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A DUALITY METHOD FOR OPTIMAL CONSUMPTION AND INVESTMENT UNDER SHORT-SELLING PROHIBITION. II. CONSTANT MARKET COEFFICIENTS[1]

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A continuous-time, consumption/investment problem with constant market coefficients is considered on a finite horizon. A dual problem is defined along the lines of Part 1. The value functions for both problems are proved to be solutions to the corresponding Hamilton–Jacobi–Bellman equations and are provided in terms of solutions to linear, second-order, partial differential equations. As a consequence, a mutual fund theorem is obtained in this market, despite the prohibition of short-selling. If the utility functions are of power form, all these results take particularly simple forms.

1. Introduction. This paper treats a consumption/portfolio decision problem for a single agent, endowed with some initial wealth, who can consume the wealth at some rate $C(t)$ and invest it in any of the $d+1$ available assets. The agent may not, however, sell any stock short. The agent is attempting to maximize a linear combination of two quantities, namely:

1. $E\int_0^T e^{-\delta t}U_1(C(t))\,dt$, the total expected discounted utility from consumption over the time interval $[0, T]$;
2. $Ee^{-\delta T}U_2(X(T))$, the expected discounted utility from terminal wealth.

There are $d+1$ assets or securities available to the agent. One of them is a bond, a security with constant instantaneous rate of return r. The other assets are stocks, risky securities whose prices are modelled by the lognormal process with mean rate of return b_i and dispersion coefficients σ_{ij}. We assume that the market is complete in the sense of Harrison and Pliska (1981) and Bensoussan (1984), except that short-selling of the stocks is prohibited. We also assume that our agent is a small investor in that his decisions do not influence the asset prices, which are treated as exogenous.

Single agent consumption/investment problems have been investigated by a number of authors. We refer the reader to the introduction of Xu and Shreve (1992), hereafter referred to as Part 1, for a fuller account. When short-selling of stocks is prohibited, Part 1 develops a duality and martingale method for the case of general (random, time-dependent) market coefficients. It is shown there how to construct a dual problem and how to characterize the optimal

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solution of the original primal problem in terms of the solution to the dual problem.

In this paper, we specialize the model of Part 1 to the case of constant market coefficients. We write down the Hamilton–Jacobi–Bellman (HJB) equations for both problems. The HJB equation for the dual problem can be solved by exploiting its linearity and this leads to a solution of the (nonlinear) HJB equation for the primal problem. Although this approach is made possible by the relationship established in Part 1 between the primal and dual problems, the verification in Theorem 3.3 that we have actually solved the HJB equation for the primal problem does not rely on Part 1. Thus this paper is nearly independent of Part 1 and can be read without first reading Part 1.

In addition to providing explicit formulas for the optimal consumption and portfolio processes, both for power and general utility functions, we show in this paper that despite the short-selling prohibition, a mutual fund theorem holds (Remark 3.6).

2. Primal and dual problems. In this section we formulate the problems under study and introduce notation. A more leisurely account appears in Part 1. We also state the existence of optimal solutions to these problems.

2.1. *Basic notation.* Uncertainty in the market is modelled by a d-dimensional Brownian motion $\{w(t), \mathscr{F}(t); 0 \le t \le T\}$, where T is a fixed, finite time horizon and $\{\mathscr{F}(t)\}$ is the augmentation by null sets of the filtration generated by w. Let $p_0(t)$ ($p_i(t)$, $1 \le i \le d$) be the price of the bond (ith risky asset, $1 \le i \le d$, respectively). We assume that

$$\frac{dp_0(t)}{p_0(t)} = r\,dt, \qquad 0 \le t \le T, \tag{2.1}$$

$$\frac{dp_i(t)}{p_i(t)} = b_i\,dt + \sum_{j=1}^{d} \sigma_{ij}\,dw_j(t), \qquad 0 \le t \le T, i = 1, \dots, d. \tag{2.2}$$

Denote by b the column vector of constants $(b_1, \dots, b_d)^T$ and denote by σ the $d \times d$ matrix of constants (σ_{ij}). The interest rate r is also constant. We assume that σ is nonsingular, and we define the *relative risk coefficient*

$$\theta \triangleq \sigma^{-1}(b - r\mathbf{1}), \tag{2.3}$$

where $\mathbf{1}$ is the vector with each component equal to 1.

Assume that at time $t \in [0, T]$, our agent is endowed with some initial wealth $x \ge 0$ and he must choose a *consumption/portfolio process pair* $(C(s), \pi(s))$, $t \le s \le T$ (see Definition 2.1 of Part 1). In the problem at hand, such a pair (C, π) is admissible if and only if:

(i) stocks are not sold short, that is,

$$\pi_i(s) \ge 0, \qquad t \le s \le T, i = 1, \dots, d; \tag{2.4}$$

(ii) wealth remains nonnegative, that is,

$$X^{(t,x)}(s) \geq 0, \qquad t \leq s \leq T, \tag{2.5}$$

where $X^{(t,x)}$ is the solution to the *wealth equation*

$$X^{(t,x)} = x, \tag{2.6}$$

$$\begin{aligned} dX^{(t,x)}(s) &= \left[rX^{(t,x)}(s) - C(s)\right] ds + \pi^T(s)(b - r\mathbf{1})\, ds \\ &\quad + \pi^T(s)\sigma(s)\, dw(s), \qquad t \leq s \leq T. \end{aligned} \tag{2.7}$$

We denote by $A(t, x)$ the set of all admissible pairs (C, π) for the problem with endowment x at initial time $t \in [0, T]$ and by $\tilde{A}$ the set of all portfolio processes defined and satisfying (2.4) for all $s \in [0, T]$.

