathbf{G}^*] + \frac{2}{n^2}\sum_{\nu=1}^{n-1}\sum_{\iota=\nu+1}^{n}\mathrm{Cov}_\pi[\gamma_j(\nu), \gamma_j(\iota)|Z_n \in \mathbf{G}^*].$$

First, $\mathrm{Var}_\pi[\gamma_j(\nu)|Z_n \in \mathbf{G}^*] \leq 1$ implies

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{\nu=1}^{n} \mathrm{Var}_\pi[\gamma_j(\nu)|Z_n \in \mathbf{G}^*] = 0.$$

Second, denoting by

$$\chi_{\pi,\,j}(\nu, n) = \mathbf{E}_\pi[\gamma_j(\nu)|Z_n \in \mathbf{G}^*] = P_\pi(Z_\nu = j|Z_n \in \mathbf{G}^*),$$

we obtain

$$\begin{aligned}
&\mathrm{Cov}_\pi[\gamma_j(\nu), \gamma_j(\iota)|Z_n \in \mathbf{G}^*] \\
&\quad = \chi_{\pi,\,j}(v, n)\chi_{\pi,\,j}(\iota, n)\ P_\pi(Z_\nu \neq j, Z_\iota \neq j|Z_n \in \mathbf{G}^*) \\
&\qquad + \cdots + \{1 - \chi_{\pi,\,j}(\nu, n)\}\{1 - \chi_{\pi,\,j}(\iota, n)\}\ P_\pi(Z_\nu = j, Z_\iota = j|Z_n \in \mathbf{G}^*) \\
&\qquad - \cdots - \{1 - \chi_{\pi,\,j}(\nu, n)\}\ \chi_{\pi,\,j}(\iota, n)\ P_\pi(Z_\nu = j, Z_\iota \neq j|Z_n \in \mathbf{G}^*) \\
&\qquad - \cdots - \ \chi_{\pi,\,j}(\nu, n)\ \{1 - \chi_{\pi,\,j}(\iota, n)\}\ P_\pi(Z_\nu \neq j, Z_\iota = j|Z_n \in \mathbf{G}^*).
\end{aligned}$$

Now, as above, there is $K_{\pi,\,j} \in (0;\infty)$ such that, if we denote $\kappa_{\pi,\,j}(n) = K_{\pi,\,j}(1-\varepsilon)^n$,

$$\begin{aligned}
|P_\pi(Z_\iota \neq j|Z_\nu \neq j, Z_n \in \mathbf{G}^*) - 1 + u_j v_j| &\leq \kappa_{\pi,\,j}[\min(\iota - \nu; n - \iota)], \\
|P_\pi(Z_\iota \neq j|Z_\nu = j, Z_n \in \mathbf{G}^*) - 1 + u_j v_j| &\leq \kappa_{\pi,\,j}[\min(\iota - \nu; n - \iota)], \\
|P_\pi(Z_\iota = j|Z_v \neq j, Z_n \in \mathbf{G}^*) - u_j v_j| &\leq \kappa_{\pi,\,j}[\min(\iota - \nu; n - \iota)]
\end{aligned}$$

and

$$|P_\pi(Z_\iota = j|Z_\nu = j, Z_n \in \mathbf{G}^*) - u_j v_j| \leq \kappa_{\pi,\,j}[\min(\iota - \nu; n - \iota)].$$

Since, for instance,

$$\begin{aligned}
&P_\pi(Z_\iota \neq j, Z_\nu \neq j|Z_n \in \mathbf{G}^*) \\
&\quad = P_\pi(Z_\iota \neq j|Z_\nu \neq j, Z_n \in \mathbf{G}^*)\{1 - \chi_{\pi,\,j}(\nu, n)\},
\end{aligned}$$

we then obtain

$$\begin{aligned}
&\mathrm{Cov}_\pi(\gamma_j(\nu), \gamma_j(\iota)|Z_n \in \mathbf{G}^*) \\
&\quad = \xi_{\pi,\,j}(\iota, \nu, n) + \{\chi_{\pi,\,j}(\iota, n)(1 - u_j v_j) \\
&\qquad - (1 - \chi_{\pi,\,j}(\iota, n))u_j v_j\}[\chi_{\pi,\,j}(\nu, n)\{1 - \chi_{\pi,\,j}(\nu, n)\} \\
&\qquad - \{1 - \chi_{\pi,\,j}(\nu, n)\}\chi_{\pi,\,j}(\nu, n)]
\end{aligned}$$

where $|\xi_{\pi,\,j}(\iota, \nu n)| \leq 4\kappa_{\pi,\,j}[\min(\iota - \nu; n - \iota)]$. Because the last term of the above equation is null, we get

$$|\mathrm{Cov}_\pi[\gamma_j(\nu), \gamma_j(\iota)|Z_n \in \mathbf{G}^*]| \leq 4\kappa_{\pi,\,j}[\min(\iota - \nu; n - \iota)].$$

