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A BRANCHING RANDOM WALK WITH A BARRIER

BY J. D. BIGGINS, BORIS D. LUBACHEVSKY, ADAM SHWARTZ AND ALAN WEISS

University of Sheffield, Technion Israel Institute of Technology, and AT&T Bell Laboratories

Suppose that a child is likely to be weaker than its parent and a child who is too weak will not reproduce. What is the condition for a family line to survive? Let b denote the mean number of children a viable parent will have; we suppose that this is independent of strength as long as strength is positive. Let F denote the distribution of the change in strength from parent to child, and define $h = \sup_\theta(-\log(\int e^{\theta t}\, dF(t)))$. We show that the situation is black or white:

1. If $b < e^h$, then $P(\text{family line dies}) = 1$.
2. If $b > e^h$, then $P(\text{family survives}) > 0$.

Define $f(x) := E(\text{number of members in the family} \mid \text{initial strength } x)$. We show that if $b < e^h$, then there exists a positive constant C such that $\lim_{x\to\infty} e^{-\alpha x} f(x) = C$, where α is the smaller of the (at most) two positive roots of $b\int e^{st}\, dF(t) = 1$. We also find an explicit expression for $f(x)$ when the walk is on a lattice and is skip-free to the left.

This process arose in an analysis of rollback-based simulation, and these results are the foundation of that analysis.

Introduction. Consider the following naïve model of reproductive prowess. An individual with a nonpositive prowess value cannot reproduce. The number of offspring of each individual is chosen independently from a fixed offspring distribution (regardless of its prowess value, provided only it is positive). The prowess value of an offspring is computed by adding a random number (independently, and from a second fixed distribution) to the prowess of its parent.

This is a standard random walk on a branching process, but with a barrier: A branch terminates when its value is nonpositive. We describe the model in greater detail (and greater modesty) in Section 2. The main result of this paper is simple necessary and sufficient conditions under which, from some generation on, no more reproduction occurs. We also obtain the asymptotics of the mean size of the nth generation and, when the process dies out, of the mean number of members in the family. Additionally, in the case when the random walk is on a lattice and skip-free downwards, and the branching has integer mean, we show how to calculate exactly the mean total population.

This model of branching random walk with a barrier arose in the authors' work on parallel simulations (Lubachevsky, Shwartz and Weiss [15, 16]). We

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conclude this section with a quick description of how this process arises and the implications of its analysis.

Consider many interconnected processing nodes simulating different components of a large system. The nodes are allowed to progress at their individual speeds, but are related in that a node, completing its tasks up to (simulated) time t, may send a job to be processed by another node. If the second node has already completed simulating up to a further time, say $t + s$, it may have to retract or roll back and reconsider its actions from time t. This rollback may start a chain reaction, where the second node sends a correction message to other nodes, forcing them to roll back and so on. The size of the rollback (s for the second node) is modeled as the originating rollback plus an independent random variable. Rollback clearly stops when it becomes nonpositive, and the resulting model is the one described above. For a more accurate description, see [15, 16]. It turns out that the results presented here make it possible to give sufficient conditions for the efficiency of a class of parallel simulation algorithms. These conditions are tight for the class under consideration.

Theorems on branching random walks *without barriers* have been presented by Biggins [4, 5, 6], Kingman [13], Nerman [18] and others. Our stability result is very close to Biggins [6]; surprisingly, the barrier adds little complication for the basic methods (for more detailed proofs of some of these results see [17]). Kesten [12] studied a branching diffusion with a barrier, which is also closely related to the present model.

