The Annals of Applied Probability
1992, Vol. 2, No. 4, 928–941

SHOCKS IN ASYMMETRIC EXCLUSION AUTOMATA

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A one-dimensional cellular automaton with conservative dynamics is studied. The automaton are a special case of the Boghosian–Levermore model, a probabilistic automaton, that has been used as a microscopic approximation for the Burgers equation. We study the stationary measures, the hydrodynamical limit and the existence of a microscopic interface.

1. Introduction. The generation of shock waves is a phenomenon which appears in many physical systems such as fluid flow [Lax (1972)] and traffic flow [Walker (1989)]. One of the most commonly used macroscopic models of shock wave generation is the Burgers equation. The microscopic structure of the shock in the Burgers equation has been studied by Spohn (1989), De Masi, Kipnis, Presutti and Saada (1988), Gärtner and Presutti (1990) and Ferrari, Kipnis and Saada (1991). In all these works the rigorous results were proven for the simple exclusion process. Boghosian and Levermore (1987) constructed a probabilistic cellular automaton as a microscopic model for the Burgers equation with viscosity; we call it BLCA for brevity. Lebowitz, Orlandi and Presutti (1989) then proved that in the weakly asymmetric kinetic limit the total density of particles of the BLCA satisfies the Burgers equation with viscosity. We study the totally asymmetric BLCA in the hydrodynamic limit. This is an automaton where the only randomness comes from the initial configuration. We show that the density obeys a nonlinear conservation law of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial r} F(u) = 0, \tag{1.1a}$$

where

$$F(u) = \begin{cases} u, & \text{if } 0 \le u \le 1, \\ 2 - u, & \text{if } 1 \le u \le 2. \end{cases} \tag{1.1b}$$

We also prove that there is a sharply defined microscopic position for the shock, and we obtain a law of large numbers and a functional central limit theorem for the position of the shock. From these results we prove the existence of a dynamical phase transition. The probabilistic cellular automaton, for which the updating rule is also random, presents the same type of phenomenon. This is being studied by Ferrari, Ravishankar and Vares (1992).

Received March 1991; revised October 1991.
AMS 1980 *subject classification*. 60K35.
Key words and phrases. Burgers' equation, cellular automata, conservation laws, hydrodynamic limit, microscopic shock.

2. The model. A configuration of the one-dimensional Boghosian–Levermore cellular automaton (BLCA) is an arrangement of particles with velocities $+1$ and -1 on $\mathbb{Z}$, satisfying the exclusion condition that there is at most one particle with a given velocity ($+1$ or -1) at each site. We denote a configuration by $\eta(x, s)$, where $x \in \mathbb{Z}$ and $s \in \{-1, +1\}$ and $\eta \in \{0, 1\}^{\mathbb{Z} \times \{-1, +1\}} := \mathbf{X}$.

Dynamics. The dynamics are discrete and are given in two steps:

1. *Collision:* For a given η let $C\eta$ be the configuration

$$C\eta(x, +1) = 1\{\eta(x, 1) + \eta(x, -1) \geq 1\},$$
$$C\eta(x, -1) = 1\{\eta(x, 1) + \eta(x, -1) = 2\}.$$

   In other words, if there is no particle or two particles at x, then nothing happens. If there is only one particle, under C this particle adopts the velocity 1.
2. *Advection:* This part of the dynamics moves each particle along its velocity to a neighboring site in unit time. The operator A is defined by

$$A\eta(x, s) = \eta(x - s, s).$$

Defining $T := AC$, the dynamics are given by

$$\eta_{t+1} = T\eta_t.$$

Cheng, Lebowitz and Speer (1990) have noticed that these dynamics act independently in the space-time sublattices $\{(x, t)$: $x + t$ is even$\}$ and $\{(x, t)$: $x + t$ is odd$\}$. We work with the even sublattice.