Let U_j, $j = 1, 2$, be *utility functions* (see Definition 2.4 of Part 1). Repeating Definition 3.1 of Part 1, we define the *convex conjugate* of U_j by

$$\tilde{U}_j(y) \triangleq \sup_{x \geq 0}\{U_j(x) - xy\} = U_j(I_j(y)) - yI_j(y), \qquad y > 0, \tag{2.8}$$

where I_j is the inverse of U_j'. Throughout this paper, we assume that there exist constants K and α such that

$$U_j(I_j(y)) + yI_j(y) \leq K(1 + y^{-\alpha} + y^{\alpha}), \qquad \forall\, y > 0,\ j = 1, 2. \tag{2.9}$$

A sufficient condition for (2.9) is that for some $\alpha > 2$,

$$\lim_{x \downarrow 0} \frac{(U_j'(x))^2}{U_j''(x)} \text{ exists}, \qquad \lim_{x \to \infty} \frac{(U_j'(x))^{\alpha}}{U_j''(x)} = 0, \qquad j = 1, 2 \tag{2.10}$$

[see the Appendix of Karatzas, Lehoczky and Shreve (1987)].

The *value function for the primal problem* is defined to be

$$\begin{aligned} V(t, x) \triangleq \sup_{(C, \pi) \in A(t, x)} E\bigg\{ &\int_t^T e^{-\delta(s-t)} U_1(C(s))\, ds \\ &+ e^{-\delta(T-t)} U_2(X^{(t,x)}(T)) \bigg\}, \end{aligned} \tag{2.11}$$

where $X^{(t,x)}$ is given by (2.6), (2.7) and δ is a nonnegative *discount factor*. The *value function for the dual problem* is

$$\begin{aligned} \tilde{V}(t, y) \triangleq \inf_{\tilde{\pi} \in \tilde{A}} E\bigg\{ &\int_t^T e^{-\delta(s-t)} \tilde{U}_1(ye^{(\delta - r)(s-t)} Z_{\tilde{\pi}}(t, s))\, ds \\ &+ e^{-\delta(T-t)} \tilde{U}_2(ye^{(\delta - r)(T-t)} Z_{\tilde{\pi}}(t, T)) \bigg\}, \end{aligned} \tag{2.12}$$

where $y \geq 0$ and $Z_{\tilde{\pi}}(t, s)$ is the local martingale (in the parameter s) given by

$$\begin{aligned} Z_{\tilde{\pi}(t,s)} \triangleq \exp\bigg\{ &-\int_t^s (\theta + \sigma^{-1}\tilde{\pi}(u))^T\, dW(u) \\ &- \tfrac{1}{2}\int_t^s \|\theta + \sigma^{-1}\tilde{\pi}(u)\|^2\, du \bigg\}, \qquad t \leq s \leq T. \end{aligned} \tag{2.13}$$

The relationship between the primal and dual problems is developed in detail in Part 1. In this paper, we use only the connection in equation (2.19).

2.2. *Existence results.*

2.1 THEOREM. *Let $\hat{\pi}$ be the unique minimizer of $\|\theta + \sigma^{-1}\tilde{\pi}\|$ over $\tilde{\pi} \in [0,\infty)^d$. The process which is identically equal to $\hat{\pi}$ attains the infimum in (2.12) for all $t \in [0,T]$ and $y \geq 0$. More specifically, with*

$$\hat{\theta} \triangleq \theta + \sigma^{-1}\hat{\pi}, \tag{2.14}$$

$$Z_{\hat{\pi}}(t,s) \triangleq \exp\left\{-\hat{\theta}(w(s) - w(t)) - \tfrac{1}{2}\|\hat{\theta}\|^2(s-t)\right\}, \tag{2.15}$$

$$\zeta(t,s) \triangleq e^{(\delta - r)(s-t)}Z_{\hat{\pi}}(t,s), \tag{2.16}$$

the dual value function is given by

$$\tilde{V}(t,y) = E\left\{\int_t^T e^{-\delta(s-t)}\tilde{U}_1(y\zeta(t,s))\,ds + e^{-\delta(T-t)}\tilde{U}_2(y\zeta(t,T))\right\}, \tag{2.17}$$
$$0 \leq t \leq T,\ y \geq 0.$$

Theorem 2.1 is proved in the Appendix. The simplicity of the dual optimal solution leads to a good understanding of the primal problem, which is the object of real interest. Two manifestations of this understanding are the following theorems. The proof of Theorem 2.2, given in the Appendix, depends on Part 1. The proof of Theorem 2.3 is given in Section 3, where the optimal consumption and investment processes are exhibited (see Theorem 3.4).

2.2 THEOREM. *The dual value function is the convex conjugate of the primal value function, that is,*

$$\tilde{V}(t,y) = \sup_{x \geq 0}\{V(t,x) - xy\}, \qquad \forall\, t \in [0,T],\ y \geq 0, \tag{2.18}$$

and moreover,

$$V(t,x) = \inf_{y \geq 0}\{\tilde{V}(t,y) + xy\}, \qquad \forall\, t \in [0,T],\ x \geq 0. \tag{2.19}$$

2.3 THEOREM. *For every $t \in [0,T]$ and $x \geq 0$, there exists an optimal consumption/investment pair $(C,\pi) \in A(t,x)$ for the primal problem.*

3. The Hamilton–Jacobi–Bellman equations. One would expect the primal value function $V(t,x)$ given by (2.11) to satisfy the Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{aligned} V_t(t,x) - \delta V(t,x) + \sup_{\substack{c \geq 0 \\ \pi \geq 0}}\Big\{&\big[(rx - c) + \pi^T(b - r\mathbf{1})\big]V_x(t,x) \\ &+ \tfrac{1}{2}\|\sigma^T\pi\|^2 V_{xx}(t,x) + U_1(c)\Big\} = 0, \end{aligned} \tag{3.1}$$
$$0 \leq t \leq T,\ x > 0,$$

along with the boundary conditions

$$V(T, x) = U_2(x), \qquad x \geq 0, \tag{3.2}$$

$$V(t, 0) = 0, \qquad 0 \leq t \leq T. \tag{3.3}$$

[In connection with (3.3), recall from Definition 2.4 of Part 1 that $U_1(0) = U_2(0) = 0$.] Indeed, if one has a nonnegative C^2 solution to (3.1)–(3.3), then one can show under a mild growth condition that this solution is the primal value function. Therefore, a common approach taken to the determination of the value function in problems like this is to attempt to directly solve the HJB equation.