1. Model. Consider a supercritical branching random walk on the real line started from a single initial ancestor at the origin (see Biggins [4, 5, 6] for a discussion). We denote by $Z(n; x)$ the number of nth generation people larger than $(-x)$. For our application it is natural to focus attention on the special case where each child receives an i.i.d. displacement from its parent. Let b denote the mean family size and F the displacement distribution. We are dealing with the supercritical case ($b > 1$). We assume that F has negative mean ν but attaches positive probability to the positive half axis, so that a random walk based on F drifts downwards but may have upward jumps. We also assume that F has an exponentially decaying right tail; that is, for some $s > 0$, we have $\int_{-\infty}^{\infty} e^{st}\, dF(t) < \infty$. In practice we do not expect this final condition to be burdensome. Without it the results can be expected to be quite different; see Durrett [10]. We define the Chernoff rate $h = h(0)$ through

$$h(a) := \sup_{\theta} \left(\theta a - \log \int_{-\infty}^{\infty} e^{\theta t}\, dF(t) \right). \tag{1}$$

Denote by θ_a^* the maximizer in (1) and set $\theta^* = \theta_0^*$ (these are unique and positive for $a > \nu$, by convexity; see [19]).

We will use the sample paths of this branching random walk to construct realizations of our model for rollback, which is a branching random walk with a barrier. We do this by deleting all people smaller than $(-x)$ and all their descendants. Any line of descent remaining is then just a random walk, started at 0, with a barrier at $(-x)$. Denote by $z(n; x)$ the number of nth generation

people remaining after the deletion (we assume throughout that $x \geq 0$). Obviously $z(n;x) \leq Z(n;x)$. We could now shift the origin to $(-x)$, so that the initial ancestor is at x and the barrier is at 0, to obtain the model for rollback as developed in [15, 16].

2. Stability conditions.

THEOREM 1. (i) $\lim_{n\to\infty}(1/n)\log Ez(n;x) = \log b - h$.
(ii) *If* $\log b - h < 0$, *then* $P(\lim_{n\to\infty} z(n;x) = 0) = 1$.
(iii) *If* $\log b - h > 0$, *then* $P(\lim_{n\to\infty} z(n;x) = \infty) > 0$.

Note that the condition $\log b - h < 0$ depends on the offspring distribution only through its mean b. Also, the condition does not depend on x and so is independent of the location of the barrier.

PROOF. Note first that, as $n \to \infty$,

$$\frac{1}{n}\log Ez(n;x) \leq \frac{1}{n}\log EZ(n;x) \to \log b - h$$

using Theorem 1 of Biggins [6]. This, and a Chebyshev bound, show that $P(z(n;x) \geq 1)$ decays geometrically when $\log b - h < 0$, proving (ii). Let w_i be i.i.d. with distribution function F and let

$$Q_n(x) = P\left(\inf_{1\leq j\leq n}\sum_{i=1}^{j} w_i > -x\right),$$

which is the probability the random walk fails to get below $(-x)$ in n steps. Then it is clear that

$$Ez(n;x) = b^n Q_n(x) \geq b^n Q_n(0).$$

By a well-known variant of Chernoff's theorem (see [14], Lemma 2.1), $\lim_{n\to\infty}(1/n)\log Q_n(0) = -h$, completing the proof of (i).

To establish part (iii), construct a lower-bounding supercritical Galton–Watson process. Let n^* be large enough that $Ez(n^*;0) = b^{n^*}Q_{n^*}(0) > 1$. The first generation of the new process is now given by $z(n^*;0)$ $(\leq z(n^*;x))$, with (independent) copies of $z(n^*;0)$ being used to construct subsequent generations. This process grows exponentially fast with positive probability and can be constructed so that its sample paths are below those of $z(n;x)$, proving the result. (More detailed examples using this technique can be found in Kingman [13] and Biggins [4, 6] for example.) □

Let B_n denote the largest member of the nth generation in a branching random walk starting at 0, and let Y_x denote the largest member of such a process, but starting at x and with a barrier at 0. Let $\gamma = \sup\{a: \log b > h(a)\}$; then it is known ([6], Corollary to Theorem 2), that $\lim_n B_n/n = \gamma$ almost surely on the survival set. To simplify the discussion of the first part of the

next theorem, we will rule out the possibility that there is a finite maximum displacement that has probability at least b^{-1}. With this proviso γ is the unique solution to $h(\gamma) = \log b$ greater than ν (there may be another below ν) and $0 < \theta_\gamma^* < \infty$. In the excluded case, on which more information can be found in Bramson [7], $B_n \leq n\gamma$ always, so part (i) of the theorem will hold, rather trivially, in this case also.