Hydrodynamics. Let $u_0(r)$ be a smooth function on $\mathbb{R}$. Let the occupation number at site x for the BLCA be noted by $\zeta_t(x)$, that is, $\zeta_t(x) = \eta_t(x, -1) + \eta_t(x, 1)$. Consider a family of initial measures on $\mathbf{X}$ that approximates the density u_0: $\{\mu^\varepsilon: \varepsilon > 0\}$ is a family of product measures on $\mathbf{X}$ with the property $\mu^\varepsilon(\zeta_0(x)) = u_0(\varepsilon x)$. Let μ_t^ε denote $\mu^\varepsilon(T^*)^t$. We are interested in describing μ_t^ε by a density profile $u(r, t)$ which is the solution at time t of some partial differential equation with initial condition u_0. We can try to guess the equation by looking at the evolution of μ^ε, which we do know. We have

$$\begin{aligned}\zeta_t(x) - \zeta_{t-1}(x) &= \eta_{t-1}(x - 1, 1) - \eta_{t-1}(x, 1)\\ &\quad + \eta_{t-1}(x - 1, -1) - \eta_{t-1}(x, -1)\\ &\quad + \eta_{t-1}(x + 1, 1)\eta_{t-1}(x + 1, -1)\\ &\quad - \eta_{t-1}(x - 1, 1)\eta_{t-1}(x - 1, -1).\end{aligned}$$

If we denote $E_{\mu^\varepsilon}(\zeta_t(x))$ by $u^\varepsilon(\varepsilon x, \varepsilon t)$ and assume that the two-point correlations factor, that is,

$$E_{\mu^\varepsilon}(\eta_t(x, 1)\eta_t(x, -1)) = E_{\mu^\varepsilon}(\eta_t(x, 1))E_{\mu^\varepsilon}(\eta_t(x, -1)),$$

then we obtain the following difference equation for u^ε. Let $u_{\pm}^\varepsilon(\varepsilon x, \varepsilon t) =$

$E_{\mu^\varepsilon}(\eta_t(x, \pm 1))$. Clearly $u^\varepsilon = u^\varepsilon_+ + u^\varepsilon_-$:

$$\begin{aligned} u^\varepsilon(\varepsilon x, \varepsilon t) &- u^\varepsilon(\varepsilon x, \varepsilon(t-1)) \\ &= u^\varepsilon(\varepsilon(x-1), \varepsilon(t-1)) - u^\varepsilon(\varepsilon x, \varepsilon(t-1)) \\ &\quad + u^\varepsilon_+(\varepsilon(x+1), \varepsilon(t-1)) u^\varepsilon_-(\varepsilon(x+1), \varepsilon(t-1)) \\ &\quad - u^\varepsilon_+(\varepsilon(x-1), \varepsilon(t-1)) u^\varepsilon_-(\varepsilon(x-1), \varepsilon(t-1)). \end{aligned} \tag{2.1}$$

Now, assume that local equilibrium is valid for this system and consider u^ε_+ and u^ε_- to be densities of a stationary product distribution (locally). We prove in Section 3 that, for all x, either $u^\varepsilon_+(x) = 1$ or $u^\varepsilon_-(x) = 0$. Hence the left-hand side of (2.1) equals

$$u^\varepsilon(\varepsilon(x-1), \varepsilon(t-1)) - u^\varepsilon(\varepsilon x, \varepsilon(t-1))$$

if $u^\varepsilon_- = 0$, or the left-hand side of (2.1) equals

$$u^\varepsilon(\varepsilon(x+1), \varepsilon(t-1)) - u^\varepsilon(\varepsilon x, \varepsilon(t-1))$$

if $u^\varepsilon_+ = 1$. Thus we obtain that, in the limit as $\varepsilon \to 0$, the density profile $u(r,t)$ for μ^ε_t is a weak solution of (1.1).

In this case a smooth increasing initial profile $u(r)$ such that $u(r) < 1$ for $r < 0$ and $u(r) > 1$ for $r > 0$ will lead to a shock in finite time. This is seen from the observation that the characteristic velocity to the left of the origin (F') is 1 (since $u < 1$) while to the right of the origin it is -1. Indeed, $u(r,t) = u_0(r - vt)$ is a solution of (1.1), where $v = (1 - (\rho + \lambda))/(1 - (\rho - \lambda))$ [see Lax (1972)].

