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# EXTREMAL CHARACTER OF THE LYAPUNOV EXPONENT OF THE STOCHASTIC HARMONIC OSCILLATOR

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We give a formula for the quadratic Lyapunov exponent of the harmonic oscillator in the presence of a finite-state Markov noise process. In case the noise process is reversible, the quadratic Lyapunov exponent is *strictly less* than that for the corresponding white-noise process obtained from the central limit theorem. An example is presented of a nonreversible Markov noise process for which this inequality is *reversed*.

**Introduction.** Many authors have studied the asymptotic behavior of the solution of the stochastic differential equation

$$x''(t) + (\gamma + \varepsilon N(t))x(t) = 0,$$

where $\gamma > 0$, $\varepsilon > 0$ and $[N(t)\colon t \geq 0]$ is either a white-noise process or a centered function of an ergodic finite-state Markov process [1, 2, 6]. The latter model is referred to as the *real-noise-driven oscillator process*.

The Lyapunov exponent is the exponential growth rate, defined as

$$\lambda(\varepsilon,\gamma) = \lim_{t\uparrow\infty} (2t)^{-1} \log\left[x'(t)^2 + \gamma x(t)^2\right].$$

For the models studied here the Lyapunov exponent depends neither on the initial conditions $(x(0), x'(0))$ nor on the sample realization of the noise process $[N(t)\colon t \geq 0]$. Our interest in this paper is to compare the effects of white noise versus real noise for a general Markov dependence. This will be accomplished by means of an expansion in the parameter $\varepsilon \downarrow 0$.

Lyapunov exponents have appeared in a number of applied areas in the recent past. Perhaps the most basic of these is *stochastic stability theory*, where we study the temporal exponential growth/decay of solutions of linear stochastic systems which generalize the damped random oscillator equation $x''(t) + \beta x'(t) + (\gamma + \varepsilon N(t))x(t) = 0$ of which the current model is a special case. A general reference for these equations of stochastic stability is [2]. Another source of interest in Lyapunov exponents lies in the field of *random media* where the Lyapunov exponent gives the rate of spatial exponential decay of the solution of the Schrödinger equation with random potential. These developments are covered in [2], [3] and [4]. The connections between stochastic Lyapunov stability and the subject of *random evolution* is described in the author's recent monograph [7].

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Our goal in this paper is to obtain asymptotic expansions of the Lyapunov exponent of the form

$$\lambda(\varepsilon, \gamma) = \varepsilon^2 \lambda_2(\gamma) + O(\varepsilon^2), \qquad \varepsilon \downarrow 0,$$

where $\lambda_2(\gamma)$ is the so-called *quadratic Lyapunov exponent*, which will be computed for both the case of white noise and real noise. In order to make a meaningful comparison between the two cases, we will define below the notion of *associated white-noise process*. This is a formal derivative of a Wiener process whose variance parameter is obtained from the central limit theorem for Markov chains [5], applied to the real-noise process.

**1. Formulation of results.** We begin with a finite-state Markov process $[\xi(t): t \geq 0]$. The transition matrix and infinitesimal matrix are defined by

$$p_{ij}(t) = \operatorname{Prob}[\xi(t) = j | \xi(0) = i], \qquad 1 \leq i, j \leq N,$$

$$q_{ij} = \lim_{t \downarrow 0} \left(p_{ij}(t) - \delta_{ij}\right)/t, \qquad 1 \leq i, j \leq N.$$

