The Annals of Applied Probability
2001, Vol. 11, No. 1, 239–260

A MODEL FOR INVESTMENT DECISIONS WITH SWITCHING COSTS

BY KATE DUCKWORTH AND MIHAIL ZERVOS

King's College London

We address the problem of determining in an optimal way the sequence of times at which a firm can enter or exit an economic activity. In particular, we consider an investment model which involves production scheduling as well as a sequence of entry and exit decisions. The pricing of an investment conforming with this model gives rise to a stochastic impulse control problem that we explicitly solve. Our solution takes qualitatively different forms, depending on the problem's data.

1. Introduction. The past two decades have seen the emergence of new techniques in the area of investment pricing. These approaches extend the more classical discounted cash flow approaches to asset pricing such as the net present value approach, and can account for the value of managerial flexibility which is associated with every real-life investment. Their development has been facilitated by recent advances in the theories of stochastic optimal control and mathematical finance. The new research area has been termed real options and has been documented in several references including the books by Dixit and Pindyck [8] and Trigeorgis [19].

In this paper, we formulate and study, on a rigorous mathematical basis, an investment model that can be described informally as follows. An investment project can produce a single commodity whose price is modelled by a geometric Brownian motion. The project can operate in two modes: active and passive. In the active mode, the project yields payoff at a rate which depends on the commodity price as well as on the choice of a production rate which is a decision variable. In the passive mode, the project incurs losses at a constant rate. The transition from one mode to the other can be realised immediately at certain fixed costs, and constitutes an additional decision strategy. There is no limit to the number of times the project's mode can be changed. In practice, activation of the project can correspond to the case where a company expands its position in the market by developing a production unit, whereas rendering the project passive can correspond to the case where the company abandons the unit under consideration.

Variants of this model have attracted the interest of several researchers. The model itself was proposed by Brennan and Schwartz [5] as an approximation to a realistic model that they developed for the life cycle of an investment in the natural resource industry such as a copper mine or an oil field. Dixit [7] analysed the special case which arises if there is no choice of a production

Received October 1999; revised March 2000.
AMS 2000 *subject classifications*. Primary 93E20; secondary 60G40, 90A16.
Key words and phrases. Stochastic impulse control, real options.

rate, the rate of payoff that the investment yields in its active mode is a linear function and the investment yields neither profits nor losses in its passive mode (see also Dixit and Pindyck [8], Section 7.1). His analysis was developed using heuristic arguments and is based on deep intuition and insight. A rigorous analysis, though still not a complete one and of somewhat less explicit nature, of the same special case was developed by Shirakawa [17]. Both of these authors failed to unveil the full spectrum of optimal actions. Also, the same special case, but with the additional feature that the investment can produce at most a given amount of the commodity, was solved by Brekke and Øksendal [3, 4]. At this point, observe that allowing for much more general choices of the running payoff function is interesting because it offers more flexibility when modelling and pricing actual investments. Also, it is worth observing that more general choices of the running payoff function give rise to further qualitatively different optimal strategies (see also Remark 1 in Section 4). A further investment model of the same nature was analysed by Trigeorgis [18] under the assumption that the commodity price is modelled by a binomial tree. Other related models have been considered in the economics literature by Paddock, Siegel and Smith [14], who considered investments in the natural resource industry, Pindyck [15], who also considered the problem of capacity choice, Cortazar and Schwartz [6], who also considered existence of intermediate inventories, and McDonald and Siegel [13]. These, as well as a number of other models, can also be found in the books by Dixit and Pindyck [8] and Trigeorgis [19].

The problem that we solve is also closely related with the models analysed by Knudsen, Meister and Zervos [11] and Duckworth and Zervos [9]. These authors solve the problems which arise if there is no option of multiple entry and exit decisions, but the project can be totally abandoned or it can be activated and then totally abandoned at discretionary times, respectively. From the perspective of mathematical analysis, the solution of these problems proceeds through their reduction to appropriate optimal stopping problems. In contrast, the problem that we consider here has the structure of an explicitly solvable stochastic impulse control problem. In fact, it can easily be reduced to a special case of the most general model that Brekke and Øksendal [4] analyse. Of course, the maximal generality that these authors adopt cannot lead to results of such an explicit nature as the ones obtained in this paper.

With regard to economics considerations relating to the theory of real options, we address the problem of pricing the project considered from the perspective of the so-called dynamic programming approach. However, our model conforms with all of the assumptions made by Knudsen, Meister and Zervos [12]. As a consequence, our analysis also yields an expression for the so-called contingent claim price of the investment in a straightforward way.

The paper is organised as follows. Section 2 is concerned with the formulation of the investment model. In Section 3, we establish and discuss the associated dynamic programming differential equation. In Section 4, we explicitly solve the resulting stochastic impulse control problem under the assumption that the commodity price is a geometric Brownian motion, whereas, in the Appendix, we collect the proofs of some of the more technical results.

2. Problem formulation. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{C}$ the set of all $(\mathcal{F}_t)$-progressively measurable processes U with values in a compact subset of the real line $\mathcal{U}$ and denote by $\mathcal{Q}$ the family of all $(\mathcal{F}_t)$-adapted, finite variation, càglàd processes Q with values in $\{0, 1\}$.

We consider an investment that can produce a single commodity. We model the *commodity price* by the solution of the stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x > 0, \tag{1}$$

where the following assumption holds.

ASSUMPTION A1. b, σ: $\mathbb{R}_+ \to \mathbb{R}$ are given functions such that (1) has a unique strong solution with values in $]0, \infty[$, P-a.s.

Specific assumptions under which (1) has a unique solution can be found in any book that treats the subject of SDEs (e.g., see Corollary 5.5.16 and Proposition 5.5.17 in Karatzas and Shreve [10] or Theorem IX.3.5 in Revuz and Yor [16]), whereas Feller's test for explosions (Karatzas and Shreve [10], Theorem 5.5.29) yields necessary and sufficient conditions for the solution of (1) to have values in $]0, \infty[$.

The investment can operate in two modes, say "open" and "closed." The transition from one operating mode to the other is immediate and forms a sequence of decisions made by the management. These decisions form an *intervention strategy* and we model them by a process $Q \in \mathcal{Q}$. Specifically, $Q_t = 0$ means that the investment is closed at time t, whereas $Q_t = 1$ means that the investment is open at time t. Also, the stopping times at which the jumps of Q occur are the *intervention times* at which the investment's mode is changed. At time 0, we assume that the investment can be in either of the two modes that we denote by $q \in \{0, 1\}$.

Given that the investment is open, the management can decide on a *production rate* that we model by a process $U \in \mathcal{C}$. Assuming that the investment is open, if, for some $u \in \mathcal{U}$, $U_s = u$, for all s in a given time interval $[t, t + \Delta t]$, then the investment produces an amount of the commodity equal to $u\Delta t$ during this time. Also, we make the assumption that the production rate can be changed instantly to any value within the set of allowable values $\mathcal{U}$ without cost.

We define the set of *admissible strategies* to be

$$\Pi_q = \{(Q, U)\colon Q \in \mathcal{Q}, Q_0 = q, U \in \mathcal{C}\}.$$

With each admissible strategy $(Q, U) \in \Pi_q$, we associate the performance criterion

$$J_{q,x}(Q, U) = E\bigg[\int_0^\infty e^{-rs}\big[\bar{h}(X_s, U_s)Q_s - C(1 - Q_s)\big]ds - \sum_{0 \le s} e^{-rs}\big[K_I(\Delta Q_s)^+ + K_O(\Delta Q_s)^-\big]\bigg],$$

where $\Delta Q_t = Q_{t+} - Q_t$ and $(\Delta Q_t)^\pm = \max\{\pm \Delta Q_t, 0\}$. Here, $\bar{h}$: $]0, \infty[\times \mathscr{U} \to \mathbb{R}$ models the running payoffs/costs resulting from an open investment. If in a time interval $[t, t + \Delta t]$ the investment is open, the commodity price is x and the production rate is set at u, then, depending on whether $\bar{h}(x, u)$ is positive or negative, the investment yields a profit or incurs a loss equal to $\bar{h}(x, u)\Delta t$. The constant C models running "standby" costs resulting from a closed investment. Also, the constants K_I, K_O are the costs resulting from "switching" the investment from the closed to the open mode and vice versa, respectively.