In Section 6 we show that, for initial profiles which are step functions [i.e., $u(r) = (1+\lambda)1\{r > 0\} + \rho 1\{r \le 0\}$, $0 \le \lambda$, $\rho \le 1$], the appropriate rescaled density converges to the unique entropic solution of (1.1). In Section 4 we show that the shock (which at time zero is located at the origin) stays sharp. More precisely, we show that there exists a position $X(t)$ such that, as seen from $X(t)$, the system at time t has the same distribution as at time zero (with left density ρ and right density $1 + \lambda$). The shock satisfies a law of large numbers, that is, $X(t)/t$ converges as $t \to \infty$ to v, the macroscopic velocity of the travelling wave of (1.1). This and the shock fluctuations are studied in Section 5. There we also show that the shock position—in the diffusive limit—converges to a Brownian motion. The limiting diffusion coefficient is explicitly computed. Indeed, the position of the shock is represented by a random sum of independently distributed random variables. Using the results mentioned above we show in Section 6 that at the average position of the shock, the system converges weakly to a mixture of the stationary measures ν_ρ and $\nu_{1+\lambda}$, where ν_α is a Bernoulli product measure with $\nu_\alpha(\zeta(x)) = \alpha$. Due to the discontinuity of the derivative of F, decreasing profiles also present shocks. But in this case they do not fluctuate. In the next section we characterize the stationary and translation invariant measures.

3. Stationary measures. We say that a measure μ is stationary for the process if $\mu T^* = \mu$. The set of stationary measures is convex and compact, hence it can be described by the set of extremal stationary measures (those stationary measures that cannot be written as a nontrivial convex combination of other stationary measures). Next we characterize the set of translation invariant stationary measures.

THEOREM 1. *Any measure μ which is translation invariant and satisfies either*

$$\begin{aligned}&\mu(\eta: \eta(x,1) = 1) = 1 \quad \textit{and} \quad 0 \le \mu(\eta: \eta(x,-1) = 1) \le 1 \quad \textit{or}\\ &\mu(\eta: \eta(x,-1) = 1) = 0 \quad \textit{and} \quad 0 \le \mu(\eta: \eta(x,1) = 1) \le 1\end{aligned} \tag{3.1}$$

is stationary. Conversely, if μ is extremal stationary and translation invariant, then μ must satisfy (3.1).

PROOF. Let τ_x be the translation by x defined by $\tau_x\eta(y) := \eta(y - x)$. If $\mu(\eta\colon\ \eta(x, 1) = 1) = 1$, then $T\eta = \tau_{-1}\eta$. Hence, by translation invariance of μ we get stationarity. On the other hand, when $\mu(\eta\colon\ \eta(x, -1) = 1) = 0$, $T\eta = \tau_1\eta$.

In order to prove the converse, take μ stationary and translation invariant. We want to prove that either $\mu(\eta(x, 1)) = 1$ or $\mu(\eta(x, -1)) = 0$. We have that

$$\begin{aligned}\mu(\eta(x,1) = 1) &= \mu(\eta(x+1,1) = 1) = (\mu T)(\eta(x+1,1) = 1)\\ &= (\mu C)(\eta(x,1) = 1).\end{aligned}$$

But $(\mu C)(\eta(x, 1) = 1) = \mu(\eta(x, 1) = 1) + \mu(\eta(x, 1) = 0,\ \eta(x, -1) = 1)$. Therefore we have that $\mu(\eta(x, 1) = 0,\ \eta(x, -1) = 1) = 0$ for all $x \in \mathbb{Z}$. Let $\mathbf{X}_B = \{\eta \in \mathbf{X}\colon\ \eta(x, 1) = 0,\ \eta(x, -1) = 1, \text{ for some } x \in \mathbb{Z}\}$. Clearly $\mu(\mathbf{X}_B) = 0$. Let $\mathbf{X}' = \mathbf{X} \setminus \mathbf{X}_B$. Let $\mathbf{X}_1 = \{\eta \in \mathbf{X}'\colon\ \eta(x, 1) + \eta(x, -1) < 2, \text{ for all } x \in \mathbb{Z}\}$. Clearly $\tau_x\mathbf{X}_1 = \mathbf{X}_1$ for all $x \in \mathbb{Z}$ and $T\mathbf{X}_1 = \mathbf{X}_1$. Therefore an extremal μ is supported either on $\mathbf{X}_1$ or on $\mathbf{X}_2 = \mathbf{X}' \setminus \mathbf{X}_1$. If μ is supported on $\mathbf{X}_1$, then $\mu(\eta(x, -1) = 1) = 0$, for all $x \in \mathbb{Z}$. The extremal measures supported on $\mathbf{X}_1$ are those which are ergodic with respect to translations.