We assume that the state space $[1, \ldots, N]$ consists of a single ergodic class with no transient states. It follows that zero is a simple eigenvalue of $Q = (q_{ij})$, where the unique (up to a constant) right and left eigenvectors are displayed as

$$Q1 = 0, \qquad \pi Q = 0.$$

The invariant distribution $\pi = (\pi_i)$ is obtained as the limit of the transition matrix for large time: $\pi_j = \lim_{t \uparrow \infty} p_{ij}(t)$, $1 \leq i \leq N$, where the convergence is exponentially fast. The Markov process is said to be *reversible* if we have identically

$$\pi_i q_{ij} = \pi_j q_{ji}, \qquad 1 \leq i, j \leq N.$$

Reversibility is equivalent to the statement that for any $t_1 < \cdots < t_p$ and any $p$-tuple $i_1, \ldots, i_p$,

$$\operatorname{Prob}\left[\xi(t_1) = i_1, \ldots, \xi(t_p) = i_p\right] = \operatorname{Prob}\left[\xi(t_p) = i_1, \ldots, \xi(t_1) = i_p\right].$$

Any Markov process with $N = 2$ states is reversible, but there exist nonreversible processes with $N = 3$ (see the final section of this paper).

The "real noise" is introduced into the oscillator equations by means of a real-valued function $F(\xi) \not\equiv 0$, which satisfies the condition that $\sum_{i=1}^{N} F(i)\pi_i = 0$. The *real-noise-driven stochastic oscillator process* is defined as the solution of the stochastic initial-value problem

$$x''(t) + (\gamma + \varepsilon F(\xi(t)))x(t) = 0, \qquad x(0) = x_1,\ x'(0) = x_2,$$

where $\gamma, \varepsilon > 0$. The *real-noise Lyapunov exponent* is defined as

$$\lambda^{\text{real}}(\varepsilon, \gamma) = \lim_{t \uparrow \infty} (2t)^{-1} \log\left[x'(t)^2 + \gamma x(t)^2\right].$$

The *white-noise-driven stochastic oscillator process* is defined as the solution of the stochastic initial-value problem

$$x''(t) = (\gamma + \varepsilon w'(t))x(t) = 0, \qquad x(0) = x_1,\ x'(0) = x_2.$$

Here $[w(t)\colon t \geq 0]$ is a Wiener process with $Ew(t) = 0$, $Ew(t)^2 = \sigma^2 t$, where

$$\sigma^2 = 2\int_0^\infty E[F(\xi(t))F(\xi(0))]\,dt.$$

The formal derivative process $[w'(t)\colon t \geq 0]$ is, by definition, the *associated white-noise process*. The second-order differential equation is interpreted rigorously as a system of Itô stochastic equations (see Section 2). The *white-noise Lyapunov exponent* $\lambda^{\text{white}}(\varepsilon,\gamma)$ is defined by the preceding formulas as in the case of real noise.

THEOREM 1.1. *Suppose that $[\xi(t)\colon t \geq 0]$ is a finite-state ergodic Markov process on the state space $[1,\dots,N]$ with invariant distribution $\pi = (\pi_i)$ and $F \not\equiv 0$ is a function on the state space with mean zero: $E_\pi F := \sum_{i=1}^N F(i)\pi_i = 0$. Then the preceding Lyapunov exponents have the asympotic developments*

$$\lambda^{\text{real}}(\varepsilon,\gamma) = \varepsilon^2\lambda_2^{\text{real}}(\gamma) + O(\varepsilon^3), \qquad \varepsilon \downarrow 0,$$

$$\lambda^{\text{white}}(\varepsilon,\gamma) = \varepsilon^2\lambda_2^{\text{white}}(\gamma) + O(\varepsilon^3), \qquad \varepsilon \downarrow 0,$$

*where*

$$\lambda_2^{\text{real}}(\gamma) = \frac{1}{4\gamma}\int_0^\infty \cos\bigl(2\sqrt{\gamma}\,t\bigr)E[F(\xi(t))F(\xi(0))]\,dt > 0,$$

$$\lambda_2^{\text{white}}(\gamma) = \frac{\sigma^2}{8\gamma} = \frac{1}{4\gamma}\int_0^\infty E[F(\xi(t))F(\xi(0))]\,dt > 0.$$