Our approach is more circumspect. We first solve the HJB equation associated with the dual value function and then use (2.19) to determine V. Once V has thus been found, one can verify that it satisfies (3.1)–(3.3) (see Theorem 3.3).

The HJB equation for the dual problem is

$$\begin{aligned} &\tilde{V}_t(t, y) + (\delta - r) y \tilde{V}_y(t, y) - \delta \tilde{V}(t, y) \\ &\quad + \inf_{\tilde{\pi} \in [0, \infty)^d} \left\{ \tfrac{1}{2} \| \theta + \sigma^{-1} \tilde{\pi} \|^2 y^2 \tilde{V}_{yy}(t, y) \right\} \\ &\quad + \tilde{U}_1(y) = 0, \qquad 0 \leq t \leq T,\, y > 0, \end{aligned} \tag{3.4}$$

along with the boundary condition

$$\tilde{V}(T, y) = \tilde{U}_2(y), \qquad y > 0. \tag{3.5}$$

While the HJB equation (3.1) for the primal problem is nonlinear, the HJB equation (3.4) for the dual problem turns out to be linear. This is because, as we show in Theorem 3.1, $\tilde{V}(t, \cdot\,)$ inherits convexity from $\tilde{U}_1$ and $\tilde{U}_2$, and so

$$\inf_{\tilde{\pi} \in [0, \infty)^d} \left\{ \tfrac{1}{2} \| \theta + \sigma^{-1} \tilde{\pi} \|^2 y^2 \tilde{V}_{yy}(t, y) \right\} = \tfrac{1}{2} \| \hat{\theta} \|^2 y^2 \tilde{V}_{yy}(t, y). \tag{3.6}$$

The fact that the infimum on the left-hand side of (3.6) is obtained by $\hat{\pi}$ independently of t and y accounts for the simplicity of the dual optimal solution as set forth in Theorem 2.1.

We now begin a rigorous analysis of $\tilde{V}$. The Markov process $\zeta(t, \cdot\,)$ of (2.16) has differential generator $\frac{1}{2} \| \hat{\theta} \|^2 y^2 (\partial^2 / \partial y^2) + (\delta - r) y (\partial / \partial y)$ and we may use this fact and (2.17) to derive a linear partial differential equation satisfied by $\tilde{V}$. It is convenient to apply this Feynman–Kac derivation to two functions whose difference is $\tilde{V}$, rather than to $\tilde{V}$ itself. For $(t, y) \in [0, T] \times (0, \infty)$, define

$$\begin{aligned} G(t, y) \triangleq E \bigg\{ &\int_t^T e^{-\delta(s-t)} U_1(I_1(y \zeta(t, s)))\, ds \\ &+ e^{-\delta(T-t)} U_2(I_2(y \zeta(t, T))) \bigg\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} S(t, y) \triangleq E \bigg\{ &\int_t^T e^{-\delta(s-t)} y \zeta(t, s) I_1(y \zeta(t, s))\, ds \\ &+ e^{-\delta(T-t)} y \zeta(t, T) I_2(y \zeta(t, T)) \bigg\}. \end{aligned} \tag{3.8}$$

According to Lemma 7.1 of Karatzas, Lehoczky and Shreve (1987), under our assumption (2.9), G and S are finite and continuous on $[0, T] \times (0, \infty)$, of class $C^{1,2}$ on $[0, T) \times (0, \infty)$, and are the unique solutions to the respective linear Cauchy problems

$$\left(\frac{\partial}{\partial t} + L\right)G(t,y) + U_1(I_1(y)) = 0, \qquad 0 \le t < T, y > 0, \tag{3.9}$$

$$G(T,y) = U_2(I_2(y)), \qquad y > 0, \tag{3.10}$$

and

$$\left(\frac{\partial}{\partial t} + L\right)S(t,y) + yI_1(y) = 0, \qquad 0 \le t < T, y > 0, \tag{3.11}$$

$$S(T,y) = yI_2(y), \qquad y > 0, \tag{3.12}$$

where L is the second-order differential operator defined by

$$L\varphi = \tfrac{1}{2}\|\hat{\theta}\|^2 y^2 \varphi_{yy} + (\delta - r)y\varphi_y - \delta\varphi. \tag{3.13}$$

From (2.8) and (2.17), we have

$$\tilde{V}(t,y) = G(t,y) - S(t,y), \qquad 0 \le t \le T, y > 0, \tag{3.14}$$

so $\tilde{V}$ is continuous on $[0, T] \times (0, \infty)$, of class $C^{1,2}$ on $[0, T) \times (0, \infty)$ and solves the Cauchy problem

$$\left(\frac{\partial}{\partial t} + L\right)\tilde{V}(t,y) + \tilde{U}_1(y) = 0, \qquad 0 \le t < T, y > 0, \tag{3.15}$$

$$\tilde{V}(T,y) = \tilde{U}_2(y), \qquad y > 0. \tag{3.5}$$

3.1 THEOREM. *The dual value function $\tilde{V}$ is given by* (3.14) *and satisfies the HJB equation* (3.4).

PROOF. We need only show that

$$\tilde{V}_{yy}(t,y) > 0, \qquad 0 \le t \le T, y > 0, \tag{3.16}$$

for this will imply (3.6) and then (3.4) reduces to (3.15). We begin by differentiating (2.17) with respect to y. Because of the convexity of $\tilde{U}_2$, we have for $y > 0$ and $\eta \in (-\frac{1}{2}y, 0) \cup (0, \infty)$,

$$\frac{1}{|\eta|}|\tilde{U}_2((y+\eta)\zeta(t,T)) - \tilde{U}_2(y\zeta(t,T))| \le 2\left|\tilde{U}_2\left(\frac{y}{2}\zeta(t,T)\right) - \tilde{U}_2(y\zeta(t,T))\right|.$$

