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EXPECTED ABSOLUTE RANDOM DETERMINANTS AND ZONOIDS[1]

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For matrices with iid columns, the expectations of the title are examined using ideas from convex geometry. In particular, a representation is shown which involves random realizations from the class of convex bodies called zonoids. This result is used to derive bounds and comparisons of various types.

1. Introduction. Random matrices arise in a variety of contexts, ranging from statistical applications to the asymptotics of neural networks [e.g., Wilks (1960), Grenander (1978), Silverstein (1984) and Lindsay (1989a, b)]. In particular, various moments of their determinants have been intensively studied [Girko (1988)]. Our purpose here is to introduce a novel approach from stochastic geometry for looking at expected absolute determinants (eads). We make use of particular convex figures, called zonoids, in a random form and more precisely their expectations. The notion of an expected figure (or set-valued expectation) is based on the integral of a set-valued function [Aumann (1965)] and has proved useful in other contexts [Artstein and Vitale (1975), Mecke (1987) and Vitale (1987a, b, 1988)]. For other examples of the utility of convex-geometric ideas in applications, the reader can see Anderson (1955), Egorychev (1981) and Falikman (1981).

In the next section, we set notation and preliminaries. After interpreting zonoids in probabilistic terms, Section 3 presents our key result, which expresses eads in terms of zonoids. Bounds and comparisons of various types appear in later sections.

2. Preliminaries. We shall consider the ead of a $d \times d$ matrix M_Y, which has as columns iid copies of the random d-vector Y. $N.B.$ The individual entries of the matrix are not required to be independent (as is the case for some other approaches).

We regard R^d as equipped with the usual inner product $\langle \cdot , \cdot \rangle$, norm $\| \cdot \|$, closed unit ball B, unit sphere S^{d-1} and collection $\mathscr{K}$ of nonempty compact, convex subsets or *convex bodies*. Distance between convex bodies is given by the Hausdorff metric $\rho(K, L) = \inf\{\varepsilon > 0 | K \subseteq L + \varepsilon B,\ L \subseteq K + \varepsilon B\}$. Here "+" means vector addition of sets and $\alpha K \equiv \{\alpha x | x \in K\}$. The *norm* of a

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convex body K is $\|K\| = \max\{\|x\| \,|x \in K\} = \rho(K, \{0\})$. For other general background, see Eggleston (1969) and Guggenheimer (1977).

It is often the case that geometric arguments for convex bodies can be replaced by simpler function-theoretic proofs by using the support function identification $K \leftrightarrow h_K$, $h_K(x) \equiv \max\{\langle x, y\rangle | y \in K\}$, $u \in S^{d-1}$, which has the following features:

$$K \subseteq L \Leftrightarrow h_K \le h_L \quad \text{(pointwise)}, \tag{2.1}$$

$$K = L \Leftrightarrow h_K = h_L, \tag{2.2}$$

$$h_{\alpha K} = \alpha h_K, \qquad \alpha \ge 0, \tag{2.3}$$

$$h_{K+L} = h_K + h_L. \tag{2.4}$$

We shall also deal with the restriction of h_K to S^{d-1}, with the further properties:

$$|h_K(u) - h_K(u')| \le \|K\| \, \|u - u'\|, \qquad u, u' \in S^{d-1}, \tag{2.5}$$

$$\rho(K, L) = \max\{|h_K(u) - h_L(u)| \, | u \in S^{d-1}\}, \tag{2.6}$$

$$K_n \to K \Leftrightarrow h_{K_n}(u) \to h_K(u), \qquad \forall\, u \in S^{d-1}. \tag{2.7}$$

A *random convex body* X is a Borel measurable map from a probability space into $\mathscr{K}$. Among several equivalent ways of defining the expectation of X [Aumann (1965), Debreu (1967) and Artstein and Vitale (1975)], we give a quick sketch of one based on support functions: If $E\|X\| < \infty$, an easy application of the Cauchy–Schwarz inequality gives $E|h_X(x)| < \infty$ and hence the existence of $Eh_X(x)$, $\forall\, x \in R^{d-1}$. A convex combination of support functions is again a support function [(2.3) and (2.4)], and this generalizes to show that $Eh_X(\cdot)$ is a support function. Its associated convex body is defined to be the *expectation* EX of X: $h_{EX}(x) \equiv Eh_X(x)$, $x \in R^{d-1}$.