The objective is to maximise the performance criterion $J_{q,x}(Q, U)$ over Π_q. Accordingly, we define the value function v by

$$v(q, x) = \sup_{(Q, U) \in \Pi_q} J_{q,x}(Q, U).$$

The following general assumption ensures that the optimisation problem is well posed, in the sense that there are no integrability problems and there are no admissible strategies with payoff equal to ∞.

ASSUMPTION A2. The running payoff function $\bar{h}$ is upper semicontinuous and if h: $]0, \infty[\to \mathbb{R}$ is the function defined by

$$h(x) = \max_{u \in \mathscr{U}} \bar{h}(x, u), \tag{2}$$

then

$$E\int_0^\infty e^{-rt}|h(X_t)|dt < \infty \tag{3}$$

for every initial condition $x > 0$. Also, $K_I + K_O > 0$.

Note that, since $\bar{h}$ is upper semicontinuous, h is upper semicontinuous as well (see Bertsekas and Shreve [2], Proposition 7.32). Moreover, Proposition 7.33 in [2] implies that there exists a Borel measurable u: $]0, \infty[\to \mathscr{U}$ such that

$$h(x) = \bar{h}(x, u(x)). \tag{4}$$

At this point, observe that although we have used C, K_I, K_O to model costs, we only require that $K_I + K_O > 0$. The reason is that this condition is sufficient for the development of our analysis. To some extent, negative values of these constants offer more flexibility in the modelling. For instance, a negative value of K_O can correspond to a situation where the cost of K_I, which now has to be strictly positive, is not totally sunken, but can be partly recovered when switching the investment to its closed mode.

On the other hand, we cannot dispense with the assumption that $K_I + K_O > 0$. If this condition failed, then the management would be able to generate arbitrarily high profits by rapidly changing the investment's operating mode, which would be unrealistic. From the perspective of mathematical modelling, this condition ensures that any intervention strategy which requires an infinite number of interventions within a finite time interval, on a set of positive probability, has payoff equal to $-\infty$. Therefore, every intervention strategy which is associated with a finite payoff can be modelled by a process in $\mathscr{D}$, and so the problem is well posed.

3. The Hamilton–Jacobi–Bellman (HJB) equations. Since the choice of a production rate does not affect the dynamics of the problem's state variable, namely the commodity price, we can expect that the optimal production rate process is indistinguishable from $u \circ X$, where u is a function satisfying (4). In light of this observation, we can see that the problem reduces to the choice of the intervention times at which the investment's operating mode is changed. Therefore, our problem can be viewed as a stochastic impulse control problem; for a treatment of impulse control theory, see Bensoussan and Lions [1].

With reference to standard results of the theory of impulse control, we expect that the value function v should satisfy the HJB equation which takes the form of the quasivariational inequality

$$\begin{aligned}\max\bigl\{&\tfrac{1}{2}\sigma^2(x)w_{xx}(q,x) + b(x)w_x(q,x) - rw(q,x) + qh(x) - (1-q)C,\\ &w(1-q,x) - w(q,x) - qK_O - (1-q)K_I\bigr\} = 0.\end{aligned} \tag{5}$$

To obtain some qualitative feeling about the origins of this equation, suppose that at time 0, the investment is open, that is, $q = 1$. In view of the foregoing discussion regarding the optimal choice of a production rate, the management's immediate decisions consist of choosing between two options. The first one is to produce according to the optimal rate $u \circ X$ for a short time Δt and then continue optimally. With reference to Bellman's principle of optimality, we must have

$$v(1,x) \geq E\biggl[\int_0^{\Delta t} e^{-rs}h(X_s)\,ds + e^{-r\Delta t}v(1, X_{\Delta t})\biggr]$$

for every $\Delta t > 0$. Here, we can have strict inequality because this is not necessarily the best option. Assuming that $v(1,\cdot)$ is sufficiently smooth, we can apply Itô's formula to the last term and divide by Δt before letting $\Delta t \downarrow 0$, to obtain

$$\tfrac{1}{2}\sigma^2(x)v_{xx}(1,x) + b(x)v_x(1,x) - rv(1,x) + h(x) \leq 0. \tag{6}$$

The second option is to pay the cost of K_O to switch the investment to its closed mode and then continue optimally. Therefore, we must have

$$v(1, x) \geq v(0, x) - K_O. \tag{7}$$

Since these two are the only options available, we can conclude that, given any value of $x > 0$, one of (6) or (7) should hold with equality. Therefore, we expect that the value function $v(1, \cdot)$ satisfies

$$\begin{aligned} \max \big\{ \tfrac{1}{2}\sigma^2(x) v_{xx}(1, x) + b(x) v_x(1, x) - r v(1, x) \\ + h(x),\ v(0, x) - v(1, x) - K_O \big\} = 0. \end{aligned} \tag{8}$$

Using similar reasoning, we can conclude that the value function $v(0, \cdot)$ satisfies

$$\begin{aligned} \max \big\{ \tfrac{1}{2}\sigma^2(x) v_{xx}(0, x) + b(x) v_x(0, x) \\ - r v(0, x) - C,\ v(1, x) - v(0, x) - K_I \big\} = 0. \end{aligned} \tag{9}$$

Now, combining (8) and (9), we conclude that the value function v of the control problem under consideration should be a solution of the HJB equation (5). In general, this equation can have uncountably many solutions even with $w(q, \cdot) \in C^\infty(]0, \infty[)$, $q = 0, 1$, as the following example reveals.

EXAMPLE 1. Suppose that $\sigma(x) = \sqrt{2}x$, $b(x) = x$, $r = 4$, $h(x) = x+4$, $C = 2$ and $K_I = K_O = 1$. It is straightforward to verify that each of the functions defined by

$$w(q, x) = Ax^2 + Bx^{-2} + \tfrac{1}{3}x + q,$$

where $A, B \in \mathbb{R}$, satisfies (8)–(9).

On the other hand, it turns out that, in general, the functions $v(q, \cdot)$, $q = 0, 1$, are not twice continuously differentiable. For this reason, we have to consider functions in a Sobolev space $W^{2,p}_{\text{loc}}(]0, \infty[)$ for some $p \geq 1$. (Note that the elements of this function space have sufficient smoothness to be able to apply Itô's formula.)

The following verification theorem provides conditions under which a solution of the HJB equation (5) can be identified with the value function v of our control problem. This is of the same nature as Theorem 3.4 in Brekke and Øksendal [4]. However, the assumptions that we make in the context of our special case can be verified more easily for the value function of the problem explicitly solved in the next section. On the other hand, it does not follow from the results of Bensoussan and Lions [1], because they assumed that $-h$ is lower bounded, which is not a reasonable assumption to impose in the context of our economic application.