The right-hand side has finite expectation, so the dominated convergence theorem and (2.8) imply that

$$\frac{\partial}{\partial y}E\tilde{U}_2(y\zeta(t,T)) = E\left[\zeta(t,T)\tilde{U}_2'(y\zeta(t,T))\right] = -E\left[\zeta(t,T)I_2(y\zeta(t,T))\right].$$

A similar analysis applies to $\tilde{U}_1$ and we thus obtain from differentiation of

(2.17) that

$$
\begin{aligned}
-\tilde{V}_y(t,y) &= E\left\{\int_t^T e^{-\delta(s-t)}\zeta(t,s)I_1(y\zeta(t,s))\,ds \right. \\
&\qquad \left. + e^{-\delta(T-t)}\zeta(t,T)I_2(y\zeta(t,T))\right\} \\
&= \frac{1}{y}S(t,y), \qquad 0 \le t \le T,\ y > 0.
\end{aligned}
\tag{3.17}
$$

Because I_1 and I_2 are strictly decreasing mappings from $(0,\infty)$ onto $(0,\infty)$, $-\tilde{V}_y(t,\cdot)$ is also a strictly decreasing mapping from $(0,\infty)$ onto $(0,\infty)$. Indeed, the mean value theorem implies that for $(t,y) \in [0,T] \times (0,\infty)$ and $0 < \varepsilon < 1$,

$$
\begin{aligned}
\frac{1}{\varepsilon}\left[\tilde{V}_y(t,y+\varepsilon) - \tilde{V}_y(t,y)\right] \ge E\left\{\int_t^T e^{-\delta(s-t)}\zeta^2(t,s) \min_{\eta \in [y,y+1]} |I_1'(\eta\zeta(t,s))|\,ds \right. \\
\left. + e^{-\delta(T-t)}\zeta^2(t,T) \min_{\eta \in [y,y+1]} |I_2'(\eta\zeta(t,T))|\right\} > 0.
\end{aligned}
$$

This proves (3.16). □

The proof of Theorem 3.1 shows that $-\tilde{V}_y(t,\cdot)$ has an inverse $Y(t,\cdot)$ mapping $(0,\infty)$ onto $(0,\infty)$, that is,

$$
\begin{aligned}
-\tilde{V}_y(t,Y(t,x)) = x, \qquad Y\big(t, -\tilde{V}_y(t,y)\big) = y, \\
0 \le t \le T,\ x > 0,\ y > 0.
\end{aligned}
\tag{3.18}
$$

The function Y is of class C^1 and

$$
Y_x(t,x) = -\frac{1}{\tilde{V}_{yy}(t,Y(t,x))} < 0, \qquad 0 \le t \le T,\ x > 0. \tag{3.19}
$$

The expression $\tilde{V}(t,y) + xy$ is minimized over $y > 0$ by $Y(t,x)$, so by (2.19),

$$
V(t,x) = \tilde{V}(t,Y(t,x)) + xY(t,x), \qquad 0 \le t \le T,\ x > 0. \tag{3.20}
$$

Differentiation in (3.20), coupled with (3.18) and (3.19), yields

$$
\begin{aligned}
V_t(t,x) = \tilde{V}_t(t,Y(t,x)), \qquad V_x(t,x) = Y(t,x), \\
V_{xx}(t,x) = -\frac{1}{\tilde{V}_{yy}(t,x)}, \qquad 0 \le t \le T,\ x > 0.
\end{aligned}
\tag{3.21}
$$

Substitution of (3.18), (3.20) and (3.21) into (3.15) results in the equation

$$
\begin{aligned}
V_t(t,x) - \delta V(t,x) + rxV_x(t,x) + \tilde{U}_1(V_x(t,x)) \\
- \frac{1}{2}\|\hat{\theta}\|^2 \frac{V_x^2(t,x)}{V_{xx}(t,x)} = 0, \qquad 0 \le t \le T,\ x > 0.
\end{aligned}
\tag{3.22}
$$

To see that (3.22) is the HJB equation for the primal stochastic control problem, we need the following lemma.

3.2 LEMMA. *For every nonnegative number a, the unique minimizer of*

$$g(\pi) \triangleq \tfrac{1}{2}\pi^T\sigma\sigma^T\pi - a\pi^T(b - r\mathbf{1}) \tag{3.23}$$

over $\pi \in [0,\infty)^d$ *is* $a(\sigma^T)^{-1}\hat{\theta}$. *Furthermore,* $\hat{\pi}^T(\sigma^T)^{-1}\hat{\theta} = 0$ *and*

$$g\left(a(\sigma^T)^{-1}\hat{\theta}\right) = -\tfrac{1}{2}a^2\|\hat{\theta}\|^2. \tag{3.24}$$

PROOF. The Kuhn–Tucker conditions for the minimization of f in (5.6) over $[0,\infty)^d$ imply the existence of a vector $\lambda \in [0,\infty)^d$ such that $\lambda = \nabla f(\hat{\pi}) = (\sigma^T)^{-1}\hat{\theta}$, $\lambda^T\hat{\pi} = 0$. For $\pi \in [0,\infty)^d$, we have

$$\begin{aligned} g(\pi) &= g(a\lambda) + \tfrac{1}{2}(\pi - a\lambda)^T\sigma\sigma^T(\pi - a\lambda) \\ &\quad + (\pi - a\lambda)^T\left[\sigma\sigma^T a\lambda - a(b - r\mathbf{1})\right] \\ &= g(a\lambda) + \tfrac{1}{2}\|\sigma^T(\pi - a\lambda)\|^2 + a(\pi - a\lambda)^T\hat{\pi}. \end{aligned}$$