*If either $Q$ is reversible or $\gamma$ is sufficiently large, then we have the strict inequalities*

$$0 < \lambda_2^{\text{real}}(\gamma) < \lambda_2^{\text{white}}(\gamma).$$

This result can be interpreted as providing an upper bound for the quadratic Lyapunov exponent of all Markov-driven oscillator processes with the same asymptotic variance parameter. It will be shown (see Proposition 4.2) that in an appropriate "white-noise limit" we have $\lim \lambda_2^{\text{real}} = \lambda_2^{\text{white}}$. In the final section we present an example of a nonreversible Markov process for which the inequality is reversed. The precise statement is:

PROPOSITION 1.1. *There exists a nonreversible Markov process with $N = 3$ for which $0 < \lambda_2^{\text{white}}(\gamma) < \lambda_2^{\text{real}}(\gamma)$.*

**2. White-noise Lyapunov exponent computation.** In this section we give a self-contained treatment of the computation in the white-noise case.

The stochastic oscillator process is obtained as a solution of the system of Itô stochastic equations

$$dx_1 = x_2\,dt, \qquad dx_2 = -\gamma x_1\,dt + \varepsilon x_1\,dw = -\gamma x_1\,dt + \varepsilon x_1 \circ dw,$$

where $[w(t)\colon t \geq 0]$ is a Wiener process with mean zero and variance $\sigma^2 t$ and the small circle indicates Stratonovich multiplication. This process is most conveniently studied by introducing logarithmic polar coordinates $(\rho, \theta)$ through the equations

$$x_1\sqrt{\gamma} = e^{\rho}\cos\theta, \qquad x_2 = e^{\rho}\sin\theta.$$

We first solve the Stratonovich system

$$d\theta = -\sqrt{\gamma}\,dt + \epsilon h(\theta)\circ dw, \qquad d\rho = \varepsilon q(\theta)\circ dw,$$

where $h(\theta) = \cos^2\theta/\sqrt{\gamma}$ and $q(\theta) = \sin\theta\cos\theta/\sqrt{\gamma}$.

This defines a diffusion process on $\mathbb{R}^2$ with the infinitesimal generator

$$L = -\sqrt{\gamma}\,\partial/\partial\theta + \tfrac{1}{2}(\varepsilon\sigma)^2\big(h(\theta)\partial/\partial\theta + q(\theta)\partial/\partial\rho\big)^2.$$

To obtain the quadratic Lyapunov exponent, we introduce the function

$$J(\rho,\theta) = \rho + \varepsilon^2 J_2(\theta),$$

where $J_2(\theta)$ is a $2\pi$-periodic solution of the equation

$$-\sqrt{\gamma}\,\partial J_2/\partial\theta + \tfrac{1}{2}\sigma^2 h(\theta)q'(\theta) = \lambda_2$$

and $\lambda_2$ is to be determined. This equation has a solution if and only if the constant $\lambda_2$ is determined by

$$\lambda_2 = \left(\frac{1}{2\pi}\right)\frac{1}{2}\int_0^{2\pi}\sigma^2 h(\theta)q'(\theta)\,d\theta = \frac{\sigma^2}{8\gamma}.$$

The solution $J_2$ is then obtained as $J_2(\theta) = \sigma^2(\sin 2\theta + \frac{1}{4}\sin 4\theta)/8\gamma^{3/2}$. Substitution of this function into the generator yields the inequality $|LJ - \lambda_2\varepsilon^2| \leq \text{const.}\,\varepsilon^4$. Now we can apply Itô's formula for stochastic integrals to the function $J(\rho(t), \theta(t))$:

$$\rho(t) + \varepsilon^2 J_2(\theta(t)) = \text{const.} + M(t) + \int_0^t LJ(\rho(s),\theta(s))\,ds,$$

where $M(t)$ is an Itô stochastic integral of a bounded function, hence $\lim_{t\uparrow\infty} M(t)/t = 0$. Dividing by $t$ and taking the limit we have

$$\limsup_{t\uparrow\infty}\rho(t)/t \leq \lambda_2\varepsilon^2 + \text{const.}\,\varepsilon^4,$$

$$\liminf_{t\uparrow\infty}\rho(t)/t \geq \lambda_2\varepsilon^2 - \text{const.}\,\varepsilon^4.$$

These inequalities give the required information on the quadratic Lyapunov exponent in the white-noise case.