THEOREM 2. (i) *There exists a constant* $C > 0$ *such that the random variable* $\sup_n(B_n - \gamma n - C \log n)$ *possesses an exponential right tail.*

(ii) *Suppose* $\log b - h < 0$. *There exists a constant* $D > 0$ *such that*

$$P(Y_x - x > t) \leq De^{-\theta^* t}.$$

PROOF. Let $x(n)$ have the distribution of the size of a typical person at level n. For any deterministic sequence s_n and any $\theta > 0$,

$$\begin{aligned} P\Big(\sup_n (B_n - s_n) > t\Big) &\leq \sum_n P(B_n - s_n > t) \\ &\leq \sum_n b^n P(x(n) > t + s_n) \\ &\leq \sum_n b^n e^{-\theta(t+s_n)} E e^{\theta x(n)} \\ &= \sum_n b^n e^{-\theta(t+s_n)} \Big(\int e^{\theta r}\, dF(r)\Big)^n \end{aligned}$$

using Chebyshev's inequality. Take $s_n = n\gamma + C \log n$ and $\theta = \theta_\gamma^* > 0$ to obtain (i). For (ii), set $s_n = 0$ with $\theta = \theta^* > 0$ and note that the largest member of the process without a barrier exceeds the largest member of the process with a barrier. □

It is probably worth noting that in part (ii), θ^* can be replaced by ϕ provided that $\phi \in \{\theta\colon b\int e^{\theta t}\, dF(t) < 1\}$. (Of course D changes.)

Part (ii) immediately gives $\limsup_{x\to\infty}(Y_x/x) = 1$ w.p.1. This is important in our application since the size of a rollback also represents the amount of memory required to support the rollback. Now the design of a rollback-based simulation has to take memory limitations into account and our analysis says that if a (large) initial rollback can be supported, then so can the whole rollback tree it generates.

EXTENSIONS. The general model of branching random walks allows for correlations between the step sizes of siblings and their number [6]. Furthermore, if the random walk is modulated by a Markov chain, then the variant of

Chernoff's theorem used above would be replaced by

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log P\Big(\inf_{1 \le i \le n} w_1 + \cdots + w_i + \varepsilon n > 0 \Big)$$

$$= \lim_{n \to \infty} \frac{1}{n} \log P(w_1 + \cdots + w_n > 0)$$

and results on Markov modulated random walks could be applied (e.g. [9]). Finally, a more realistic model of rollback should allow for the distribution F to depend on the current size of the rollback; Azencott and Ruget's large deviation theory applies to this case [3].

3. Examples. In studying rollback, we found random walks on binary trees to be important. They model very closely the rollback process on a shuffle-exchange network [15, 16]. It turns out that the stability region as computed by naïve stationary considerations is wrong by almost an order of magnitude (see below). Since our analysis is in very close agreement with simulation results, we work out some simple examples below. Consider a *binary tree* ($b = 2$) with $P(w = 1) = q$, $P(w = -K) = 1 - q$. Then for $K = 1$, consider a naïve one-step stationary analysis: both children of a node will have smaller prowess than the parent with probability $(1 - q)^2$. Hence we might expect that the tree is finite when $(1 - q)^2 > \frac{1}{2}$ or $q < 0.3$. This naïve analysis is off by more than a factor of four; elementary calculus yields that the stability condition $b < e^h$ is satisfied when $q < (2 - \sqrt{3})/4 \doteq 0.066987$.

When $q = \frac{1}{2}$, then $Ew = (1 - K)/2$ and $h = -\ln(\frac{1}{2}K^{1/(K+1)}(1 + (1/K)))$. So $be^{-h} = K^{1/(K+1)}(1 + (1/K)) > 1$. That is, this way of adding negative drift is *ineffective* in stabilizing the process. In fact, it is easy to show that as $K \to \infty$, the critical value of q is about $\frac{1}{2}(1 - (\ln K/K))$.

The following are some values of the critical value of q, calculated numerically.