Now let μ be a measure supported on $\mathbf{X}_2$. We claim that $\mu(\eta(2y, 1) = 1,\ y \le x | \eta(2x, 1) + \eta(2x, -1) = 2) = 1$. Let $B_k(x) = \{\eta \in \mathbf{X}_2\colon\ \eta(2x, 1) + \eta(2x, -1) = 2,\ \eta(2(x - n), 1) = 1,\ 0 \le n < k,\ \eta(2(x - k), 1) = 0\}$. Then

$$\begin{aligned}&T^k B_k(x) \subset \mathbf{X}_B,\\ &\mu(B_k(x)) \le \mu(T^{-k}X_B) = \mu(X_B) = 0.\end{aligned}$$

This proves the claim. Thus we have that if $\eta \in \mathbf{X}_2$, then there exists $x(\eta)$ such that $\eta(2y, 1) = 1$ if $y \le x(\eta)$ almost surely.

Let $A_n = \{\eta \in \mathbf{X}_2\colon\ \eta(2x, 1) = 1, \text{ if } x \le n\}$. Clearly $\{A_n\}_n$ is a decreasing family of sets and

$$\mu\Big(\bigcap_{n \in \mathbb{Z}} A_n\Big) = \lim_{n \to -\infty} \mu(A_n) = 1. \tag{3.2}$$

Now we observe that $T^2A_n \subset A_{n+1}$, which implies that $A_n \subset T^{-2}A_{n+1}$. Thus $\mu(T^{-2}A_{n+1}) = \mu(A_{n+1}) \ge \mu(A_n)$. Since $A_{n+1} \subset A_n$, we have that $\mu(A_{n+1}) \le$

$\mu(A_n)$. So we conclude that $\mu(A_n) = \mu(A_{n+1})$, for all $n \in \mathbb{Z}$, which together with (3.2) implies that $\mu(A_n) = 1$ for all $n \in \mathbb{Z}$. This proves that if $\eta \in \mathbf{X}_2$, then $\eta(2x, 1) = 1$ for all $x \in \mathbb{Z}$ almost surely. The extremal measures are the ones which are ergodic with respect to translations. □

REMARK. As noted earlier, the dynamics act independently on odd and even space-time sublattices. Therefore it would be of interest to prove a corresponding theorem if we studied the system at even times and at even sites; that is, a theorem classifying measures which are stationary with respect to T^2 and are invariant with respect to τ_{2x}. Such a theorem could be proved proceeding along the same lines as in Theorem 1.

There are other stationary measures which, in analogy with the simple exclusion process, we call blocking measures. Let η^n be the configuration $\eta^n(x, s) = 1$ for all $x \geq n$, $s = \pm 1$, and let $\xi^n := T\eta^n$. Notice that $T\xi^n = \eta^n$. Define $\nu^n := \frac{1}{2}\delta_{\eta^n} + \frac{1}{2}\delta_{\xi^n}$. It is easy to check that ν^n is stationary. We say that each one of these measures represents a microscopic shock: The measure ν^n has density 2 at sites $x > n$, density 0 at sites $x < n - 1$ and, due to the periodicity of the motion, has density $\frac{1}{2}$ at $n - 1$ and $\frac{3}{4}$ at n.

In contrast with the simple exclusion process for which the ergodic stationary measures are product measures, Theorem 1 tells us that the set of stationary measures is much richer. It is known that for the simple exclusion process the extremal stationary measures are either product measures or blocking measures [Liggett (1985)]. One might guess that for the totally asymmetric BLCA the extremal stationary measures are either measures satisfying Theorem 1 or blocking measures. We show that this is not true by exhibiting an extremal stationary measure which is neither a product measure nor a blocking measure. (This construction was pointed out to us by a referee.)

Consider the following configuration:

$$\eta(x, 1) = \begin{cases} 0, & \text{if } x = 2K + 1,\ K \in \mathbb{Z},\ x = -2 - 4n,\ n \in \mathbb{N}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\eta(x, -1) = \begin{cases} 1, & \text{if } x = 2 + 4n,\ n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that $T^4\eta = \eta$. Therefore a measure which puts equal mass on η, $T\eta$, $T^2\eta$ and $T^3\eta$ is stationary. We can easily see that this measure cannot be obtained as a convex combination of measures satisfying the conditions of Theorem 1 and blocking measures. We know that there exist an infinite family of such configurations.