Among convex bodies, we focus on finite sums of line segments, or *zonotopes*, and their limits in the Hausdorff metric called *zonoids* [Bolker (1969) and Schneider and Weil (1983)].

3. Zonoids and eads. To fix ideas [and incidentally answering a query of Bolker (1971)], we begin with a probabilistic interpretation of zonoids.

THEOREM 3.1. *$Z \in \mathscr{K}$ is a zonoid if and only if there is some $x \in R^d$ and random vector Y with $E\|Y\| < \infty$ such that $Z = x + E\overline{OY}$.*

PROOF. For the only if part, rewrite a typical zonotope $\overline{x_1y_1} + \overline{x_2y_2} + \cdots + \overline{x_ny_n}$ as

$$p_1\overline{u_1v_1} + p_2\overline{u_2v_2} + \cdots + p_n\overline{u_nv_n}, \tag{3.1}$$

where $p_i = \|\overline{x_iy_i}\|/\Sigma\|\overline{x_jy_j}\|$ and $\overline{u_iv_i} = (\Sigma\|\overline{x_jy_j}\|/\|\overline{x_iy_i}\|)\overline{x_iy_i}$. Localize $\overline{u_iv_i}$ by fixing $2d$ signed unit vectors $e_1, \dots, e_{2d}$ and observing that $\|\overline{x_jy_j}\| \le$

$\Sigma_k h_{\overline{x_j y_j}}(e_k)$ so that

$$\sum_j \|\overline{x_j y_j}\| \le \sum_j \sum_k h_{\overline{x_j y_j}}(e_k) = \sum_k \sum_j h_{\overline{x_j y_j}}(e_k) \le \sum_k \|\overline{x_j y_j}\| = 2d\|\overline{x_j y_j}\|.$$

Thus each $\overline{u_i v_i}$ lies in the ball $(2d\Sigma\|\overline{x_j y_j}\|)B$. Thinking of (3.1) as $E\overline{UV}$, we conclude that if Z is a zonoid, it is the limit of zonotopes $\{Z_N = E\overline{U_N V_N}\}_{N=1}^{\infty}$, where $U_1, V_1, U_2, V_2, \ldots$ are uniformly bounded random vectors. Further, without loss of generality, $(U_N, V_N) \to_d (U, V)$ so that, for each $u \in S^{d-1}$,

$$\begin{aligned} h_{Z_N}(u) &= E\max(\langle u, U_N\rangle, \langle u, V_N\rangle) \to E\max(\langle u, U\rangle, \langle u, V\rangle) \\ &= \langle u, EU\rangle + E\max(0, \langle u, V-U\rangle), \end{aligned}$$

which together with (2.7) and setting $x = EU$ and $Y = V - U$ gives the desired representation. For the if part, assume without loss of generality that $x = 0$. Given $\varepsilon > 0$, let Y_ε be a random vector with finite support such that $E\|Y - Y_\varepsilon\| < \varepsilon$. Then $E\overline{OY_\varepsilon}$ is a zonotope and by elementary estimates $h_{\overline{OY_\varepsilon}}(u) \to h_{\overline{OY}}(u)$ for each $u \in S^{d-1}$ as $\varepsilon \to 0$. By (2.7), $\overline{OY} = \lim_{\varepsilon\to 0}\overline{OY_\varepsilon}$ and is hence a zonoid. □

Our point of departure for examining eads is the following representation theorem. Recall that M_Y signifies a $d \times d$ matrix whose columns are iid copies of Y.