Because $a(\pi - a\lambda)^T\hat{\pi} = a\pi\hat{\pi} \geq 0$, we see that g attains its minimum at $\pi = a\lambda = a(\sigma^T)^{-1}\hat{\theta}$. Furthermore,

$$g\left(a(\sigma^T)^{-1}\hat{\theta}\right) = \tfrac{1}{2}a^2\|\hat{\theta}\|^2 - a^2\hat{\theta}^T\sigma^{-1}(b - r\mathbf{1} + \hat{\pi}) + a^2\lambda^T\hat{\pi} = -\tfrac{1}{2}a^2\|\hat{\theta}\|^2. \quad \square$$

3.3 THEOREM. *The primal value function V is given by*

$$V(t,x) = G(t, Y(t,x)), \qquad 0 \leq t \leq T,\ x > 0, \tag{3.25}$$

and satisfies the Hamilton–Jacobi–Bellman equation (3.1).

PROOF. Equation (3.25) follows from (3.20), (3.14), (3.17) and (3.18). From Lemma 3.2 with $a = -(V_x(t,x)/V_{xx}(t,x))$, we have

$$\begin{aligned} &\sup_{\pi \geq 0}\left\{\pi^T(b - r\mathbf{1})V_x(t,x) + \frac{1}{2}\|\sigma^T\pi\|^2 V_{xx}(t,x)\right\} \\ &\quad = V_{xx}(t,x)\inf_{\pi \geq 0}\left\{\frac{1}{2}\pi^T\sigma\sigma^T\pi + \frac{V_x(t,x)}{V_{xx}(t,x)}\pi^T(b - r\mathbf{1})\right\} \\ &\quad = -\frac{1}{2}\frac{V_x^2(t,x)}{V_{xx}(t,x)}\|\hat{\theta}\|^2. \end{aligned} \tag{3.26}$$

Therefore, equation (3.1) is equivalent to (3.22). $\square$

The supremum in the HJB equation (3.1) is attained by

$$c = I_1(V_x(t,x)) = I_1(Y(t,x)) \tag{3.27}$$

$$\pi = -\frac{V_x(t,x)}{V_{xx}(t,x)}(\sigma^T)^{-1}\hat{\theta} = -\frac{Y(t,x)}{Y_x(t,x)}(\sigma^T)^{-1}\hat{\theta}. \tag{3.28}$$

We have thus obtained optimal consumption and portfolio processes in feedback form, a fact we now state precisely and verify properly.

3.4 THEOREM. *Let $(t,x) \in [0,T] \times (0,\infty)$ be given. The optimal wealth process for the consumption/portfolio problem with initial time t and initial wealth x is*

$$X^{(t,x)}(s) \triangleq \frac{S(s, Y(t,x)\zeta(t,s))}{Y(t,x)\zeta(t,s)}, \qquad t \le s \le T. \tag{3.29}$$

This process satisfies (2.6), (2.7) *with $C(\cdot)$ and $\pi(\cdot)$ replaced by*

$$\begin{aligned} C^*(s) &\triangleq I_1(Y(s, X^{(t,x)}(s))), \\ \pi^*(s) &\triangleq -\frac{Y(s, X^{(t,x)}(s))}{Y_x(s, X^{(t,x)}(s))}(\sigma^T)^{-1}\hat{\theta}, \qquad t \le s \le T. \end{aligned} \tag{3.30}$$

PROOF. To simplify notation, we define the function H: $[0,T] \times (0,\infty) \to (0,\infty)$ by $H = -\tilde{V}_y$, that is, [see (3.17)]

$$H(t,y) = \frac{1}{y}S(t,y), \qquad 0 \le t \le T, y > 0. \tag{3.31}$$

Because S is continuous on $[0,T] \times (0,\infty)$ and of class $C^{1,2}$ on $[0,T) \times (0,\infty)$, H has these properties as well. From (3.11), (3.12), we derive the formulas

$$\begin{aligned} &H_t(t,y) + \tfrac{1}{2}\|\hat{\theta}\|^2 y^2 H_{yy}(t,y) + (\delta - r + \|\hat{\theta}\|^2) y H_y(t,y) \\ &\qquad - rH(t,y) + I_1(y) = 0, \qquad 0 \le t \le T, y > 0, \end{aligned} \tag{3.32}$$

$$H(T,y) = I_2(y), \qquad y > 0. \tag{3.33}$$

In terms of H, (3.29) becomes

$$X^{(t,x)}(s) = H(s, Y(t,x)\zeta(t,s)), \qquad t \le s \le T. \tag{3.34}$$

Because $d\zeta(t,s) = (\delta - r)\zeta(t,s)\,ds - \zeta(t,s)\hat{\theta}^T\,dw(s)$, Itô's lemma and (3.32) imply

$$\begin{aligned} dX^{(t,x)}(s) &= \left[H_s + \tfrac{1}{2}\|\hat{\theta}\|^2 Y^2(t,x)\zeta^2(t,s)H_{yy} + (\delta - r)Y(t,x)\zeta(t,s)H_y\right] ds \\ &\quad - Y(t,x)\zeta(t,s)H_y\hat{\theta}^T\,dw(s) \\ &= \left[rH(s, Y(t,x)\zeta(t,s)) - I_1(Y(t,x)\zeta(t,s))\right] ds \\ &\quad - \|\hat{\theta}\|^2 Y(t,x)\zeta(t,s)H_y(s, Y(t,x)\zeta(t,s))\,ds \\ &\quad - Y(t,x)\zeta(t,s)H_y(s, Y(t,x)\zeta(t,s))\hat{\theta}^T\,dw(s), \qquad t \le s \le T. \end{aligned} \tag{3.35}$$