4. Microscopic shocks. We consider the BLCA starting from an initial measure $\nu_{\rho,\lambda}$ with $0 < \lambda \leq 1$ and $0 \leq \rho \leq 1$, the measure defined as follows. The marginals of $\nu_{\rho,\lambda}$ are

$$\nu_{\rho,\lambda}(\eta(x,1) = 1) = 1,\ \nu_{\rho,\lambda}(\eta(x,-1) = 1) = \lambda, \quad \text{for } x > 0,$$

$$\nu_{\rho,\lambda}(\eta(x,1) = 1) = \rho,\ \nu_{\rho,\lambda}(\eta(x,-1) = 1) = 0, \quad \text{for } x < 0,$$

$$\nu_{\rho,\lambda}(\eta(0,-1) = a, \eta(0,1) = b) = m(a,b),$$

where $m(0,0) = m(1,1) = \lambda(1-\rho)/(1+\lambda-\rho)$, $m(0,1) = (1-\lambda)(1-\rho)/(1+\lambda-\rho)$ and $m(1,0) = \lambda\rho/(1+\lambda-\rho)$. The marginals at the origin have this unnatural definition for reasons that will be clear later. We assume also that $\nu_{\rho,\lambda}$ is a product measure in the following sense. Let $A \subset (\mathbb{Z} \setminus \{0\}) \times \{-1, 1\}$. Then

$$\nu_{\rho,\lambda}(\eta(x,i) = 1, (x,i) \in A; \eta(0,-1) = a, \eta(0,1) = b)$$
$$= \prod_{(x,i)\in A} \nu_{\rho,\lambda}(\eta(x,i) = 1) \times m(a,b).$$

In order to define the shock, let η be any configuration and define η' as the configuration that differs from η only at site 0, for only one velocity that depends on $(a,b) = (\eta(0,-1), \eta(0,1))$. (The precise dependence is given below.) The reader can check that at later times the two configurations will differ at only one site for one of the two velocities. Let $X(t)$ be the site where the two configurations differ:

$$X(t) = x \quad \text{iff } \eta_t'(x,s) = 1 - \eta_t'(x,s), \quad \text{for } s = 1 \text{ or } s = -1.$$

THEOREM 2. *If η_0 has distribution $\nu_{\rho,\lambda}$, then $\tau_{X(t)}\eta_t$ also has distribution $\nu_{\rho,\lambda}$ for all $t \geq 0$.*

We define more carefully the microscopic interface. Let η be a configuration and define η' as follows: Let $(a,b) = (\eta(0,-1), \eta(0,1))$. Then $\eta'(x,s) = \eta(x,s)$ for all $x \neq 0$; if $(a,b) = (1,1)$ or $(a,b) = (0,0)$, then $\eta'(0,1) = \eta(0,1)$, $\eta'(0,-1) = 1 - \eta(0,-1)$; if $(a,b) = (1,0)$ or $(a,b) = (0,1)$, then $\eta'(0,1) = 1 - \eta(0,1)$, $\eta'(0,-1) = \eta(0,-1)$.

Let η_t and η_t' be the evolutions of η and η', respectively. Let $X(t)$ and $V(t)$ be the site and velocity where η and η' differ:

$$\eta_t'(x,s) = \begin{cases} 1 - \eta_t(x,s), & \text{if } x = X(t) \text{ and } s = V(t), \\ \eta_t(x,s), & \text{otherwise.} \end{cases}$$

The motion of $(X(t), V(t))$, the "second-class particle," is the following:

If $V(t) = 1$ and

$$\text{if } \eta_t(X(t), -1) = 0 \quad \text{then} \quad V(t+1) = V(t) = 1$$

and

$$X(t+1) = X(t) + 1;$$

$$\text{if } \eta_t(X(t), -1) = 1, \quad \text{then} \quad V(t+1) = -V(t) = -1$$

and

$$X(t+1) = X(t) - 1.$$

If $V(t) = -1$ and

$$\text{if } \eta_t(x+1, 1) = 1, \quad \text{then} \quad V(t+1) = V(t) = -1$$

and

$$X(t+1) = X(t) - 1;$$

$$\text{if } \eta_t(x+1,1) = 0, \quad \text{then} \quad V(t+1) = -V(t) = 1$$

and

$$X(t+1) = X(t) + 1.$$

In words, if the second-class particle has velocity 1, it continues at velocity 1 until it meets a particle with velocity -1. Similarly the second-class particle moves to the left until it meets a hole with velocity 1, at which time it reverses velocity.