THEOREM 3.2. *Let Y be a random d-vector with $E\|Y\| < \infty$. Then*

$$E|\det M_Y| = d!\,\mathrm{vol}(E\overline{OY}). \tag{3.2}$$

PROOF. Consider an infinite sequence $Y_1, Y_2, \ldots$ of iid copies of Y, and for each $n = 1, 2, \ldots$ form the zonoid $Z_n = (1/n)[\overline{OY_1} + \overline{OY_2} + \cdots + \overline{OY_n}]$. Since $E\|Y\| < \infty$, we have $E\|\overline{OY}\| < \infty$. This suffices to invoke the relevant strong law of large numbers [Artstein and Vitale (1975)] to conclude that $Z_n \to E\overline{OY}$ a.s. in the Hausdorff metric. Since the volume functional is continuous,

$$\mathrm{vol}(Z_n) \to \mathrm{vol}(E\overline{OY}) \quad \text{a.s.} \tag{3.3}$$

On the other hand, Shephard (1974) provides the expansion

$$\mathrm{vol}(Z_n) = \frac{1}{n^d} \sum_{i_1 < i_2 < \cdots < i_d} \left|\det M(Y_{i_1}, \ldots, Y_{i_d})\right|, \tag{3.4}$$

where we have displayed particular columns of the matrix. By Hadamard's determinant theorem [e.g., Roberts and Varberg (1973), page 205], the displayed term in (3.4) is bounded above by $\|Y_{i_1}\|\,\|Y_{i_2}\| \cdots \|Y_{i_d}\|$ and so has finite expectation. It follows from Hoeffding's strong law for U-statistics [e.g., Serfling (1980)] that $\mathrm{vol}(Z_n) \to (d!)^{-1}E|\det M_Y|$ a.s., which upon comparison with (3.3) completes the proof. □

REMARKS. 1. Note that (3.2) exhibits a kind of interchange between volume evaluation and taking of expectations.

2. Representations for zonoids and versions of (3.2) with no probabilistic connotation can be found in the convex geometry literature [e.g., Schneider and Weil (1983), page 307].

4. Bounds. In a few special cases, for example, an isotropic (or even elliptically contoured) distribution for Y, vol($\overline{EOY}$) can be evaluated directly to provide a value for (3.2). It is, however, precisely in these cases when $E|\det M_Y|$ can likely be evaluated by other methods as well. It is in more general situations that (3.2) appears to be useful and then not for exact evaluation but for bounds. The idea is to parlay some information about the distribution of Y into information about the shape (and size) of $\overline{EOY}$.

For example, Hadamard's bound in the form $|\det M_Y| \le \|Y_1\| \cdot \|Y_2\| \cdots \|Y_d\|$ upon taking expectations, yields the bound $E|\det M_Y| \le (E\|Y\|)^d$. One would expect to get a better bound by relaxing the point-wise requirement, and this was already indicated in a preliminary study [Vitale (1988)].

THEOREM 4.1. *Let Y be a random d-vector with $E\|Y\| < \infty$. Then $E|\det M_Y| \le \alpha_d(E\|Y\|)^d$, where*

$$\alpha_d = \frac{\Gamma(d+1)}{\Gamma(d/2+1)}\left[\frac{\Gamma(d/2)}{2\Gamma((d+1)/2)}\right]^d$$

satisfies $\alpha_d^{1/d} \to e^{-1/2}$ as $d \to \infty$.

[Note: In Vitale (1988), page 203, the expressions for γ_d and the later asymptotic bound carry an incorrect divisor of 2.]

Theorem 4.1 can be improved by making use of a remarkable and in some ways definitive result of Lutwak (1975). The sharpening can be explained by noting that Theorem 4.1 is based on the inequality of Urysohn [e.g., Burago and Zalgaller (1988)] which asserts that the volume of a convex body K (e.g., $\overline{EOY}$) is bounded above by the volume of a ball which shares the same mean width (i.e., average separation of parallel supporting hyperplanes): $E[h_K(U) + h_K(-U)]$ for U uniform on S^{d-1}. Lutwak showed that the assertion of Urysohn can be sharpened by replacing mean width with "$(-p)$-mean width" $w_{-p} = \{E[h_K(U) + h_K(-U)]^{-p}\}^{-1/p}$.