We now examine the three terms on the right-hand side of (3.35). The functions $H(s,\cdot)$ and $Y(s,\cdot)$ are inverses [see (3.18)], so (3.34) can be rewritten as

$$Y(s, X^{(t,x)}(s)) = Y(t,x)\zeta(t,s), \qquad t \le s \le T. \tag{3.36}$$

Therefore,

$$C^*(s) = I_1(Y(t,x)\zeta(t,s)), \tag{3.37}$$

$$rH(s, Y(t,x)\zeta(t,s)) - I_1(Y(t,x)\zeta(t,s)) = rX^{(t,x)}(s) - C^*(s). \tag{3.38}$$

Because $H(t,\cdot)$ and $Y(t,\cdot)$ are inverses, we also have

$$H_y(s,y) = \frac{1}{Y_x(s, H(s,y))}. \tag{3.39}$$

From the equality $\hat{\pi}^T(\sigma^T)^{-1}\hat{\theta} = 0$ obtained in Lemma 3.2, we see that

$$\|\hat{\theta}\|^2 = \hat{\theta}^T\sigma^{-1}(b - r\mathbf{1} + \hat{\pi}) = \hat{\theta}^T\sigma^{-1}(b - r\mathbf{1}). \tag{3.40}$$

Therefore,

$$-\|\hat{\theta}\|^2 Y(t,x)\zeta(t,s)H_y(s, Y(t,x)\zeta(t,s)) = (\pi^*)^T(s)(b - r\mathbf{1}). \tag{3.41}$$

Finally,

$$-Y(t,x)\zeta(t,s)H_y(s, Y(t,x)\zeta(t,s))\hat{\theta}^T = (\pi^*)^T(s)\sigma. \tag{3.42}$$

Substituting (3.38), (3.41) and (3.42) into (3.35), we verify the last sentence in the theorem.

To verify optimality for the problem with initial time t and initial wealth x, we first note that $X^{(t,x)}(t) = H(t, Y(t,x)) = x$. According to (3.33),

$$X^{(t,x)}(T) = H(T, Y(t,x)\zeta(t,T)) = I_2(Y(t,x)\zeta(t,T)). \tag{3.43}$$

By (3.37), (3.43), (3.7) and (3.25), the utility associated with (C^*, π^*) is

$$E\left\{\int_t^T e^{-\delta(s-t)}U_1(I_1(Y(t,x)\zeta(s,t)))\,ds + e^{-\delta(T-t)}U_2(I_2(Y(t,x)\zeta(t,T)))\right\}$$
$$= G(t, Y(t,x)) = V(t,x). \qquad \square$$

3.5 Remark. Karatzas, Lehoczky and Shreve (1987) obtain the analogues of Theorems 3.3 and 3.4 for the problem with no prohibition on short-selling. Under the additional assumption that the utility functions are of class C^3, Proposition 7.3 of Karatzas, Lehoczky and Shreve provides integral formulas for G and S; these formulas can be adapted to our model by replacing θ in them by $\hat{\theta}$.

3.6 Remark. The form of the optimal portfolio process in Theorem 3.4 shows that even though the market is rendered incomplete by the investor's inability to sell stocks short, a so-called mutual fund theorem prevails. Let δ_k be the kth component of $(\sigma^T)^{-1}\hat{\theta}$; from Lemma 3.2, we have $\delta_k \geq 0$, $k = 1, \dots, d$. If $\Sigma_{j=1}^d \delta_j = 0$, then the agent should not invest in the stocks. If $\Sigma_{j=1}^d \delta_j \neq 0$, then a mutual fund is constructed which, through continuous rebalancing, always holds fraction $\delta_k / \Sigma_{j=1}^d \delta_j$ of its assets in stock k, $k = 1, \dots, d$. The agent can realize the optimal portfolio by investing only in the bond and in this mutual fund. The constants δ_k, $k = 1, \dots, d$, are independent of preferences (utility functions), wealth of the investor and time horizon.

4. Power utility functions. In this subsection, we specialize the formulas of the previous subsection to the case of power utility functions

$$U_1(x) = \frac{1}{p_1} x^{p_1}, \qquad U_2(x) = \frac{1}{p_2} x^{p_2}, \qquad x > 0, \tag{4.1}$$

where $p_1, p_2 \in (0, 1)$. We have then

$$\tilde{U}_1(y) = \frac{1}{q_1} y^{-q_1}, \qquad \tilde{U}_2(y) = \frac{1}{q_2} y^{-q_2}, \qquad y > 0, \tag{4.2}$$

where $q_j = p_j/(1 - p_j)$, $j = 1, 2$. Direct evaluation of (2.17) yields

$$\tilde{V}(t, y) = \frac{a_1(t)}{q_1} y^{-q_1} + \frac{a_2(t)}{q_2} y^{-q_2}, \qquad 0 \leq t \leq T, y > 0, \tag{4.3}$$

where

$$a_1(t) \triangleq \begin{cases} \dfrac{1}{k(p_1)}[1 - e^{-k(p_1)(T-t)}], & \text{if } k(p_1) \neq 0, \\ T - t, & \text{if } k(p_1) = 0, \end{cases} \tag{4.4}$$

$$a_2(t) \triangleq e^{-k(p_2)(T-t)}, \tag{4.5}$$

$$k(p) \triangleq \frac{1}{1-p}\left[\delta - rp - \frac{p\|\hat{\theta}\|^2}{2(1-p)}\right]. \tag{4.6}$$

It is straightforward to verify that $\tilde{V}$ solves the Cauchy problem (3.15), (3.5).