THEOREM 1. *Consider the control problem described in Section* 2 *and assume that* A1 *and* A2 *hold. Suppose that the HJB equation* (5) *has a solution*

w: $\{0,1\}\times]0,\infty[\to\mathbb{R}$ *such that* $w(q,\cdot)\in W^{2,p}_{\mathrm{loc}}(]0,\infty[)$ *for some* $p\in[1,\infty]$, *for* $q=0,1$, *such that*

$$\liminf_{t\to\infty} e^{-rt}E|w(Q_t,X_t)|=0 \qquad \forall Q\in\mathcal{Q}, \tag{10}$$

and the process M^Q *defined by*

$$M^Q_t=\int_0^t e^{-rs}\sigma(X_s)w_x(Q_s,X_s)dW_s, \qquad t\geq 0, \tag{11}$$

is a martingale, for all $Q\in\mathcal{Q}$. *Given any initial condition* $(q,x)\in\{0,1\}\times]0,\infty[$:

(a) $v(q,x)\leq w(q,x)$, *and*
(b) *if there exists a process* $Q^*\in\mathcal{Q}$ *such that*

$$\begin{aligned}&\tfrac{1}{2}\sigma^2(X_t)w_{xx}(Q^*_t,X_t)+b(X_t)w_x(Q^*_t,X_t)\\&\quad -rw(Q^*_t,X_t)+Q^*_th(X_t)-(1-Q^*_t)C=0,\end{aligned} \tag{12}$$

$$\big[w(Q^*_{t+},X_t)-w(Q^*_t,X_t)-K_I\big](\Delta Q^*_t)^+=0, \tag{13}$$

$$\big[w(Q^*_{t+},X_t)-w(Q^*_t,X_t)-K_O\big](\Delta Q^*_t)^-=0, \tag{14}$$

for all $t\geq 0$, P-*a.s., then* $v(q,x)=w(q,x)$ *and the optimal strategy is* (Q^*,U^*), *where* $U^*_t=u(X_t)$ *and* u *satisfies* (4).

PROOF. (a) Fix any admissible strategy $(Q,U)\in\Pi_q$ and define the sequence of $(\mathcal{F}_t)$-stopping times (τ_n) by $\tau_1=\inf\{t>0: Q_t\neq q\}$ and $\tau_{n+1}=\inf\{t>\tau_n: Q_t\neq Q_{\tau_n+}\}$, with the usual convention that $\inf\emptyset=\infty$. Note that the assumption that Q is a finite variation process implies that $\tau_n\to\infty$, P-a.s. Using the Itô–Tanaka formula (see Theorem IV.1.5, Corollary IV.1.6, and the remarks thereafter in Revuz and Yor [16]), we can see that

$$\begin{aligned}e^{-rt}w(Q_t,X_t)&=e^{-rt}w(Q_t,X_t)1_{\{t\leq\tau_1\}}\\&\quad+\sum_{n=1}^{\infty}\Bigg[e^{-rt}w(Q_t,X_t)-e^{-r\tau_n}w(Q_{\tau_n+},X_{\tau_n})\\&\qquad+\sum_{j=1}^{n-1}\big[e^{-r\tau_{j+1}}w(Q_{\tau_{j+1}},X_{\tau_{j+1}})-e^{-r\tau_j}w(Q_{\tau_j+},X_{\tau_j})\big]\\&\qquad+e^{-r\tau_1}w(Q_{\tau_1},X_{\tau_1})\\&\qquad+\sum_{j=1}^{n}e^{-r\tau_j}\big[w(Q_{\tau_j+},X_{\tau_j})-w(Q_{\tau_j},X_{\tau_j})\big]\Bigg]1_{\{\tau_n<t\leq\tau_{n+1}\}}\\&=w(q,x)+\int_0^t e^{-rs}\bigg[\tfrac{1}{2}\sigma^2w_{xx}+bw_x-rw\bigg](Q_s,X_s)ds\\&\qquad+\int_0^t e^{-rs}\sigma(X_s)w_x(Q_s,X_s)dW_s\\&\qquad+\sum_{0\leq s<t}e^{-rs}\big[w(Q_{s+},X_s)-w(Q_s,X_s)\big].\end{aligned}$$

This implies

$$\begin{aligned}\int_0^t e^{-rs}\big[h(X_s)Q_s - C(1-Q_s)\big]ds - \sum_{0\le s<t} e^{-rs}\big[K_I(\Delta Q_s)^+ + K_O(\Delta Q_s)^-\big]\\ = w(q,x) - e^{-rt}w(Q_t,X_t) + \int_0^t e^{-rs}\Big[\tfrac{1}{2}\sigma^2(X_s)w_{xx}(Q_s,X_s) + b(X_s)w_x(Q_s,X_s)\\ - rw(Q_s,X_s) + h(X_s)Q_s - C(1-Q_s)\Big]ds\\ + M_t^Q + \sum_{0\le s<t} e^{-rs}\big[w(Q_{s+},X_s) - w(Q_s,X_s) - K_I\big](\Delta Q_s)^+\\ + \sum_{0\le s<t} e^{-rs}\big[w(Q_{s+},X_s) - w(Q_s,X_s) - K_O\big](\Delta Q_s)^-.\end{aligned}$$

Since w satisfies (5), this implies

$$\begin{aligned}&\int_0^t e^{-rs}\big[h(X_s)Q_s - C(1-Q_s)\big]ds\\ &\quad - \sum_{0\le s<t} e^{-rs}\big[K_I(\Delta Q_s)^+ + K_O(\Delta Q_s)^-\big]\\ &\qquad \le w(q,x) - e^{-rt}w(Q_t,X_t) + M_t^Q.\end{aligned} \tag{15}$$

Taking expectations and noting that the stochastic integral has expectation 0, we obtain

$$\begin{aligned}E\Big[&\int_0^t e^{-rs}\big[h(X_s)Q_s - C(1-Q_s)\big]ds\\ &- \sum_{0\le s<t} e^{-rs}\big[K_I(\Delta Q_s)^+ + K_O(\Delta Q_s)^-\big]\Big]\\ &\le w(q,x) - e^{-rt}E[w(Q_t,X_t)].\end{aligned}$$

In view of (10), we can let $t \to \infty$ through an appropriate subsequence to obtain

$$\begin{aligned}E\Big[&\int_0^\infty e^{-rs}[h(X_s)Q_s - C(1-Q_s)]ds\\ &- \sum_{0\le s} e^{-rs}\big[K_I(\Delta Q_s)^+ + K_O(\Delta Q_s)^-\big]\Big] \le w(q,x).\end{aligned}$$

Here, we have used the dominated convergence theorem in conjunction with (3), as well as the monotone convergence theorem. Now, observing that the left-hand side of this inequality dominates $J_{q,x}(Q,U)$, because of (2), we can conclude that $J_{q,x}(Q,U) \le w(q,x)$.

(b) If Q^* satisfies (12)–(14), then it follows that (15) holds with equality. By following the same arguments as before, we can then see that $J_{q,x}(Q^*,U^*) = w(q,x)$, and the proof is complete. □

4. The solution of the control problem when the commodity price is a geometric Brownian motion. We now solve the stochastic optimal control problem formulated in Section 2 under the following assumption.

ASSUMPTION A3. The functions b and σ are given by $b(x) = bx$ and $\sigma(x) = \sqrt{2}\sigma x$ for some constants b and σ. The function $\bar{h}$ satisfies Assumption A2 and the function h defined by (2) is nondecreasing and satisfies $\lim_{x\to\infty} h(x) = \infty$. Also, $K_I + K_O > 0$.

Note that, since the function h is the best rate of payoff that the investment can yield in its open mode, the additional assumptions on h are very reasonable as far as the modelling of the application that we consider is concerned.

To solve the problem, we are going to construct a solution of the HJB equation (5) which satisfies the requirements of the verification theorem in the previous section. In the case that we consider here, the HJB equation takes the form of the pair of coupled quasivariational inequalities

$$\max\left\{\sigma^2 x^2 w_I''(x) + bxw_I'(x) - rw_I(x) + h(x),\, w_O(x) - K_O - w_I(x)\right\} = 0, \tag{16}$$

$$\max\left\{\sigma^2 x^2 w_O''(x) + bxw_O'(x) - rw_O(x) - C,\, w_I(x) - K_I - w_O(x)\right\} = 0. \tag{17}$$

To simplify the notation, we write w_I and w_O in place of $w(1,\cdot)$ and $w(0,\cdot)$, respectively, throughout this section.