**3. Real-noise Lyapunov exponent calculation.** The real-noise model is defined as follows. $Q = (q_{ij})$ is an $N \times N$ matrix which defines a continuous parameter Markov process $[\xi(t)\colon t \geq 0]$ on the state space $[1, \dots, N]$. We specifically assume that this process has a single ergodic class and no transient states. In terms of the $Q$ matrix this implies the properties

$$q_{ij} \geq 0,\ i \neq j, \qquad \sum_{j=1}^{N} q_{ij} = 0, \quad \text{zero is a simple eigenvalue of } Q.$$

The invariant distribution is written $\pi = (\pi_j)$, solution of the equation $\sum_{i=1}^{N} \pi_i q_{ij} = 0$ for $1 \leq j \leq N$. This can be obtained as $\lim_{t\uparrow\infty} e^{tQ}$ where the convergence is exponentially fast. Denoting by $\Pi = 1 \otimes \pi = (\pi_j)$, we have $\int_0^\infty |e^{tQ} - \Pi|\, dt < \infty$. In particular we may define the inverse operator

$$H \doteq H^{(0)} = \int_0^\infty (e^{tQ} - \Pi)\, dt,$$

which satisfies $QH = HQ = \Pi - I$. The inner product is denoted $\langle v, w\rangle_\pi = \sum_{j=1}^{N} \pi_j v_j w_j$.

We note for future reference a very simple fact about the bilinear form defined by a $Q$-matrix.

PROPOSITION 3.1. *For any vector $v = (v_i)$ we have the identity*

$$\langle Qv, v\rangle_\pi = -\frac{1}{2} \sum_{i,j} \pi_i q_{ij} (v_j - v_i)^2.$$

*In particular $\langle Qv, v\rangle_\pi \leq 0$ with equality if and only if $v = \text{const.}(1, \dots, 1)^T$.*

PROOF. The identity is a straightforward calculation using the definition of $\pi$ and the properties of $Q$. If $\langle Qv, v\rangle_\pi = 0$, then all of the terms in the sum are zero. By the ergodic properties of $Q$ we conclude that $v_i - v_j \equiv 0$ and the result follows. □

The stochastic oscillator process is the solution of the equation

$$x''(t) + \big(\gamma + \varepsilon F(\xi(t))\big)x(t) = 0, \qquad t > 0, \qquad x(0) = x_1,\ x'(0) = x_2,$$

where $F(i)$ is a real-valued function on the state space satisfying $\sum_{i=1}^{N} \pi_i F(i) = 0$.

This is most conveniently analyzed through the polar coordinates $\rho, \theta$ introduced in Section 2 by solving the equations

$$\theta'(t) = -\sqrt{\gamma} + \frac{\varepsilon}{\sqrt{\gamma}} F(\xi(t)) \cos^2(\theta(t)),$$

$$\rho'(t) = \frac{\varepsilon}{\sqrt{\gamma}} F(\xi(t)) \sin(\theta(t)) \cos(\theta(t)).$$

The triple $(\rho(t), \theta(t), \xi(t))$ is a Markov process on the space $\mathbb{R}^2 \times [1, \dots, N]$

with infinitesimal generator

$$L = \left(Q - \sqrt{\gamma}\,\partial/\partial\theta\right) + \left(\varepsilon/\sqrt{\gamma}\right)F(\xi)\cos^2\theta\,\partial/\partial\theta + \left(\varepsilon/\sqrt{\gamma}\right)F(\xi)\sin\theta\cos\theta\,\partial/\partial\rho.$$