Now we study the motion of $(X(t), V(t))$ when η_0 is distributed according to $\nu_{\rho,\lambda}$. Let $V(1) = -1$ [the case $V(1) = +1$ is similar]. We define L_1 as the first time after 0 that the velocity is reversed to $+1$, and analogously R_1 is the first time that the velocity is reversed again to -1. More precisely, set $L_0 = R_0 = 0$ and

$$L_n(\eta) := \min\{t > L_{n-1}(\eta): V(t) = -1, V(t+1) = 1\},$$
$$R_n(\eta) := \min\{t > R_{n-1}(\eta): V(t) = 1, V(t+1) = -1\}.$$

Now we observe that the dynamics on even sites are independent of the dynamics on the odd sites [Cheng, Lebowitz and Speer (1990)]. This justifies the following definitions:

$$D_n^-(\eta) = 2\sum_{k=1}^{n} L_k - R_{k-1} = \sup\{2x < D_{n-1}^-(\eta): \eta_0(2x,1) = 0\};$$

$$D_n^+(\eta) = 2\sum_{k=1}^{n} R_k - L_k = \inf\{2x > D_{n-1}^+(\eta): \eta_0(2x,-1) = 1\};$$

$$D_0^+ = D_0^- = 0.$$

If η has distribution $\nu_{\rho,\lambda}$, then $D_{n-1}^- - D_n^-$ and $D_n^+ - D_{n-1}^+$, $n \geq 1$, are geometric random variables with parameters $(1-\rho)/2$ and $\lambda/2$, respectively. Furthermore, they depend on different regions of the space for the initial configuration. This implies that $D_n^- - D_{n-1}^-$ and $D_n^+ - D_{n-1}^+$, $n \geq 1$, are mutually independent.

Proof of Theorem 2. For $x < 0$ and $y > 0$, let $H_{x,y}$ be the operator which when applied to a configuration η deletes the part of the configuration that is between x and y and translates the configuration to the left of x and to the right of y to the origin. In other words,

$$H_{x,y}\eta(z,s) := \begin{cases} \eta(z+x,s), & \text{if } z < 0, \\ \eta(z+y,s), & \text{if } z \geq 0. \end{cases}$$

Let $\overline{\mathbf{X}} = \{0,1\}^{(\mathbb{Z}\setminus\{0\})\times\{-1,1\}}$. Observe that if the distribution of η restricted to $\overline{\mathbf{X}}$ has distribution $\nu_{\rho,\lambda}$ (restricted to $\overline{\mathbf{X}}$), then, for any $x < 0$ and $y > 0$, the distribution of $H_{x,y}\eta$ restricted to $\overline{\mathbf{X}}$ is also $\nu_{\rho,\lambda}$ (restricted to $\overline{\mathbf{X}}$).

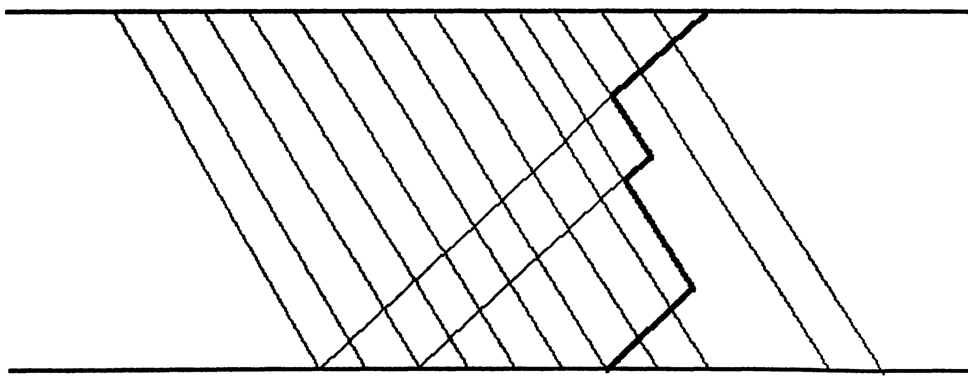

FIG. 1. *The trajectory of $X(t)$ is darker. We only draw the trajectories of particles.*