K	1	2	5	10	20	50	100	200	500	1000	∞
q	0.0670	0.147	0.270	0.348	0.405	0.452	0.472	0.484	0.493	0.496	0.5

4. Growth rate of the process. It is possible to find *asymptotic* expressions for the expected number of nodes in the tree (when $\log b - h < 0$ so that this number is finite), and for the total weight of the tree, and *exact* expressions in special cases. The results presented below range from most general (and least informative) to most special (and most informative).

4.1. *Asymptotics.* In this section we assume that $\log b - h < 0$ so that, for some $s > 0$, $b\int e^{st}\, dF(t) < 1$. Therefore there is a smallest α (necessarily > 0) such that

$$b\int e^{\alpha t}\, dF(t) = 1 \tag{2}$$

(by convexity there are at most two solutions to this equation). If F were

concentrated on $(-\infty, 0)$ this would be (essentially) the classical definition of the Malthusian parameter for the branching process. The case of general F has not often been considered, but some information relevant here can be found in Biggins [5]. In particular note that

$$\frac{d}{ds} b \int e^{st}\, dF(t)\Big|_{s=\alpha} < 0. \tag{3}$$

Let

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} Ez(n;x), & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \tag{4}$$

so $f(x)$ is the mean total family size when the initial prowess is x.

THEOREM 3. *Suppose* $\log b - h < 0$ *and that* F *is nonlattice. Let* α *be the smallest positive root of* (2). *Then for some* $C > 0$, $\lim_{x\to\infty} e^{-\alpha x} f(x) = C$.

REMARK. A similar result holds in the lattice case.

PROOF. Since the mean number of nodes in a tree starting from root size x is one plus the mean number of children times their subtree's mean size, f satisfies

$$f(x) = \begin{cases} 1 + b\int_{-\infty}^{\infty} f(x+t)\, dF(t), & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases} \tag{5}$$

This can be viewed as a renewal equation with a barrier. If the constant 1 were removed in (5) the resulting equation (of Deny type) falls within the study of Alzaid, Lau, Rao and Shanbhag [1]. Our discussion of the asymptotics of f has some features in common with that study.

Define $H(x) := e^{-\alpha x} f(x)$ and $d\mu(t) := be^{\alpha t}\, dF(t)$. Note that, by the definition of α, μ is a probability measure. Let $\{Z_i, i \ge 1\}$ be i.i.d. random variables distributed according to μ and let $Z_0 = S_0 = 0$, $S_n := \sum_{j=1}^{n} Z_j$. Note that by (3), $EZ_i < 0$. Let U denote the renewal measure corresponding to $\{Z_i\}$. Now

$$\begin{aligned} H(x) &= e^{-\alpha x} \sum_{n=0}^{\infty} Ez(n;x) \le e^{-\alpha x} \sum_{n=0}^{\infty} EZ(n;x) \\ &= e^{-\alpha x} \sum_{n=0}^{\infty} \int_{-x}^{\infty} e^{-\alpha t} \mu^{n*}(dt) \qquad (6) \\ &= \int_{-x}^{\infty} e^{-\alpha(x+t)} U(dt), \end{aligned}$$

which is bounded, using Theorem 1 (iii) of VI.10 in Feller [11].

Define $\mathbf{1}(x) := 1$ if $x > 0$, and zero otherwise. From (5), $H(x)$ satisfies

$$H(x) = e^{-\alpha x}\mathbf{1}(x) + \int_{-\infty}^{\infty} H(x+t)\,d\mu(t)\mathbf{1}(x)$$
$$= e^{-\alpha x}\mathbf{1}(x) + EH(x+Z_1)\mathbf{1}(x).$$

Recursive substitutions yield

$$\begin{aligned} H(x) &= e^{-\alpha x}\mathbf{1}(x) + Ee^{-\alpha(x+Z_1)}\mathbf{1}(x)\mathbf{1}(x+Z_1) \\ &\quad + EH(x+Z_1+Z_2)\mathbf{1}(x)\mathbf{1}(x+Z_1) \\ &= \sum_{n=0}^{\infty} e^{-\alpha x}E\left(\prod_{j=0}^{n} e^{-\alpha Z_j}\mathbf{1}(x+S_j)\right) + \lim_{n\to\infty} E\left(H(x+S_n)\prod_{i=0}^{n}\mathbf{1}(x+S_i)\right). \end{aligned} \tag{7}$$

Since $EZ_i < 0$, dominated convergence implies that the final term is zero.