In order to find the quadratic Lyapunov exponent in this case we construct a function of the form

$$J(\rho,\theta,\xi) = \rho + \varepsilon f_1(\theta,\xi) + \varepsilon^2 f_2(\theta,\xi),$$

where the functions $f_1(\theta,\xi)$, $f_2(\theta,\xi)$ are smooth $2\pi$-periodic in $\theta$ and satisfy the equations

$$(3.1)\qquad \left(Q - \sqrt{\gamma}\,\partial/\partial\theta\right)f_1 + F(\xi)\cos\theta\sin\theta/\sqrt{\gamma} = 0,$$

$$(3.2)\qquad \left(Q - \sqrt{\gamma}\,\partial/\partial\theta\right)f_2 + F(\xi)\cos^2\theta\,\partial f_1/\partial\theta/\sqrt{\gamma} = \lambda_2$$

for some constant $\lambda_2$. Conditions (3.1) and (3.2) are equivalent to the asymptotic statement that $LJ = \lambda_2\varepsilon^2 + O(\varepsilon^3)$, $\varepsilon \downarrow 0$.

To solve (3.1) and (3.2) we introduce the *resolvent operators*

$$H^{(n)} = \int_0^\infty (e^{tQ} - \Pi)e^{-int\sqrt{\gamma}}\,dt, \qquad i = \sqrt{-1}.$$

It is immediately checked that these satisfy the equations $(Q - in\sqrt{\gamma}I)H^{(n)} = \Pi - I$.

The solution of (3.1) is then obtained by writing $\sin\theta\cos\theta = (1/4i)[e^{2i\theta} - e^{-2i\theta}]$ to obtain

$$-f_1(\theta,\xi) = \frac{1}{4i\sqrt{\gamma}}\left[(H^{(2)}F)(\xi)e^{2i\theta} - (H^{(-2)}F)(\xi)e^{-2i\theta}\right].$$

Computing directly we have

$$\cos^2\theta\,\partial f_1/\partial\theta = \tfrac{1}{4}(2 + e^{2i\theta} + e^{-2i\theta})\left(\tfrac{1}{2}\right)(e^{2i\theta}H^{(2)}F + e^{-2i\theta}H^{(-2)}F).$$

We substitute this in (3.2), multiply by the invariant distribution $\pi(\xi)$, integrate over $(0, 2\pi)$, sum over $\xi$ and use the orthogonality relations for $e^{in\theta}$ to obtain

$$\lambda_2 = \frac{1}{8\gamma}\left[\langle H^{(2)}F, F\rangle_\pi + \langle H^{(-2)}F, F\rangle_\pi\right].$$

This can be written more directly by noting that $\Pi F = 0$ and thus $\langle H^{(n)}F, F\rangle = \int_0^\infty \langle e^{tQ}F, F\rangle_\pi e^{-int\sqrt{\gamma}}\,dt$; hence

$$\lambda_2 = \lambda_2^{\text{real}}(\gamma) = \frac{1}{4\gamma}\int_0^\infty \cos\left(2t\sqrt{\gamma}\right)\langle e^{tQ}F, F\rangle_\pi\,dt.$$

**4. Comparison of the real-noise and white-noise results.** We are now in position to complete the proof of Theorem 1.1. In terms of the preceding resolvent operators the quadratic white-noise Lyapunov exponent is expressed as

$$\lambda_2^{\text{white}}(\gamma) = \frac{\sigma^2}{8\gamma} = \frac{1}{4\gamma}\langle H^{(0)}F, F\rangle_\pi,$$

while the quadratic real-noise exponent is written

$$\lambda_2^{\text{real}}(\gamma) = \frac{1}{8\gamma}\left[\langle H^{(2)}F, F\rangle_\pi + \langle H^{(-2)}F, F\rangle_\pi\right].$$

Letting $u_j = H^{(j)}F$ we have

$$\left(Q - 2ij\sqrt{\gamma}\right)u_j = -F, \qquad j = 0, \pm 1, \pm 2, \dots .$$