THEOREM 4.2. *Let Y be a random d-vector with $E\|Y\| < \infty$. Then $E|\det M_Y| \le d!\, w_{-d}^d \beta_d$, where β_d is the volume of the d-ball and $w_{-d} = \{E\{E[|\langle U, Y\rangle| \,|U]\}^{-d}\}^{-1/d}$. Equality occurs iff $\overline{EOY}$ is an ellipsoid.*

PROOF. $\overline{EOY}$ has width $E|\langle u, Y\rangle|$ in the direction u. Lutwak's result then applies together with his assertion that the bound is tightest for $p = d$ in which case the indicated condition for equality holds. □

For comparison, we give another approach to bounding (3.2). As can be read from the condition for equality, this bound is sharper than that in Theorem 4.2 if $\overline{EOY}$ is close to a rectangular parallelepiped.

THEOREM 4.3. *Under the same conditions as Theorem* 4.1,

$$E|\det M_Y| \le d!\min \prod_{i=1}^{d} E|\langle u_i, Y\rangle|, \tag{4.1}$$

where the min is taken over orthonormal bases $\{u_1, \dots, u_d\}$. *Equality occurs iff there is an orthonormal basis* $\{u_1^*, u_2^*, \dots, u_d^*\}$ *such that the event* $\bigcup_{k=1}^{d}\{Y = \langle Y, u_k^*\rangle u_k^*\}$ *occurs with probability* 1.

PROOF. The width of $\overline{EOY}$ in the direction $u \in S^{d-1}$ is $E|\langle u, Y\rangle|$. If $u_1, \dots, u_d$ form an orthonormal basis, it follows that some translate of $\overline{EOY}$ lies in $[0, E|\langle u_1, Y\rangle|] \times [0, E|\langle u_2, Y\rangle|] \times \cdots \times [0, E|\langle u_d, Y\rangle|]$. This establishes (4.1). Equality occurs iff the translate of $\overline{EOY}$ coincides with the indicated set. In this case $\overline{EOY}$ is a zonotope and Y must behave as indicated. □

Finally, we note that bounds and expressions for $E|\det M_Y|^2$ have been discussed [Wilks (1960), Grenander (1978), pages 416–421, and Lindsay (1989a)]. Via Chebyshev's inequality, this leads to statements about $E|\det M_Y|$, which we shall take up elsewhere.

5. Comparisons.

In addition to providing bounds for eads, we can also make various comparisons. The following is a consequence of Theorem 3.2.

THEOREM 5.1. *Suppose that* Y *and* Y' *are random* d*-vectors with* $\max\{E\|Y\|, E\|Y'\|\} < \infty$. *If there is a vector* $a \in R^d$ *such that* $E|\langle u, Y\rangle| \le E|\langle u, Y'\rangle| + \langle u, a\rangle$ *for all* $u \in S^{d-1}$, *then* $E|\det M_Y| \le E|\det M_{Y'}|$.

PROOF. It is clear from (3.2) that the conclusion holds if some translate of $\overline{EOY}$ lies in $\overline{EOY'}$. In terms of support functions, this means that $E\langle u, Y\rangle_+ \le E\langle u, Y'\rangle_+ + \langle u, a\rangle$ for some $a \in R^d$ [recall (2.1) and the fact that translation of bodies appears in support functions as a linear term]. An alternative form follows by taking expectations on the identity $\langle u, Y\rangle_+ = |\langle u, Y/2\rangle| + \langle u, Y/2\rangle$ to get $h_{\overline{EOY}} = h_{1/2E\,\overline{-Y,Y}} + \langle\,\cdot\,, EY/2\rangle$, which asserts that $1/2E\,\overline{-Y,Y}$ is a translate of $\overline{EOY}$. The same holds for $1/2E\,\overline{-Y', Y'}$ and $\overline{EOY'}$, which provides the asserted sufficient condition. □

We use this result to show that making a distribution more diffuse in a certain way increases the ead.