Define for an interval I,

$$\psi_x(I) := \sum_{n=0}^{\infty} P(S_i + x > 0 \text{ for } 0 \le i \le n; S_n + x \in I).$$

For each x, ψ_x is a positive measure, and it is finite for bounded intervals I since Z_1 has negative mean. By (7), we have

$$H(x) = \int_0^{\infty} e^{-\alpha t}\,d\psi_x(t) = \alpha\int_0^{\infty} e^{-\alpha t}\psi_x([0,t])\,dt,$$

using integration by parts. Consider now a random walk starting at $x > t$ and with steps Z_i, and let L_x denote the location of the first step below t. Then by standard renewal arguments, L_x converges weakly. But $\psi_x([0,t])$ is precisely the mean number of steps, starting at L_x, that fall between 0 and t before first crossing 0 downwards. By the nonlattice assumption, this depends smoothly on the distribution of the starting point L_x, hence $\psi_x([0,t])$ has a limit as $x \to \infty$ for each t. Since $\psi_x([0,t]) \le C_1 t + C_2$ for some constants that are independent of $x > 0$, dominated convergence applies to prove that, for some constant C, $H(x) \to C$. □

Estimates on C are easily obtained as follows. Recall that U is the renewal measure for the random variables $\{Z_i\}$ and let V be the renewal measure for the descending ladder variables associated with $\{Z_i\}$ (see, e.g., Feller [11], page 391). Then U is not restricted to have all partial sums above $(-x)$, while V counts only decreasing partial sums. Hence

$$V([-x, -x+t]) \le \psi_x([0,t]) \le U([-x, -x+t])$$

for all $x > t > 0$. But ([11], page 381) $U([-x, -x+t]) \to t/(EZ_1)$, so that an application of dominated convergence as above yields $C \le (\alpha EZ_1)^{-1}$. Since the probability of an upward jump is strictly positive, this inequality is in fact strict. Similar considerations provide a lower bound on C in terms of the mean of the descending ladder variables.

The same analysis gives a related interesting result. With a slight shift in notation, let $z(n;x)$ now denote the point process of all nth generation persons in a branching random walk started at x with a barrier at 0. Let g be the total weight, that is,

$$g(x) = \begin{cases} \sum_{n=0}^{\infty} E\int_0^{\infty} yz(n;x)(dy), & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

Then Theorem 3 holds with g in place of f. In the context of rollback, g represents the total memory, over all processors, used by one rollback tree. This is of interest if rollback is supported by memory servers.

4.2. *Exact calculations.* We can find exact expressions for $f(x)$ in the following particular case:

1. $b > 1$ is an integer.
2. F is supported on $-1, 0, 1, 2, \ldots$.

It is easy to see that $f(x)$ depends only on the mean of the branching factor. Thus, in deriving an expression for $f(x)$, we assume without loss of generality that branching is deterministic, each person having exactly b children. Now $f(x)$ need only be calculated for $x = 0, 1, 2, \ldots$ since $f(x) = f(\lceil x \rceil)$ where $\lceil x \rceil$ is the "round up" or ceiling function.