From the definition of $H^{(j)}$, it follows that $u_j$ and $u_{-j}$ are complex conjugates. The positivity of $\lambda_2^{\text{real}}(\gamma)$ follows from the equivalent representation

$$\begin{aligned} &8\gamma\lambda_2^{\text{real}}(\gamma)\\ &\quad = \langle u_2 + u_{-2}, F\rangle_\pi\\ &\quad = -(1/2)\left(\left\langle u_2 + u_{-2}, Q(u_2 + u_{-2}) - 2i\sqrt{\gamma}(u_2 - u_{-2})\right\rangle_\pi\right)\\ &\quad = (-1/2)\left\langle u_2 + u_{-2}, Q(u_2 + u_{-2})\right\rangle_\pi + i\sqrt{\gamma}\langle u_2 + u_{-2}, u_2 - u_{-2}\rangle_\pi. \end{aligned}$$

The first term is nonnegative from Proposition 3.1. The second term is also nonnegative, which can be seen by writing $u_2 = A + iB$, $u_{-2} = A - iB$. The equations $Qu_{\pm 2} = \pm 2i\sqrt{\gamma}u_{\pm 2} - F$ immediately imply that

$$Q(u_2 - u_{-2}) = 2i\sqrt{\gamma}(u_2 + u_{-2}).$$

Therefore,

$$\begin{aligned} 2i\sqrt{\gamma}\langle u_2 + u_{-2}, u_2 - u_{-2}\rangle_\pi &= \left\langle Q(u_2 - u_{-2}), u_2 - u_{-2}\right\rangle_\pi\\ &= \langle Q(2iB), 2iB\rangle_\pi\\ &\geq 0 \end{aligned}$$

by Proposition 3.1, from which we obtain $\lambda_2^{\text{real}}(\gamma) \geq 0$. If equality occurs, then from Proposition 3.1 we must have $u_2 + u_{-2} = c_1(1, \dots, 1)^T$ and $B = c_2(1, \dots, 1)^T$ for constants $c_1$ and $c_2$. Applying $Q$ to the first of these, we see that $-2F + 4\sqrt{\gamma}B \equiv 0$, hence $F$ is a constant, which must be zero by the normalization, a contradiction. Therefore $\lambda_2^{\text{real}}(\gamma) > 0$.

In order to establish the inequality $\lambda_2^{\text{real}}(\gamma) < \lambda_2^{\text{white}}(\gamma)$, we first note the following identity.

LEMMA 4.1. *The function* $U = u_0 - \frac{1}{2}(u_2 + u_{-2})$ *satisfies the equation*

$$(Q^2 + 4\gamma)QU = -4\gamma F.$$

PROOF. Direct calculation using the preceding equations for $u_0, u_2, u_{-2}$ yields the result. □

Using this we can write the difference of the Lyapunov exponents in the form

$$\lambda_2^{\text{white}}(\gamma) - \lambda_2^{\text{real}}(\gamma) = \frac{1}{4\gamma}\langle U, F\rangle_\pi = -\frac{1}{16\gamma^2}\left[\langle Q^3U, U\rangle_\pi + 4\gamma\langle QU, U\rangle_\pi\right].$$

The second term in the bracket is clearly negative. If $Q$ is also reversible, then the first term may be written as $\langle Q(QU), QU\rangle_\pi$ to demonstrate its nonpositivity. In the general case we may choose $\gamma$ sufficiently large so that the matrix $Q^3 + 4\gamma Q$ has positive off-diagonal elements, hence is the $Q$ matrix of a Markov chain, from which the nonpositivity follows, which proves the theorem. □

Finally, we give the *white-noise limit* of the real-noise Lyapunov exponent.