THEOREM 5.2. *If Y and Y' are independent random d-vectors with* $\max\{E\|Y\|, E\|Y'\|\} < \infty$, $EY' = 0$, *then* $E|\det M_Y| \le E|\det M_{Y+Y'}|$.

PROOF.

$$
\begin{aligned}
E|\langle u, Y\rangle| &= E(\operatorname{sgn}\langle u, Y\rangle\langle u, Y\rangle)\\
&= E[\operatorname{sgn}\langle u, Y\rangle\langle u, Y+Y'\rangle + \operatorname{sgn}\langle u, Y\rangle\langle u, -Y'\rangle]\\
&\le E|\langle u, Y+Y'\rangle| + E\operatorname{sgn}\langle u, Y\rangle\langle u, -Y'\rangle\\
&\le E|\langle u, Y+Y'\rangle| + E\operatorname{sgn}\langle u, Y\rangle \cdot E\langle u, -Y'\rangle\\
&\le E|\langle u, Y+Y'\rangle|. \qquad \square
\end{aligned}
$$

We show next that moving the center of a symmetric distribution away from the origin results in a larger ead [cf. Anderson (1955) and Vitale (1990) for a similar statement about multivariate densities].

THEOREM 5.3. *Suppose that $x_0 \in R^d$, $x_0 \ne 0$, is fixed and that Y is a random d-vector which is symmetrically distributed about the origin and with* $E\|Y\| < \infty$. $E|\det M_{Y+\lambda x_0}|$ *is an increasing function of the positive parameter λ.*

PROOF. From Theorem 5.1, it suffices to show that for $0 < \lambda < 1$,

$$E|\langle u, Y+\lambda x_0\rangle| \le E|\langle u, Y+x_0\rangle|, \qquad \forall\, u \in S^{d-1}.$$

Employing $0 < \theta < 1$ such that $Y + \lambda x_0 = \theta(Y + x_0) + (1-\theta)(Y - x_0)$, we have

$$
\begin{aligned}
E|\langle u, Y+\lambda x_0\rangle| &= E|\langle u, \theta(Y+x_0) + (1-\theta)(Y-x_0)\rangle|\\
&\le \theta E|\langle u, Y+x_0\rangle| + (1-\theta)E|\langle u, Y-x_0\rangle|\\
&\le \theta E|\langle u, Y+x_0\rangle| + (1-\theta)E|\langle u, -Y-x_0\rangle|\\
&\le E|\langle u, Y+x_0\rangle|. \qquad \square
\end{aligned}
$$

The next result, a concavity feature under mixtures, resembles the classical determinantal inequality of Minkowski [Roberts and Varberg (1973), page 205].

THEOREM 5.4. *Suppose the Y is a mixture of two random d-vectors, each with finite expectation: $Y = Y_1$ with probability p, $Y = Y_2$ with probability $1 - p$. Then*

$$\left(E|\det M_Y|\right)^{1/d} \ge p\left(E|\det M_{Y_1}|\right)^{1/d} + (1-p)\left(E|\det M_{Y_2}|\right)^{1/d}.$$

PROOF. The mixture model implies that $E\overline{OY} = pE\overline{OY_1} + (1-p)E\overline{OY_2}$. An appeal to (3.2) and an application of the Brunn–Minkowski inequality [Eggleston (1969)] then yields the result. $\square$

We close with a comparison of taking mixtures and convex combinations.

THEOREM 5.5. *Let Y, Y_1 and Y_2 be as in the last theorem, and let* $\tilde{Y} = pY_1 + (1-p)Y_2$. $E|\det M_{\tilde{Y}}| \le E|\det M_Y|$.

PROOF. Note that almost surely

$$\overline{O\tilde{Y}} = \overline{pOY_1 + (1-p)OY_2} \subseteq p\overline{OY_1} + (1-p)\overline{OY_2}$$

so that

$$E\overline{O\tilde{Y}} \subseteq E\left[p\overline{OY_1} + (1-p)\overline{OY_2}\right] = pE\overline{OY_1} + (1-p)E\overline{OY_2} = E\overline{OY}$$

and the result then follows from (3.2). □

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