Then, using exactly the same arguments as in page 74 of Reddingius (1971), we reach

$$\lim_{n\to\infty} \mathrm{Var}_\pi[b_j(n)|Z_n \in \mathbf{G}^*] = 0.$$

This implies that $b_j(n)$ conditional on $Z_n \in \mathbf{G}^*$ tends to $u_j v_j$ in mean square, from which we easily deduce that it converges to $u_j v_j$ in probability, that is,

$$\lim_{n\to\infty} P_\pi[|b_j(n) - u_j v_j| > \alpha | Z_n \in \mathbf{G}^*] = 0 \quad \text{for any } \alpha > 0.$$

This result holds for every j in $\mathbf{G}^*$. We now prove that this implies the first result in (c); that is,

$$\lim_{n\to\infty} P_\pi\left[\sum_{j\in\mathbf{G}^*} |b_j(n) - u_j v_j| > \alpha | Z_n \in \mathbf{G}^*\right] = 0 \quad \text{for any } \alpha > 0,$$

which is not obvious when $\mathbf{G}^*$ is infinite. First, notice that, conditional on $Z_n \in \mathbf{G}^*$, both $\{b_j(n)\}_{j\in\mathbf{G}^*}$ and $\{u_j v_j\}_{j\in\mathbf{G}^*}$ are probability distributions. Fixing $\alpha \in (0;1)$, there is therefore a finite subset $\mathbf{G}' = \{i_1, i_2, \dots, i_g\}$ of $\mathbf{G}^*$ such that

$$\sum_{j\in\mathbf{G}^*-\mathbf{G}'} u_j v_j < \frac{\alpha}{4},$$

for which

$$P_\pi\left[\sum_{j\in\mathbf{G}'} |b_j(n) - u_j v_j| > \frac{\alpha}{4} | Z_n \in \mathbf{G}^*\right]$$
$$\leq \sum_{s=1}^{g} P_\pi\left[|b_{i_s}(n) - u_{i_s} v_{i_s}| > \frac{\alpha}{4g} | Z_n \in \mathbf{G}^*\right].$$

From the above, we get

$$\lim_{n\to\infty} P_\pi\left[\sum_{j\in\mathbf{G}'} |b_j(n) - u_j v_j| > \frac{\alpha}{4} \middle| Z_n \in \mathbf{G}^*\right] = 0.$$

Now, still conditional on $Z_n \in \mathbf{G}^*$, if $\sum_{j\in\mathbf{G}'} |b_j(n) - u_j v_j| \leq \alpha/4$, it results from $\sum_{j\in\mathbf{G}^*-\mathbf{G}'} u_j v_j < \alpha/4$ that

$$1 - \frac{\alpha}{2} \leq \sum_{j\in\mathbf{G}'} b_j(n) \leq 1 \quad \text{and} \quad \sum_{j\in\mathbf{G}^*} |b_j(n) - u_j v_j| \leq \alpha.$$

We hence reach

$$\lim_{n\to\infty} P_\pi\left[\sum_{j\in\mathbf{G}^*} |b_j(n) - u_j v_j| > \alpha \middle| Z_n \in \mathbf{G}^*\right]$$
$$\leq \lim_{n\to\infty} p_\pi\left[\sum_{i\in G'} |b_j(n) - u_j v_j| > \frac{\alpha}{4} \middle| Z_n \in \mathbf{G}^*\right] = 0,$$

which proves the first part of (c). □

7. Final remarks. According to Seneta and Vere-Jones (1966), the spectral theory, upon which this paper heavily relied, "would give more detailed information about a somewhat narrower class of problems" than iterated function techniques. In view of the above results, this remark is both true and false.

On the one hand, the results in Sections 3 and 4, which generalize the results known for BGW branching processes [see Athreya and Ney (1972) and Asmussen and Hering (1983)] and absorbing Markov chains having a R-positive irreducible matrix Q [see Seneta and Vere-Jones (1966) and Buiculescu (1975)] to more general Markov chains under supplementary conditions [essentially (3.I) and (3.II)] are more restrictive than previous results. They indeed apply to "a somewhat narrower class of problems," in the case of these two kinds of absorbing Markov chains but also of BGW branching processes in a random environment [see Athreya and Karlin (1971)] and multitype BGW branching processes [see Joffe and Spitzer (1967) and Buiculescu (1975), Gosselin (1997, 1998b)]. For instance, considering BGW branching processes, the present results are not as general as classical ones since they further require $\sum_{k\in\mathbf{N}} k^2 p_k < \infty$ (Section 4.3).