Now, every solution of the homogeneous ordinary differential equation (ODE)

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) = 0 \tag{18}$$

which is associated with (16)–(17) is given by

$$w(x) = Ax^m + Bx^n$$

for some constants $A, B \in \mathbb{R}$, where the constants $m < 0 < n$ are defined by

$$m, n = \frac{1}{2\sigma^2}\left[\sigma^2 - b \pm \sqrt{(b-\sigma^2)^2 + 4\sigma^2 r}\right].$$

Also, a special solution of the ODE

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) + h(x) = 0 \tag{19}$$

is the nondecreasing function given by

$$w_p(x) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1}h(s)ds + x^n \int_x^\infty s^{-n-1}h(s)ds\right]. \tag{20}$$

In particular, a special solution of the ODE

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) - C = 0 \tag{21}$$

is equal to the constant $-C/r$. For future reference, also note that w_p admits the probabilistic characterisation

$$w_p(x) = E\int_0^\infty e^{-rs}h(X_s)ds \tag{22}$$

and satisfies

$$\lim_{x\to\infty} w_p(x) = \infty, \tag{23}$$

$$\lim_{t\to\infty} e^{-rt} E|w_p(X_t)| = 0 \tag{24}$$

and

$$E\int_0^t e^{-2rs} X_s^2 |w_p'(X_s)|^2 ds = a(t) \tag{25}$$

for some increasing function a. Note that (25) follows immediately from Lemma 3.4 and Proposition 4.1.c in [11]. All of the other assertions, as well as a number of related results, are proved in [11], Section 4.)

Back to the control problem, we can make the following observations. In view of (22), if the investment is open at time 0, then producing according to the optimal production rate $u \circ X$ (see Theorem 1) and never switching it to its closed operating mode yields a payoff equal to $w_p(x)$. On the other hand, if the investment is closed at time 0, then never switching it to its open operating mode incurs a total cost equal to $-C/r$. Now, since w_p is nondecreasing and satisfies (23), we can expect the following to be true. First, it is always part of the optimal scenario to switch the investment from its closed to its open mode if the commodity price is sufficiently high. Second, if switching the investment from its open to its closed mode is ever part of the optimal strategy, then this should occur only if the commodity price is sufficiently low.

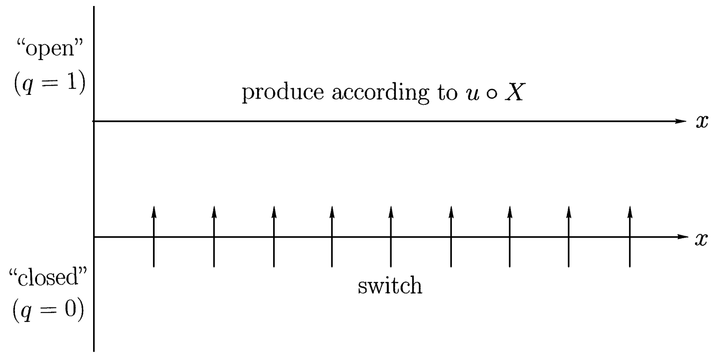
Bearing in mind these observations, we can distinguish several possibilities. The first arises if the investment is very profitable operating in open mode, and it is not worth keeping it in its closed mode for any length of time, that is, if $w_p(x)$ is large relative to the costs C, K_I for every $x > 0$. In such a situation, the optimal strategy takes the following form. If the investment is closed at time 0, it is optimal to switch it to its open operating mode immediately. Once in its open operating mode, it is optimal to produce according to the rate $u \circ X$ and never switch the investment to closed mode. This strategy can be depicted as in Figure 1a. In view of (22), if this strategy is indeed optimal, the functions defined by $w_I(x) = w_p(x)$ and $w_O(x) = w_p(x) - K_I$ for all $x > 0$ should satisfy the HJB equations (16)–(17) (see Figure 1b).

The next lemma proves a necessary and sufficient condition under which these functions constitute a solution of (16)–(17).

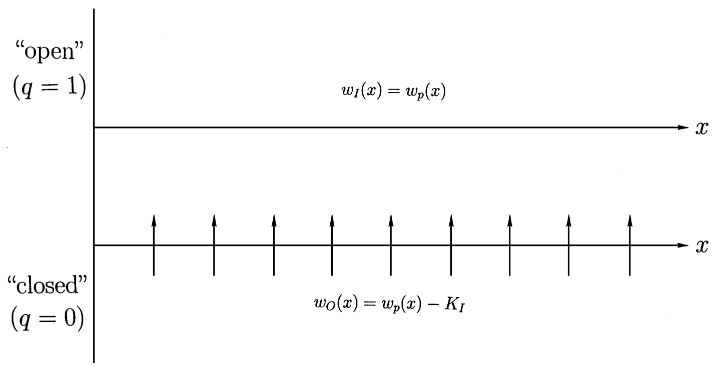
LEMMA 2. *The functions* $w_I, w_O \in W^{2,\infty}_{\text{loc}}(]0,\infty[)$ *defined by*

$$w_I(x) = w_p(x), \qquad w_O(x) = w_p(x) - K_I$$

satisfy the HJB equations (16)–(17) *if and only if* $h(x) \geq -C + rK_I$ *for all* $x > 0$.



(a)



(b)

FIG. 1.

We collect in the Appendix the proofs of those results which are not developed in the text.

A second possibility arises if it is never optimal to switch the investment from its open to its closed mode, but it may be optimal to wait for the commodity price to rise before switching it from its closed to its open mode. In such a case, the optimal strategy is composed by the following actions. If the investment is originally closed, then it is optimal to wait if the commodity price is below a certain level, say z, and switch the investment to its open operating mode as soon as the commodity price exceeds z. Once in open operating mode, it is optimal to produce according to the rate $u \circ X$ and never switch the investment to its closed mode. This strategy can be depicted as in Figure 2a. If this strategy is indeed optimal, we should find a solution of the HJB equations (16)–(17) such that $w_I(x) = w_p(x)$ for all $x > 0$, $w_O(x) = w_p(x) - K_I$ for $x \geq z$ and w_O satisfies the ODE (21) for $x < z$, that is,

$$w_O(x) = \Delta x^m + Bx^n - C/r \qquad \text{for } x < z, \tag{26}$$

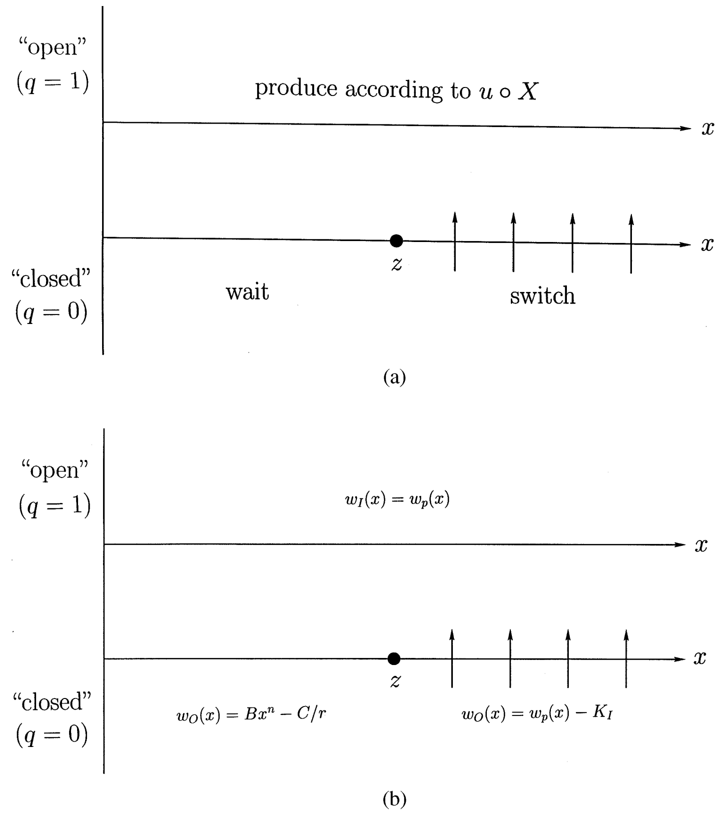


FIG. 2.