Now we leave to the reader to prove that, for all η_0,

$$\tau_{X(R_n)}\eta_{R_n}(z,s) = \begin{cases} H_{D_n^-, D_n^+}\eta_0(z,s), & \text{for } z \neq 0, \\ \eta_1(0,-1), & \text{for } z \neq 0 \text{ and } s = -1, \\ 1, & \text{for } z = 0 \text{ and } s = 1, \end{cases}$$

and that

$$D_n^- - D_{n+1}^- = 2(L_{n+1} - R_n),$$
$$D_n^+ - D_{n-1}^+ = 2(R_n - L_n)$$

(see Figure 1). This implies that (recall that D_n^- is negative),

$$X(R_n) = (D_n^+ + D_n^-)/2, \tag{4.1}$$

which is independent of $H_{D_n^-, D_n^+}\eta_0$. In other words $X(R_n)$ and $\tau_{X(R_n)}\eta_{R_n}$ are independent. This proves that for $t = R_n$, the measure outside the origin is $\nu_{\rho,\lambda}$. With the same argument we can prove the same result for $t = L_n$. For the intermediate t's, say $L_n < t < R_n$, one observes that to determine the configuration to the right of the shock one erases $t - L_n$ sites and translates the configuration at L_n while the configuration to the left remains unchanged. Something similar is done for $R_n < t < L_{n+1}$. Finally, the measure at the origin $m(a,b)$ is just the stationary measure for the Markov chain on $\{0,1\}^{\{-1,1\}}$ with transition probabilities

$$p((1,1),(0,0)) = p((1,0),(0,0)) = 1 - \rho,$$
$$p((1,1),(1,1)) = p((1,0),(1,1)) = \rho,$$
$$p((0,0),(0,1)) = p((0,1),(0,1)) = 1 - \lambda,$$
$$p((0,0),(1,0)) = p((0,1),(1,0)) = \lambda,$$
$$p((a,b),(c,d)) = 0, \quad \text{otherwise}.$$

This Markov chain governs the evolution of the configuration at the shock. This completes the proof of Theorem 2. □

5. Law of large numbers and central limit theorem. In this section we study the asymptotic behavior of the shock position.

THEOREM 3 (Law of large numbers for the shock). *The following holds*:

$$\lim_{t\to\infty} \frac{X(t)}{t} = v := \frac{1-(\lambda+\rho)}{1-(\rho-\lambda)} \quad a.s.$$

PROOF. We consider the renewal process R_n and call $n(t)$ the number of renewals up to time t. We have that

$$X(t) = \sum_{k=0}^{n(t)} M_k + \big(X(t) - X(R_{n(t)})\big), \tag{5.1}$$

where $M_k = X(R_k) - X(R_{k-1})$ is a difference of two geometric random variables of parameters $(1-\rho)$ and λ, respectively [see (4.1)]. Furthermore, $\{M_k\}_k$ is a family of mutually independent random variables. On the other hand, $(X(t) - X(R_{n(t)}))$ is tight [dominated by a sum of two geometric random variables with parameters $(1-\rho)$ and λ]. Hence we have

$$\lim_{t\to\infty} \frac{t}{n(t)} = \frac{1}{\lambda} + \frac{1}{1-\rho} \quad \text{a.s.},$$

and

$$\lim_{t\to\infty} \frac{R_{n(t)}}{n(t)} = \frac{1}{\lambda} \quad \text{and} \quad \lim_{t\to\infty} \frac{L_{n(t)}}{n(t)} = \frac{1}{1-\rho} \quad \text{a.s.}$$

Putting this all together we have

$$\lim_{t\to\infty} \frac{X(t)}{t} = \frac{(1/\lambda) - (1/(1-\rho))}{(1/\lambda) + (1/(1-\rho))} = \frac{1-(\rho+\lambda)}{1-(\rho-\lambda)} \quad \text{a.s.}$$

This proves the theorem. □

Properties of $X(t)$ enable us to prove also an invariance principle. Let $W^\varepsilon(t) = \varepsilon(X([\varepsilon^{-2}t]) - \varepsilon^{-2}tv)$.