On the other hand, the main advantage of the present results is that they generalize the results of Section IV.3.2 in Lebreton (1981) about the existence of a Yaglom limit for some population-size-dependent BGW branching processes, to more general population-size-dependent BGW branching processes (Section 4) and to models in which generalizing assumptions are made *simultaneously* [see Gosselin (1998b)].

Spectral techniques also allow the matrix Q to be reducible and yield very interesting results, such as the speeds of the convergence in Theorem 3.1 and the convergence in probability conditional on nonextinction at time n of the proportion of previous time steps spent in the different states j, $\{b_j(n)\}$. This last result is a new result for infinite absorbing Markov chains which has a stronger potential to be applied to a unique realization of (Z_n) that remains nonextinct long enough than the other convergences in distribution in Theorem 3.1.

We hope that the Lyapunov-type conditions required for the use of spectral techniques will be wide enough for practical applications in population extinction modelling. But this is not only a hope: this was the aim of this work. Indeed, the main difficulty in such applications will be to find a map t such that conditions (2.II), (2.IV), (3.I) and (3.II) are satisfied. But, contrary to, for example, Seneta and Vere-Jones (1966), the conditions for our results to hold [i.e., (2.I) to (3.II)] are explicit and *practical*, at least if Q is irreducible and if we seek a map t such that $\lim_{t(i)\to\infty\,(i)} m_t = 0$ instead of (3.I) [see Gosselin (1997, 1998b)].

This makes the present results resemble the results by Ferrari, Kesten and Martínez (1996) and Kesten (1995), even if the respective sufficient conditions do not seem to overlap. Indeed, Ferrari, Kesten and Martinez (1996) developed results for cellular automata, a framework that differs so much from ours that both series of results are barely comparable at the moment.

Kesten (1995) requires, among other conditions, the existence of $N < \infty$ and $\delta > 0$ such that $\lim_{i \in \mathbf{G}^*} \sup_{1 \le k \le N} p_{i,i}^{(k)} \ge \delta$, a condition not generally fulfilled by Markov chains used to model biological population extinction, which usually satisfy $\lim_{t(i) \to \infty} \sup_{1 \le k \le N} \; p_{i,i}^{(k)} = 0$ for every N. We think the conditions of this paper (2.I) to (3.II) are more relevant to model biological population extinctions [see Section VI in Gosselin (1997)].

Finally, additional results and comments of the above results can be found in Sections III.4, III.5, III.6, III.7, VI and Appendix 7 in Gosselin (1997) and in Gosselin (1998a,b). Especially, results concerning the convergence of most of the quantities in (3.1) are available when assumption (3.II) is replaced by:

(3 III) There is only one k in $\mathbf{I}$ such that $r(Q) = r(Q_{\mathbf{G}_k})$ and $Q_{\mathbf{G}_k}$ is not aperiodic.

or by:

(3.IV) There are r different k in $\mathbf{I}$ such that $r(Q_{\mathbf{G}_k}) = \max_{h \in \mathbf{I}} r(Q_{\mathbf{G}_h}) = r(Q)$, where r is in $\mathbf{N}^* - \{1\}$.
and

(3.V) For every k in $\mathbf{I}$ such that $r(Q_{\mathbf{G}_k}) = r(Q)$, the matrix $Q_{\mathbf{G}_k}$ is aperiodic.

Under (3.III), the absorbing Markov chain conditional on nonextinction embraces a periodic asymptotic behavior, while under (3.IV) and (3.V) the Markov chains behave asymptotically as the finite absorbing Markov chains studied by Mandl (1959) [see Gosselin (1998a)]. In this last case, the limits more often depend on the initial probability distribution π and the convergences are generally slower than in Theorem 3.1.

Acknowledgments. I thank very much Jean Jacod (Laboratoire de Probabilités, Université Paris 6) for his very relevant remarks that helped me to shape this manuscript, Jean-Dominique Lebreton (CEFE) for critical reading of various versions of the manuscript as well as for his constant support and Claude Millier (ENGREF) for his constant support, too. When I started this paper, I was on leave of absence from the French Institute of Forestry, Agricultural and Environmental Engineering (ENGREF, 19, Avenue du Maine, 75 732 Paris Cedex 15, France).

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