for some constants $\Delta, B \in \mathbb{R}$. Now, we must have $\Delta = 0$, because otherwise (10) cannot be satisfied. To see this, suppose that $\Delta \neq 0$ and observe that we can write

$$w_O(x) = \Delta x^m + w_p(x)1_{\{z<x\}} + \eta(x),$$

where η is a bounded function. In view of (24) and the fact that

$$e^{-rt}EX_t^k = x^k \exp\left([\sigma^2 k^2 + (b - \sigma^2)k - r]t\right) = x^k \quad \text{for } k = m, n, \tag{27}$$

we can see that

$$e^{-rt}E\big|w_O(X_t)\big| \geq e^{-rt}|\Delta|EX_t^m - e^{-rt}E\big|w_p(X_t)\big| - e^{-rt}E\big|\eta(X_t)\big| \longrightarrow x^m > 0.$$

As a consequence, if Q is defined by $Q_t = q1_{\{t=0\}}$ for all $t \geq 0$, then

$$\liminf_{t\to\infty} e^{-rt}E\big|w(Q_t, X_t)\big| = \liminf_{t\to\infty} e^{-rt}E\big|w_O(X_t)\big| > 0$$

and (10) fails. We conclude that the conjectured solution takes the form shown in Figure 2b. Furthermore, to specify the parameters B, z, we require that w_O is C^1 at the free boundary point z.

The following lemma is concerned with the construction of the function w_O and with a necessary and sufficient condition under which the functions just described provide a solution of the HJB equations (16)–(17).

LEMMA 3. *The equation*

$$f_1(z) := \int_0^z s^{-m-1}[h(s) + C - rK_I]ds = 0 \tag{28}$$

has a unique solution $z > 0$ *if and only if* $\inf_{x>0} h(x) < -C + rK_I$. *In this case, the functions* $w_I, w_O \in W^{2,\infty}_{\text{loc}}(]0, \infty[)$ *defined by*

$$w_I(x) = w_p(x), \qquad w_O(x) = \begin{cases} Bx^n - C/r, & \text{if } x \leq z, \\ w_p(x) - K_I, & \text{if } x \geq z, \end{cases} \tag{29}$$

where B *is given by* (39) *or* (40) *in the Appendix, satisfy the HJB equations* (16)–(17) *if and only if* $\inf_{x>0} h(x) \geq -C - rK_O$.

A final possibility arises if switching the investment from either of its two operating modes to the other one is part of the optimal scenario. In this case, the optimal strategy takes the following form. If the investment is originally in its closed operating mode, then it is optimal to wait as long as the commodity price is below a certain level, say z, and it is optimal to switch it to its open operating mode as soon as the commodity price rises above z. On the other hand, if the investment is originally in its open operating mode, then it is optimal to produce according to the rate $u \circ X$ as long as the commodity price is above a certain level, say y, and switch it to its closed mode as soon as the commodity price falls below y. Clearly, this strategy is well defined as long as $y < z$. It can be depicted by Figure 3a. If this strategy is indeed optimal, we should find a solution of the HJB equations (16)–(17) such that $w_I(x) = w_O(x) - K_O$ for $x \leq y$, w_I satisfies the ODE (19) for $x > y$, that is, $w_I(x) = Ax^m + \Gamma x^n + w_p(x)$ for $x > y$, for some constants $A, \Gamma \in \mathbb{R}$, w_O solves the ODE (21) for $x < z$, that is, $w_O(x) = \Delta x^m + Bx^n - C/r$ for $x < z$, for some constants $\Delta, B \in \mathbb{R}$ and $w_O(x) = w_I(x) - K_I$ for $x \geq z$. In view of (27), we can develop a line of argument similar to the one that led to the choice $\Delta = 0$ in (26) to conclude that we must have $\Gamma = \Delta = 0$ for (10) to hold. Therefore, the solution takes the form shown in Figure 3b. The parameters A, B, y, and z can be specified by the requirement that the functions w_I, w_O are C^1 at the free boundary points y, z, respectively.

In the following lemma, we construct the previously described functions w_I, w_O and we prove a necessary and sufficient condition under which they satisfy (16)–(17).

LEMMA 4. *The system of equations*

$$f(y, z) := m\int_y^z s^{-m-1}h(s)ds + y^{-m}\big[C + rK_O\big] - z^{-m}\big[C - rK_I\big] = 0, \tag{30}$$

$$g(y, z) := n\int_y^z s^{-n-1}h(s)ds + y^{-n}\big[C + rK_O\big] - z^{-n}\big[C - rK_I\big] = 0 \tag{31}$$

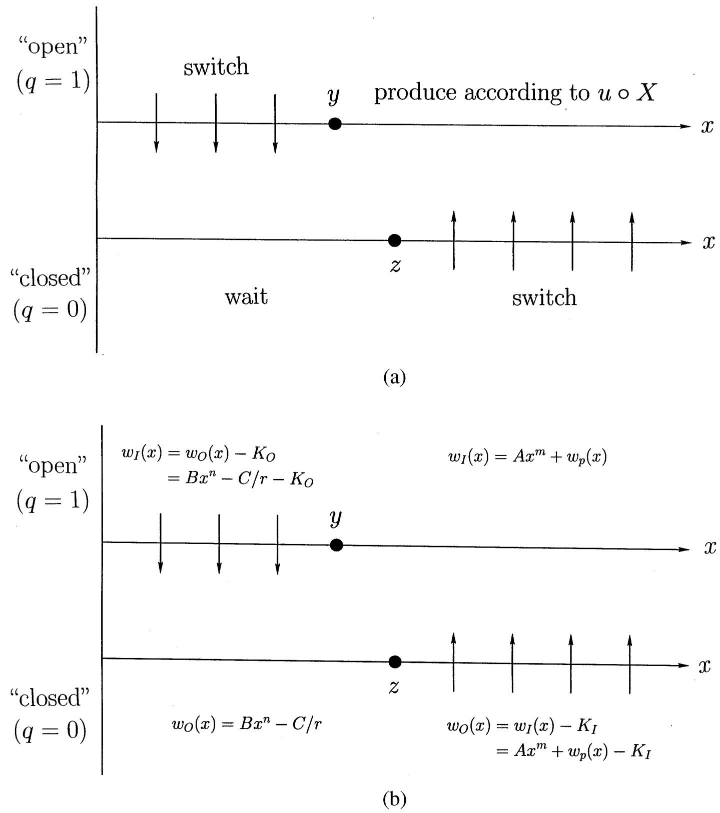


FIG. 3.

has a unique solution (y, z) *such that* $0 < y < z$ *if and only if* $\inf_{x>0} h(x) < -C - rK_O$. *In this case, the functions* $w_I, w_O \in W^{2,\infty}_{\mathrm{loc}}(]0,\infty[)$ *defined by*

$$w_I(x) = \begin{cases} Bx^n - C/r - K_O, & \text{if } x \le y, \\ Ax^m + w_p(x), & \text{if } x \ge y, \end{cases} \tag{32}$$

$$w_O(x) = \begin{cases} Bx^n - C/r, & \text{if } x \le z, \\ Ax^m + w_p(x) - K_I, & \text{if } x \ge z, \end{cases}$$

where A *and* B *are given in* (46)–(47) *and* (48)–(49), *respectively, in the Appendix, solve the HJB equations* (16)–(17).