For each $t \in [0, T]$, the function

$$-\tilde{V}_y(t, y) = a_1(t) y^{1/(p_1 - 1)} + a_2(t) y^{1/(p_2 - 1)}$$

is strictly decreasing with $\lim_{y \downarrow 0}(-\tilde{V}_y(t, y)) = \infty$ and $\lim_{y \to \infty}(-\tilde{V}_y(t, y)) = 0$. Thus, there is an inverse function $Y(t, \cdot\,)$ as in (3.18), and the value function for the primal problem and the optimal consumption and portfolio policies in feedback form are given by (3.20) and (3.30). In the special case $p_1 = p_2 = p$, these formulas become

$$Y(t,x) = \left(\frac{x}{a_1(t) + a_2(t)}\right)^{p-1}, \qquad 0 \le t \le T,\ x > 0, \tag{4.7}$$

$$V(t,x) = \left(\frac{1}{1-p}\right)(a_1(t) + a_2(t))^{p-1} x^p, \qquad 0 \le t \le T,\ x > 0, \tag{4.8}$$

$$\begin{aligned} C^*(s) &= \frac{1}{a_1(t) + a_2(t)} X^{(t,x)}(s), \\ \pi^*(s) &= \frac{1}{1-p} X^{(t,x)}(s) (\sigma^T)^{-1} \hat{\theta}, \qquad t \le s \le T. \end{aligned} \tag{4.9}$$

Note that the optimal consumption and portfolio policies are linear in wealth.

It is also possible to obtain explicit formulas when either $U_1 \equiv 0$ and $U_2(x) = (1/p)x^p$ or else $U_1(x) = (1/p)x^p$ and $U_2 \equiv 0$. In the former case, one simply omits $a_1(t)$ from (4.7)–(4.9); in the latter case, $a_2(t)$ should be omitted.

APPENDIX

This Appendix provides the proofs of Theorems 2.1 and 2.2.

5.1. *The mean comparison theorem.* In this subsection we prove a comparison theorem which will be instrumental in solving the dual problem. It is an easy consequence of Jensen's inequality for conditional expectations that if $M(\cdot)$ is a martingale and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is convex and satisfies $E|\varphi(M(t))| < \infty$ for all $0 \le t \le T$, then $\varphi(M(\cdot))$ is a submartingale. If M is only a local martingale, then $\varphi(M(\cdot))$ can fail to be a submartingale. To see this, let $M(\cdot)$ be a positive local martingale which is not a martingale (and is therefore a supermartingale) and let φ be the identity function. However, we have the following result for a convex function of a local martingale.

5.1 LEMMA. *Let $\{M(t), \mathscr{F}(t);\ 0 \le t \le T\}$ be a continuous, positive, local martingale and let $\varphi\colon (0,\infty) \to \mathbb{R}$ be a nonincreasing, lower-bounded, convex function satisfying*

$$E\varphi(M(t)) < \infty, \qquad \forall\, t \in [0,T]. \tag{5.1}$$

Then $\{\varphi(M(t)), \mathscr{F}(t);\ 0 \le t \le T\}$ is a submartingale.

PROOF. Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of stopping times converging up to T almost surely such that for each n, $\{M(t \wedge \tau_n), \mathscr{F}(t);\ 0 \le t \le T\}$ is a martingale. For $\varepsilon > 0$, define the bounded, nonincreasing, convex function φ_ε by

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(\varepsilon) + (x - \varepsilon) D^+\varphi(\varepsilon), & \text{if } 0 < x < \varepsilon, \\ \varphi(x), & \text{if } x \ge \varepsilon, \end{cases}$$

where $D^+\varphi$ denotes the right-hand derivative of φ. Then $\{\varphi_\varepsilon(M(t \wedge \tau_n)),$

$\mathscr{F}(t); 0 \le t \le T\}$ is a bounded submartingale, so for every $s \le t$ and $A \in \mathscr{F}(s)$, we have $\int_A \varphi_\varepsilon(M(s \wedge \tau_n))\,dP \le \int_A \varphi_\varepsilon(M(t \wedge \tau_n))\,dP$. Now let $n \to \infty$, using the bounded convergence theorem and then let $\varepsilon \downarrow 0$, using the monotone convergence theorem, to obtain $\int_A \varphi(M(s))\,dP \le \int_A \varphi(M(t))\,dP$. □

Hajek (1985) and Borkar (1987) have proved mean comparison theorems for solutions to stochastic differential equations. In the cited references, the dominating process is a Markov process and a martingale; in the following theorem, the situation is reversed.

5.2 Mean Comparison Theorem. *Let φ: $(0,\infty) \to \mathbb{R}$ be a nonincreasing, lower-bounded, convex function, let $\tilde{\rho}$ be an $\{\mathscr{F}(t)\}$-adapted, $\mathbb{R}^d$-valued processes satisfying $\int_0^T \|\tilde{\rho}(t)\|^2\,dt < \infty$ almost surely and let $\hat{\rho} \in \mathbb{R}^d$ be a vector such that*

$$\|\tilde{\rho}(t)\| \ge \|\hat{\rho}\|, \qquad dt \times dP\, a.e.\ on\ [0,T] \times \Omega. \tag{5.2}$$

Define

$$\tilde{Z}(t) = \exp\left\{-\int_0^t \tilde{\rho}^T(s)\,dw(s) - \frac{1}{2}\int_0^t \|\tilde{\rho}(s)\|^2\,ds\right\}, \qquad 0 \le t \le T,$$

$$\hat{Z}(t) = \exp\left\{-\hat{\rho}^T w(t) - \tfrac{1}{2}\|\hat{\rho}\|^2 t\right\}, \qquad 0 \le t \le T.$$

Then

$$E\varphi\big(\tilde{Z}(t)\big) \ge E\varphi\big(\hat{Z}(t)\big), \qquad 0 \le t \le T. \tag{5.3}$$

Proof. The process $\tilde{Z}$ is a local martingale. According to Lemma 5.1, $E\varphi(\tilde{Z}(t)) \ge \varphi(\tilde{Z}(0)) = \varphi(1)$ for all $t \in [0,T]$. If $\hat{\rho} = 0$, then $\hat{Z} \equiv 1$ and (5.3) follows.