PROPOSITION 4.2. *Suppose that the parameters of the real-noise-driven oscillator process are rescaled according to the transformation*

$$Q \to \frac{Q}{\delta^2}, \qquad F \to \frac{F}{\delta}.$$

*Let the resulting real-noise quadratic Lyapunov exponent be denoted by* $\lambda_2^{\text{real }\delta}(\gamma)$. *Then we have* $\lim_{\delta\downarrow 0}\lambda_2^{\text{real }\delta}(\gamma) = \lambda_2^{\text{white}}(\gamma)$.

PROOF. Replacing $Q$ by $Q/\delta^2$ and $F$ by $F/\delta$, we have to consider the expression

$$\int_0^\infty \cos\left(2t\sqrt{\gamma}\right)\langle e^{tQ/\delta^2}F/\delta, F/\delta\rangle_\pi\,dt = \int_0^\infty \cos\left(2t\delta^2\sqrt{\gamma}\right)\langle e^{tQ}F, F\rangle_\pi\,dt.$$

When $\delta \downarrow 0$ we can apply the dominated convergence theorem to conclude that $\lim_{\delta\downarrow 0}\lambda_2^{\text{real }\delta}(\gamma) = (1/4\gamma)\int_0^\infty\langle e^{tQ}F, F\rangle_\pi\,dt = \lambda_2^{\text{white}}(\gamma)$. □

**5. A counterexample.** We present an example of a three-state, nonreversible Markov chain for which the preceding inequality fails to hold. This example has been contributed by Stafford [8].

Let $Q = (q_{ij})$ be defined by the coefficients

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

This matrix defines an ergodic Markov chain with $\pi = \frac{1}{4}(1, 1, 2)$. For a given nonstochastic harmonic oscillator with spring constant $\gamma > 0$, let the noise process be defined by the column vector $F = (28 + 12\gamma, -4 - 20\gamma, -12 + 4\gamma)^T$. Clearly we have $\sum_{i=1}^3 \pi_i F(i) = 0$, as required.

PROPOSITION 5.1. *For this choice of* $Q, F$ *we have* $\langle(Q^3 + 4\gamma Q)U, U\rangle_\pi > 0$ *for* $\gamma < 1/11$. *In particular the quadratic Lyapunov exponents satisfy* $\lambda_2^{\text{real}}(\gamma) > \lambda_2^{\text{white}}(\gamma)$.

PROOF. Let $v$ be the vector $v = (1, -3, 2)^T$. It is immediately computed that

$$Q^3 = \begin{pmatrix} -12 & 8 & 4 \\ -4 & 0 & 4 \\ 8 & -4 & -4 \end{pmatrix}$$

and that $\langle Q^3 v, v\rangle_\pi = 2 > 0$. Further direct computation shows that $(Q^3 + 4\gamma Q)v = -F$ and that the null space of $Q^3 + 4\gamma Q$ consists of multiples of the constant vector $(1, 1, 1)^T$. From the analysis in Section 4, the required solution $U$ in Lemma 4.1 satisfies the equation $(Q^3 + 4\gamma Q)U = -4\gamma F$. Therefore the difference $U - 4\gamma v$ is a multiple of $(1, 1, 1)^T$ and we have

$$\begin{aligned} \big\langle (Q^3 + 4\gamma Q)U, U\big\rangle_\pi &= -4\gamma\langle F, U\rangle_\pi \\ &= -4\gamma\big\langle F, 4\gamma v + c(1,1,1)^T\big\rangle_\pi \\ &= -4\gamma\langle F, 4\gamma v\rangle_\pi \\ &= \big\langle (Q^3 + 4\gamma Q)v, 4\gamma v\big\rangle_\pi \\ &= 4\gamma\big\langle (Q^3 + 4\gamma Q)v, v\big\rangle_\pi. \end{aligned}$$

But a short calculation shows that $\langle (Q^3 + 4\gamma Q)v, v\rangle_\pi = 2 - 22\gamma$. Therefore if $\gamma < 1/11$, then this expression is positive and we have reversed the inequality, proving that $\lambda_2^{\text{real}}(\gamma) > \lambda_2^{\text{white}}(\gamma)$. □

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