We can now prove the main result of the paper.

THEOREM 5. *Consider the stochastic optimal control problem defined in Section* 2, *and suppose that Assumption* A3 *holds. The value function* v *is* *given by* $v(1,\cdot)=w_I$ *and* $v(0,\cdot)=w_O$*, where:*

(i) *If* $-C+rK_I\le h(x)$ *for all* $x>0$*, then* w_I*,* w_O *are given by Lemma* 2. *In this case, the optimal strategy is*

$$Q_t^* = q1_{\{t=0\}} + 1_{\{t>0\}}, \qquad U_t^* = u(X_t),\ t\ge 0. \tag{33}$$

(ii) *If* $-C-rK_O\le \inf_{x\in]0,\infty[} h(x) < -C+rK_I$*, then* w_I*,* w_O *are given by Lemma* 3. *In this case, the optimal strategy is*

$$Q_t^* = q1_{\{t\le\tau_z\}} + 1_{\{\tau_z<t\}}, \qquad U_t^* = u(X_t),\ t\ge 0, \tag{34}$$

where z *is the unique solution of* (28) *and*

$$\tau_z = \inf\{t\ge 0:\ X_t\ge z\}.$$

(iii) *If* $\inf_{x\in]0,\infty[} h(x) < -C-rK_O$*, then* w_I*,* w_O *are given by Lemma* 4. *The optimal production rate is again given by* $U_t^*=u(X_t)$*, whereas the optimal switching process* Q^* *can be constructed as in the following proof.*

PROOF. Consider any of the three cases. Each of the functions $v(q,\cdot)$, $q=0,1$, belongs to $W^{2,\infty}_{\mathrm{loc}}(]0,\infty[)$ and the candidate value function v satisfies the HJB equation (5). Also, we can write

$$|v(q,x)|\le |w_p(x)| + L_1 \qquad \forall (q,x)\in\{0,1\}\times]0,\infty[$$

for some constant L_1, which implies (10), by virtue of (24). Similarly, we can write

$$|v_x(q,x)|\le |w_p'(x)| + L_2x^{n-1} + L_3 \qquad \forall (q,x)\in\{0,1\}\times]0,\infty[$$

for some constants L_2, L_3, which, combined with (25), implies that the process M^Q defined as in (11) is a square integrable martingale if stopped at any constant time, for all $Q\in\mathscr{Q}$. These observations prove that v satisfies the assumptions of Theorem 1.

Now, in cases (i) and (ii), the processes Q^* defined by (33) and (34), respectively, clearly satisfy (12)–(14). In case (iii), if $q=1$, then we can see that the process $Q^*\in\mathscr{Q}$ defined by

$$Q_t^* = 1_{\{t=0\}} + \sum_{j=0}^{\infty} 1_{\{\tau_{2j}<t\le\tau_{2j+1}\}},$$

where $\tau_0=0$ and the stopping times τ_n, $n\in\mathbb{N}^*$, are defined recursively by

$$\tau_{2n+1} = \inf\{t\ge\tau_{2n} : X_t\le y\}, \qquad n=0,1,2,\dots,$$
$$\tau_{2n} = \inf\{t\ge\tau_{2n-1} : X_t\ge z\}, \qquad n=1,2,\dots,$$

satisfies (12)–(14). If $q=0$, a process $Q^*\in\mathscr{Q}$ can be constructed in a similar way and the proof is complete. □

REMARK 1. It is of interest to examine the form that our solution takes in the special case considered by Dixit [7] and Shirakawa [17], whose analysis resulted in only case (b) that follows. Let us assume that $\bar{h}(u, x) = x - K$ for some constant $K > 0$, $C = 0$, and $K_I, K_O > 0$. In this case, (3) can hold only if $r > b$. Indeed, if this is not true, then we can readily see that the strategy which consists of switching the investment to its open operating mode forever has infinite payoff and the problem's solution is trivial. Under these assumptions, we can have one of the following two cases:

(a) If $K < rK_O$, then we are in case (ii) of Theorem 5. The value function is given by

$$v(1, x) = \frac{x}{r-b} - \frac{K}{r}, \qquad v(0, x) = \begin{cases} Bx^n, & \text{if } x \le z, \\ \dfrac{x}{r-b} - \dfrac{K}{r} - K_I, & \text{if } x \ge z, \end{cases}$$

where

$$z = (K + rK_I)\frac{m-1}{m} > 0 \quad \text{and} \quad B = z^{-n}\left(\frac{z}{r-b} - \frac{K}{r} - K_I\right).$$

(b) If $rK_O < K$, then case (iii) of Theorem 5 holds. The value function is given by

$$v(1, x) = \begin{cases} Bx^n - K_O, & \text{if } x \le y, \\ Ax^m + \dfrac{x}{r-b} - \dfrac{K}{r}, & \text{if } x \ge y, \end{cases}$$

$$v(0, x) = \begin{cases} Bx^n, & \text{if } x \ge z, \\ Ax^m + \dfrac{x}{r-b} - \dfrac{K}{r} - K_I, & \text{if } x \ge z, \end{cases}$$

where $0 < y < z$ are the unique solutions of

$$\frac{n}{n-1}y^{-n+1} - \frac{n}{n-1}z^{-n+1} + (rK_O - K)y^{-n} + (rK_I + K)z^{-n} = 0, \tag{35}$$

$$\frac{m}{m-1}y^{-m+1} - \frac{m}{m-1}z^{-m+1} + (rK_O - K)y^{-m} + (rK_I + K)z^{-m} = 0, \tag{36}$$

and A, B are given by appropriate expressions resulting from (46)–(49).

APPENDIX

Proofs of selected results.

PROOF OF LEMMA 2. We have to show that

$$w_O(x) - K_O - w_I(x) \le 0, \tag{37}$$

$$\sigma^2 x^2 w_O''(x) + bxw_O'(x) - rw_O(x) - C \le 0. \tag{38}$$

Now, (37) is equivalent to $-K_I - K_O \le 0$, which is true. Since w_p satisfies the ODE (19), (38) holds if and only if $h(x) \ge -C + rK_I$ for all $x > 0$ and the proof is complete. □

PROOF OF LEMMA 3. The requirement that w_O is C^1 at z gives rise to the system of equations

$$Bz^n = w_p(z) - K_I + \frac{C}{r} \tag{39}$$

$$= \frac{1}{n} z w_p'(z). \tag{40}$$

Equating the right-hand sides and using the definition (20) of w_p as well as the fact that $\sigma^2 mn = -r$, we obtain (28) by simple calculations.