We now assume that $\hat{\rho} \ne 0$. Consequently, $\|\tilde{\rho}(t)\| > 0$ for all t and we can find an $\{\mathscr{F}(t)\}$-adapted, $d \times d$ orthonormal matrix-valued process $O(\cdot)$ such that

$$\frac{\tilde{\rho}(t)}{\|\tilde{\rho}(t)\|} = O(t)\frac{\hat{\rho}}{\|\hat{\rho}\|}, \qquad dt \times dP \text{ a.e. on } [0,T] \times \Omega.$$

Define $\lambda(t) \triangleq \int_0^t (\|\tilde{\rho}(s)\|^2/\|\hat{\rho}\|^2)\,ds$, $0 \le t \le T$, so $\lambda'(t) \ge 1$ for all t. The inverse function λ^{-1} is defined on $[0, \lambda(T)] \supset [0,T]$ and for each $\tau \in [0,T]$, $\lambda^{-1}(\tau)$ is an $\{\mathscr{F}(t)\}$-stopping time. Set

$$\tilde{W}(t) \triangleq \int_0^t \frac{\|\tilde{\rho}(s)\|}{\|\hat{\rho}\|} O^T(s)\,dw(s), \qquad 0 \le t \le T,$$

$$\hat{W}(\tau) \triangleq \tilde{W}(\lambda^{-1}(\tau)), \qquad 0 \le \tau \le T.$$

Relative to the filtration $\{\mathscr{F}(\lambda^{-1}(\tau))\}$, the process $\hat{W}$ is a martingale and

$$\langle \hat{W}_1, \hat{W}_j \rangle(\tau) = \delta_{ij}\int_0^{\lambda^{-1}(\tau)} \frac{\|\tilde{\rho}(s)\|^2}{\|\hat{\rho}\|^2}\,ds = \delta_{ij}\tau, \qquad 0 \le \tau \le T.$$

By Lévy's theorem [Karatzas and Shreve (1987), Theorem 3.3.16], $\{\hat{W}(\tau), \mathscr{F}(\lambda^{-1}(\tau));\ 0 \le \tau \le T\}$ is a standard, d-dimensional Brownian motion. According to Proposition 3.4.8 of Karatzas and Shreve (1987),

$$
\begin{aligned}
1 - \int_0^{\tau} \tilde{Z}(\lambda^{-1}(\nu))\hat{\rho}^T\, d\hat{W}(\nu) &= 1 - \int_0^{\lambda^{-1}(\tau)} \tilde{Z}(s)\hat{\rho}^T d\tilde{W}(s) \\
&= 1 - \int_0^{\lambda^{-1}(\tau)} \tilde{Z}(s)\frac{\|\tilde{\rho}(s)\|}{\|\hat{\rho}\|}(O(s)\hat{\rho})^T\, dw(s) \qquad (5.4) \\
&= 1 - \int_0^{\lambda^{-1}(\tau)} \tilde{Z}(s)\tilde{\rho}^T(s)\, dw(s) \\
&= \tilde{Z}(\lambda^{-1}(\tau)), \qquad 0 \le \tau \le T.
\end{aligned}
$$

But $d\hat{Z}(\tau) = -\hat{Z}(\tau)\hat{\rho}^T\, dw(\tau)$, $0 \le \tau \le T$, so

$$
\hat{Z}(\tau) = 1 - \int_0^{\tau} \hat{Z}(\tau)\hat{\rho}^T\, dw(\tau), \qquad 0 \le \tau \le T, \tag{5.5}
$$

and weak uniqueness of the solution to (5.5) implies that $\tilde{Z}(\lambda^{-1}(\cdot))$ appearing in (5.4) has the same distribution as $\hat{Z}(\cdot)$. Therefore, for any $t \in [0, T]$, $E\varphi(\hat{Z}(t)) = E\varphi(\tilde{Z}(\lambda^{-1}(t)))$. But $\lambda^{-1}(t) \le t$ and Lemma 5.1 implies that $\varphi(\tilde{Z}(\cdot))$ is a submartingale, so $E\varphi(\tilde{Z}(\lambda^{-1}(t))) \le E\varphi(\tilde{Z}(t))$. □

5.2. *The optimal dual control.* In this section we show that the optimal dual control process is identically equal to the constant vector which is the unique minimizer of

$$
f(\tilde{\pi}) \triangleq \tfrac{1}{2}\|\theta + \sigma^{-1}\tilde{\pi}\|^2 \tag{5.6}
$$

over $\tilde{\pi} \in [0, \infty)^d$. Clearly f is a continuous, strictly convex function satisfying $\lim_{\|\tilde{\pi}\| \to \infty} f(\tilde{\pi}) = \infty$. Therefore, f has a unique minimizer $\hat{\pi} \in [0, \infty)^d$, that is,

$$
\|\theta + \sigma^{-1}\hat{\pi}\| \le \|\theta + \sigma^{-1}\tilde{\pi}\|, \qquad \forall\ \tilde{\pi} \in [0, \infty)^d. \tag{5.7}
$$

Recall from (2.14) that $\hat{\theta} \triangleq \theta + \sigma^{-1}\hat{\pi}$.

Proof of Theorem 2.1. Let $\tilde{\pi} \in \tilde{A}$ be given. From (5.7) we have $\|\hat{\theta}\| \le \|\theta + \sigma^{-1}\tilde{\pi}(t)\|$, $0 \le t \le T$. Applying the mean comparison theorem to the nonincreasing, lower-bounded, convex functions $z \mapsto \tilde{U}_1(ye^{(\delta - r)(s-t)}z)$ and $z \mapsto \tilde{U}_2(ye^{(\delta - r)(T-t)}z)$ with $\tilde{\rho}(s) = \theta + \sigma^{-1}\tilde{\pi}(s)$ and $\hat{\rho}(s) \equiv \hat{\theta}$, we obtain the desired result. □

Proof of Theorem 2.2. Since the optimal dual control process is a constant process independent of y, condition (4.21) of Part 1 is satisfied. Theorem 2.2 follows from Corollaries 4.9 and 4.11 of Part 1. □

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