Let n^+ be the number of live people, that is, they and all their ancestors are above the barrier. Let n^- be the number of immediate children of these individuals that are below (or, more precisely, as F is skip-free downwards, on) the barrier. These children are therefore not alive, but they will be if the barrier is moved down one step. It is easy to see that $bn^+ = (n^+ - 1) + n^-$. Recall $f(x) = E(n^+ \mid \text{initial prowess} = x)$, so, by moving the barrier down one step, we see that

$$\begin{aligned} f(x+1) &= f(x) + E(n^-) \cdot f(1) = f(x) + [(b-1)f(x) + 1]f(1) \\ &= f(x)(1 + (b-1)f(1)) + f(1). \end{aligned}$$

This recursion can be solved easily. Let $c = 1 + (b-1)f(1)$. Then

$$f(x) = \frac{c^x - 1}{c - 1} f(1). \tag{8}$$

But by (the comment following) Theorem 3, $f(x) \sim e^{\alpha x}$, so

$$c = e^{\alpha} = 1 + (b-1)f(1). \tag{9}$$

We solve (2) for α, then find c and $f(1)$ by (9) and so find $f(x)$ using (8).

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REFERENCES

[1] ALZAID, A. A., LAU, K.-S., RAO, C. R. and SHANBHAG, D. N. (1988). Solution of Deny convolution equation restricted to a half line via a random walk approach. *J. Multivariate Anal.* **24** 309–329.

[2] ASMUSSEN, S. and HERING, H. (1983). *Branching Processes*. Birkhäuser, Boston.

[3] AZENCOTT, R. and RUGET, G. (1977). Mélanges d'équations différentielles et grands écartes à la loi des grands nombres. *Z. Wahrsch. Verw. Gebiete* **38** 1–54.

[4] BIGGINS, J. D. (1976). The first- and last-birth problem for a multitype age-dependent branching process. *Adv. Appl. Probab.* **8** 446–459.

[5] BIGGINS, J. D. (1977). Martingale convergence in the branching random walk. *J. Appl. Probab.* **14** 25–37.

[6] BIGGINS, J. D. (1977). Chernoff's theorem in the branching random walk. *J. Appl. Probab.* **14** 630–636.

[7] BRAMSON, M. (1978). Minimal displacement of branching random walk. *Z. Wahrsch. Verw. Gebiete* **45** 89–108.

[8] CHERNOFF, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statist.* **23** 493–507.

[9] DONSKER, M. D. and VARADHAN, S. R. S. (1975, 1976). Asymptotic evaluation of certain Markov process expectations for large times. *Comm. Pure Appl. Math.* **28** 279–301; **29** 389–461.

[10] DURRETT, R. (1983). Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete* **62** 165–170.

[11] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

[12] KESTEN, H. (1978). Branching Brownian motion with absorption. *Stochastic Process. Appl.* **37** 9–47.

[13] KINGMAN, J. F. C. (1975). The first birth problem for an age-dependent branching process. *Ann. Probab.* **3** 790–801.

[14] LAGARIAS, J. and WEISS, A. (1991). The $3x + 1$ problem; two stochastic models. *Ann. Appl. Probab.* To appear.

[15] LUBACHEVSKY, B., SHWARTZ, A. and WEISS, A. (1989). Rollback sometimes works ... if filtered. In *Proceedings of the Winter Simulation Conference* 630–639. IEEE, New York.

[16] LUBACHEVSKY, B., SHWARTZ, A. and WEISS, A. (1990). An analysis of rollback-based simulation. EE PUB 755, Technion, Israel.

[17] LUBACHEVSKY, B., SHWARTZ, A. and WEISS, A. (1990). The stability of branching random walks with a barrier. EE PUB 748, Technion, Israel.

[18] NERMAN, O. (1981). On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrsch. Verw. Gebiete* **57** 365–395.

[19] ROBERTS, W. and VARBERG, P. E. (1973). *Convex Functions*. Academic, New York.

J. D. BIGGINS
DEPARTMENT OF PROBABILITY AND STATISTICS
UNIVERSITY OF SHEFFIELD
P.O. BOX 597
S10 2UN
ENGLAND

BORIS LUBACHEVSKY
AT&T BELL LABORATORIES
600 MOUNTAIN AVENUE
MURRAY HILL, NEW JERSEY 07974

ADAM SHWARTZ
ELECTRICAL ENGINEERING
TECHNION—IIT
HAIFA 3200
ISRAEL

ALAN WEISS
AT & T BELL LABORATORIES
600 MOUNTAIN AVENUE
MURRAY HILL, NEW JERSEY 07974