Clearly, (28) can have a solution $z > 0$ only if $\inf_{x>0} h(x) + C - rK_I < 0$. So, suppose that this condition is true and let $\alpha := \inf\{x : h(x) + C - rK_I > 0\} > 0$ (recall that h is nondecreasing). Clearly, $f_1(z) < 0$ for all $z \le \alpha$. Now, given a constant $M > 0$, our assumptions imply that there exists a $\beta > 0$ such that $h(x) + C - rK_I \ge M$ for all $x \ge \beta$. As a consequence,

$$\lim_{z\to\infty} f_1(z) \ge \lim_{z\to\infty}\left[f_1(\beta) + \frac{M}{m}\beta^{-m} - \frac{M}{m} z^{-m}\right] = \infty,$$

because $m < 0$. Moreover, since $f_1'(z) = z^{-m-1}[h(z) + C - rK_I]$, f_1 is nonincreasing in $]0, \alpha]$ and strictly increasing in $]\alpha, \infty[$. However, these observations imply that the equation $f_1(z) = 0$ has a unique solution $z > 0$. For future reference, note that this solution satisfies

$$h(z) + C - rK_I > 0. \tag{41}$$

To show that the functions w_I, w_O defined by (29) satisfy the HJB equations (16)–(17), we have to prove that

$$w_O(x) - K_O - w_I(x) \le 0 \quad \text{for } x > 0, \tag{42}$$

$$\sigma^2 x^2 w_O''(x) + bxw_O'(x) - rw_O(x) - C \le 0 \quad \text{for } x > z, \tag{43}$$

$$w_I(x) - K_I - w_O(x) \le 0 \quad \text{for } x \le z. \tag{44}$$

For $x \ge z$, (42) is equivalent to $-K_I - K_O \le 0$, which is true. With reference to (40), we can see that for $x \le z$, (42) is equivalent to

$$g_1(x) := \frac{1}{n} z^{-n+1} w_p'(z) x^n - \frac{C}{r} - K_O - w_p(x) \le 0. \tag{45}$$

Using the integration by parts formula (see [11], Remark 4.6)

$$m\int_0^x s^{-m-1} h(s)\,ds + x^{-m} h(x) = \int_0^x s^{-m}\,dh(s)$$

and the definition (20) of w_p, we can see that

$$\frac{d}{dx}\left(x^{-n+1}w_p'(x)\right) = -\frac{1}{\sigma^2}x^{m-n-1}\int_0^x s^{-m}\,dh(s) \le 0,$$

which implies that the function $x \to x^{-n+1}w_p'(x)$ is nonincreasing. As a consequence,

$$g_1'(x) = x^{n-1}\left[z^{-n+1}w_p'(z) - x^{-n+1}w_p'(x)\right] \le 0 \quad \text{for } x \le z.$$

Now, since $g_1(z) = -K_I - K_O$ and g_1 is nonincreasing in $]0, z]$, (45) will follow if and only if $\lim_{x\downarrow 0} g_1(x) \le 0$. However, this is true if and only if $\inf_{x>0} h(x) \ge -C - rK_O$ because w_p is nonincreasing, and $\inf_{x>0} w_p(x) \ge -C/r - K_O$ is equivalent to $\inf_{x>0} h(x) \ge -C - rK_O$ (see Knudsen, Meister and Zervos [11], Lemma 5.1).

Since $g_1(z) = -K_I - K_O$ and g_1 is nonincreasing in $]0, z]$, it follows that $-K_I - K_O \le g_1(x)$ for $x \le z$. However, this implies trivially (44).

Finally, following arguments similar to the ones used in the proof of Lemma 2, we can see that (43) is equivalent to $h(x) \ge -C + rK_I$ for $x > z$. However, this is true because of (41) and the assumption that h is nondecreasing. □

PROOF OF LEMMA 4. For w_I and w_O to be C^1 at y and z, respectively, we require

$$\begin{aligned} By^n - Ay^m &= w_p(y) + \frac{C}{r} + K_O, \\ nBy^n - mAy^m &= yw_p'(y), \\ Bz^n - Az^m &= w_p(z) + \frac{C}{r} - K_I, \\ nBz^n - mAz^m &= zw_p'(z). \end{aligned}$$

This system is equivalent to

$$A = -\frac{n}{n-m}y^{-m}\left[w_p(y) - \frac{y}{n}w_p'(y) + \frac{C}{r} + K_O\right] \tag{46}$$

$$= -\frac{n}{n-m}z^{-m}\left[w_p(z) - \frac{z}{n}w_p'(z) + \frac{C}{r} - K_I\right], \tag{47}$$

$$B = -\frac{m}{n-m}y^{-n}\left[w_p(y) - \frac{y}{m}w_p'(y) + \frac{C}{r} + K_O\right] \tag{48}$$

$$= -\frac{m}{n-m}z^{-n}\left[w_p(z) - \frac{z}{m}w_p'(z) + \frac{C}{r} - K_I\right]. \tag{49}$$

Equating the right-hand sides of (46) and (47), and using the definition (20) of w_p and the identity $\sigma^2 mn = -r$, it is a matter of calculations to verify that the points y and z must satisfy (30). Similarly, (48) and (49) imply (31).

To study the solvability of the system (30)–(31), we first prove that (30) defines uniquely a mapping l: $]0, \infty[$ $]0, \infty[$ such that

$$f(y, l(y)) = 0 \qquad \text{and} \qquad l(y) > y. \tag{50}$$

To this end, fix any $y > 0$ and consider the function $\hat{f}$: $[y, \infty[\to \mathbb{R}$ defined by

$$\hat{f}(z) := f(y, z) = m \int_y^z s^{-m-1}\big[h(s) + C - rK_I\big]ds + r\big(K_I + K_O\big)y^{-m}.$$

Since h is nondecreasing and $\lim_{x\to\infty} h(x) = \infty$, there exist constants $\beta > y$, $M > 0$ such that $h(x) + C - rK_I > M$ for all $x \geq \beta$. Therefore,

$$\lim_{z\to\infty} \hat{f}(z) \leq \lim_{z\to\infty} \left[\hat{f}(\beta) + M\beta^{-m} - Mz^{-m}\right] = -\infty,$$

because $m < 0$. Also, $\hat{f}(y) = r(K_I + K_O)y^{-m} > 0$. Now, $\hat{f}'(z) = mz^{-m-1}[h(z) + C - rK_I]$, so if $\alpha := \inf\{x\colon h(x) + C - rK_I > 0\} > 0$, then $\hat{f}$ is nondecreasing in the interval $[y, \alpha]$ if $y < \alpha$ and $\hat{f}$ is strictly decreasing in the interval $]y \vee \alpha, \infty[$. From these observations, we can conclude that the equation $\hat{f}(z) = 0$ has a unique solution $z = l(y)$ which satisfies (50) as well as

$$h(z) + C - rK_I\Big|_{z=l(y)} > 0. \tag{51}$$

Furthermore, by implicit differentiation of $f(y, l(y)) = 0$, we obtain

$$l'(y) = \frac{y^{-m-1}[h(y) + C + rK_O]}{l^{-m-1}(y)[h(l(y)) + C - rK_I]}. \tag{52}$$

Now, to prove that the system of equations (30)–(31) has a unique solution (y, z) such that $0 < y < z$, we have to show that the equation $\hat{g}(y) := g(y, l(y)) = 0$ has a unique solution $y > 0$. Since

$$\hat{g}(y) = n \int_y^{l(y)} s^{-n-1}[h(s) + C - rK_I]ds + r(K_I + K_O)y^{-n},$$

h is nondecreasing and $\lim_{y\to\infty} h(y) = \infty$, we can see that

$$\exists L > 0: \qquad \hat{g}(y) > 0 \qquad \forall y \geq L. \tag{53}$$

Also, using (52), we can calculate

$$\begin{aligned}\hat{g}'(y) &= g_y(y, l(y)) + g_z(y, l(y))l'(y) \\ &= ny^{-m-1}[h(y) + C + rK_O][l^{m-n}(y) - y^{m-n}].\end{aligned}$$

In view of (50) and the fact that $m < n$, $l^{m-n}(y) - y^{m-n} < 0$. Therefore, if $\inf_{x>0} h(x) \geq -C - rK_O$, then $\hat{g}$ is nonincreasing, which, combined with (53), implies that $\hat{g}(y) = 0$ cannot have a solution $y > 0$. So, assume $\inf_{x>0} h(x) < -C - rK_O$ and let $\gamma := \sup\{x\colon h(x) + C + rK_O < 0\} > 0$. Then

(54) $\quad \hat{g}$ is strictly increasing in $]0, \gamma[$ and nonincreasing in $[\gamma, \infty[$.

Moreover, there exist constants $\delta, \varepsilon > 0$ such that $h(x) + C + rK_O \leq -\varepsilon$ for all $x \leq \delta$. For these parameters, we can calculate

$$\lim_{y\downarrow 0} \int_y^\infty s^{-n-1}[h(s) + C + rK_O]ds$$
$$\leq \lim_{y\downarrow 0}\left[-\frac{\varepsilon}{n} y^{-n} + \frac{\varepsilon}{n}\delta^{-n} + \int_\delta^\infty s^{-n-1}[h(s) + C + rK_O]ds\right] = -\infty.$$

Combining this with the fact that

$$\hat{g}(y) = n\int_y^\infty s^{-n-1}[h(s) + C + rK_O]ds - n\int_{l(y)}^\infty s^{-n-1}[h(s) + C - rK_I]ds$$

and with (51), we can see that $\lim_{y\downarrow 0} \hat{g}(y) < 0$. However, this, (53) and (54) imply that the equation $\hat{g}(y) = 0$ has a unique solution. Also, observe that this solution satisfies

$$h(y) + C + rK_O < 0. \tag{55}$$

It remains to show that w_I, w_O solve the HJB equations (16) and (17), which amounts to proving that

$$\sigma^2 x^2 w_I''(x) + bxw_I'(x) - rw_I(x) + h(x) \leq 0 \quad \text{for } x < y, \tag{56}$$

$$w_O(x) - K_O - w_I(x) \leq 0 \quad \text{for } x \geq y, \tag{57}$$

$$\sigma^2 x^2 w_O''(x) + bxw_O'(x) - rw_O(x) - C \leq 0 \quad \text{for } x > z, \tag{58}$$

$$w_I(x) - K_I - w_O(x) \leq 0 \quad \text{for } x \leq z. \tag{59}$$

Since $x \to x^n$, $x \to x^m$ and w_p solve the ODEs (18) and (19), respectively, (56) and (58) are equivalent to $h(x) \leq -C - rK_O$ for all $x < y$ and $h(x) \geq -C + rK_I$ for all $x > z$, respectively, which are true in view of (55) and (51), respectively, and the monotonicity of h.

For $x \geq z$, (57) is equivalent to $-K_I - K_O \leq 0$, which is true. Similarly, for $x \leq y$, (59) is again equivalent to $-K_I - K_O \leq 0$.

For $y \leq x \leq z$, we can use (46), (48) and the definition (20) of w_p to show that

$$\begin{aligned} g_2(x) &:= w_O(x) - K_O - w_I(x) \\ &= \frac{x^n}{\sigma^2(n-m)} \int_y^x s^{-n-1}[h(s) + C + rK_O]ds \\ &\quad - \frac{x^m}{\sigma^2(n-m)} \int_y^x s^{-m-1}[h(s) + C + rK_O]ds. \end{aligned} \tag{60}$$

Making similar calculations with (47) and (49), we can also show that

$$\begin{aligned} g_2(x) = &\frac{x^m}{\sigma^2(n-m)} \int_x^z s^{-m-1}[h(s) + C - rK_I]ds - K_I - K_O \\ &- \frac{x^n}{\sigma^2(n-m)} \int_x^z s^{-n-1}[h(s) + C - rK_I]ds. \end{aligned} \tag{61}$$

From (60), it follows that $g_2(y) = g_2'(y) = 0$, whereas from (61), it follows that $g_2(z) = -K_I - K_O$ and $g_2'(z) = 0$. Moreover, taking into account that

$$g_2'(x) = \frac{nx^{n-1}}{\sigma^2(n-m)} \int_y^x s^{-n-1}[h(s) + C + rK_O]ds - \frac{mx^{m-1}}{\sigma^2(n-m)} \int_y^x s^{-m-1}[h(s) + C + rK_O]ds,$$

$m < 0 < n$, h is upper semicontinuous and nondecreasing, and y, $z = l(y)$ satisfy (55) and (51), respectively, we can see that $g_2'(x)$ is nonincreasing as x increases from y to a point μ satisfying

$$\sup_{x<\mu} h(x) + C + rK_O \le 0 \le h(\mu) + C + rK_O$$

and is nondecreasing as x increases from μ to z. Combining this with the fact that $g_2'(y) = g_2'(z) = 0$, we can conclude that $g_2'(x) \le 0$, for all $x \in [y, z]$. These observations imply

$$-K_I - K_O \le g_2(x) \le 0 \quad \text{for } y \le x \le z.$$

However, the second inequality implies (57) for $y \le x \le z$, whereas the first inequality implies (59) for $y \le x \le z$. □

Acknowledgment. The research of Kate Duckworth was supported by a University of Newcastle Research Studentship.

REFERENCES

[1] BENSOUSSAN, A. and LIONS, J.-L. (1984). *Impulse Control and Quasivariational Inequalities*. Gauthier-Villars, Paris.

[2] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control: The Discrete Time Case*. Academic Press, New York.

[3] BREKKE, K. A. and ØKSENDAL, B. (1991). The high contact principle as a sufficiency condition for optimal stopping. In *Stochastic Models and Option Values* (D. Lund and B. Øksendal, eds.) 187–208. North-Holland, Amsterdam.

[4] BREKKE, K. A. and ØKSENDAL, B. (1994). Optimal switching in an economic activity under uncertainty. *SIAM J. Control Optim.* **32** 1021–1036.

[5] BRENNAN, M. J. and SCHWARTZ, E. S. (1985). Evaluating natural resource investments. *Journal of Business* **58** 135–157.

[6] CORTAZAR, G. and SCHWARTZ, E. S. (1993). A compound option of production and intermediate inventories. *Journal of Business* **66** 517–540.

[7] DIXIT, A. (1989). Entry and exit decisions under uncertainty. *Journal of Political Economy* **97** 620–638.

[8] DIXIT, A. K. and PINDYCK, R. S. (1994). *Investment under Uncertainty*. Princeton Univ. Press.

[9] DUCKWORTH, K. and ZERVOS, M. (2000). An investment model with entry and exit decisions. *J. Appl. Prob.* **37** 547–559.

[10] KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus*. Springer, Berlin.

[11] KNUDSEN, T. S., MEISTER, B. and ZERVOS, M. (1998). Valuation of investments in real assets with implications for the stock prices. *SIAM J. Control Optim.* **36** 2082–2102.

[12] KNUDSEN, T. S., MEISTER, B. and ZERVOS, M. (1999). On the relationship of the dynamic programming approach and the contingent claim approach to asset valuation. *Finance and Stochastics* **3** 433–449.

[13] MCDONALD, R. L. and SIEGEL, D. R. (1985). Investment and the valuation of firms when there is an option to shut down. *Internat. Econom. Rev.* **26** 331–349.

[14] PADDOCK, J. L., SIEGEL, D. R. and SMITH, J. L. (1988). Option valuation of claims on real assets: The case of offshore petroleum leases. *Quart. J. Econom.* **8** 479–508.

[15] PINDYCK, P. S. (1988). Irreversible investment, capacity choice, and the value of the firm. *Amer. Econom. Rev.* **79** 969–985.

[16] REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer, Berlin.

[17] SHIRAKAWA, H. (1997). Evaluation of investment opportunity under entry and exit decisions. *Sūrikaisekikenkyūsho Kōkyūroku* **987** 107–124.

[18] TRIGEORGIS, L. (1993). Real options and interactions with financial flexibility. *Financial Management* **22** 202–224.

[19] TRIGEORGIS, L. (1996). *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. MIT Press.

DEPARTMENT OF MATHEMATICS
KING'S COLLEGE LONDON
THE STRAND
LONDON WC2R 2LS
UNITED KINGDOM
E-MAIL: Mihail.Zervos@kcl.ac.uk