THEOREM 4. *As $\varepsilon \to 0$, $W^\varepsilon(t)/\sigma$ converges in distribution to a standard Brownian motion, where*

$$\sigma^2 = \frac{4(1-(\lambda-\rho))(1-\rho)\lambda}{(1+(\lambda-\rho))^3}.$$

PROOF. By (5.1) we have that

$$\varepsilon\big(X(\varepsilon^{-2}t) - v\varepsilon^{-2}t\big) = \varepsilon\left(\sum_{k=0}^{n(\varepsilon^{-2}t)} M_k - E\sum_{k=0}^{n(\varepsilon^{-2}t)} M_k\right) + O(\varepsilon)$$

$$= \varepsilon\left(\sum_{k=0}^{n(\varepsilon^{-2}t)} (M_k - EM_k) + \big[n(\varepsilon^{-2}t) - En(\varepsilon^{-2}t)\big]EM_1\right) + O(\varepsilon) \tag{5.2}$$

where $O(\varepsilon)/\varepsilon$ is bounded by a sum of two geometric random variables and integer parts have been taken when necessary. The right-hand side of (5.2) converges to Brownian motion by the random change of time theorem of Billingsley (1968). The diffusion coefficient of the limiting Brownian motion is given by

$$\frac{E\big((R_1 + L_1)E(R_1 - L_1) - (R_1 - L_1)E(R_1 + L_1)\big)^2}{\big(E(R_1 + L_1)\big)^3} = \sigma^2,$$

where we have used the independence of L_1 and R_1. □

REMARK. If we only assume that the initial measures are extremal stationary translation invariant (not necessarily product), then using the ergodic theorem we can still prove the law of large numbers for the shock, with the same velocity. It would also follow from the ergodic theorem that asymptotically to the right and left of the shock one sees the extremal measures with densities $1 + \lambda$ and ρ, respectively. To prove a central limit theorem one would have to assume more than ergodicity, probably some sort of mixing conditions.

6. Hydrodynamics and dynamical phase transition. Let, for all $r \in \mathbb{R}$ and $t \in \mathbb{Z}$, denoting by $[x]$ the integer part of x,

$$u^\varepsilon(r,t) = \nu_{\rho,\lambda}\big(\eta([\varepsilon^{-1}r], 1, [\varepsilon^{-1}t]) + \eta([\varepsilon^{-1}r], -1, [\varepsilon^{-1}t])\big).$$

THEOREM 5. *As $\varepsilon \to 0$, $u^\varepsilon(r,t)$ converges to $u(r,t)$, uniformly at points of continuity of $u(r,t)$, where $u(r,t)$ is the weak traveling wave entropic solution of* (1.1), *with initial conditions $u(r,0) = (1+\lambda)1\{r \geq 0\} + \rho 1\{r < 0\}$. That is, $u(r,t) = u(r - vt, 0)$, where*

$$v = \frac{1 - (\lambda + \rho)}{1 - (\rho - \lambda)}. \tag{6.1}$$

Furthermore, for any cylinder function f, for t sufficiently large,

$$\nu_{\rho,\lambda} f(\tau_{[wt]}\eta_t) = \begin{cases} \nu_{1+\lambda} f, & \text{if } w > v,\\ \nu_\rho f, & \text{if } w < v.\end{cases}$$

We note that v is the velocity of the shock of (1.1).

PROOF. Let $u(r,t) = u(r - vt, 0)$. Let $\tau_z\mu(A) = \mu(\tau_z A)$. We consider $\tau_{[wt]}(\nu_{\rho,\lambda}T^{[t]})$. From the law of large numbers and Theorem 2, it follows that

$$\lim_{t\to\infty} \tau_{[wt]}\nu_{\rho,\lambda}T^{[t]} = \begin{cases} \mu_{1+\lambda}, & \text{if } w > v, \\ \mu_{\rho}, & \text{if } v < w. \end{cases}$$

This proves Theorem 5. □

We now consider the question of what happens at the velocity of the shock.

THEOREM 6 (Dynamical phase transition). *The following limit holds*:

$$\lim_{t\to\infty} \nu_{\rho,\lambda} f(\tau_{vt}\eta_t) = \tfrac{1}{2}\nu_{1+\lambda} f + \tfrac{1}{2}\nu_{\rho} f,$$

for any cylinder function f.

PROOF. It follows from Theorem 4 that, given $k \in \mathbb{R}$,

$$\lim_{t\to\infty} \nu_{\rho,\lambda}(X(t) - vt > k) = \lim_{t\to\infty} \nu_{\rho,\lambda}(X(t) - vt < k) = \tfrac{1}{2}.$$

If f is a cylinder function, then the result of Theorem 6 follows from Theorem 2. □

Other initial conditions. Now we describe briefly the behavior of the system under other initial conditions.

1. Left density $\rho < 1$, right density $\lambda < 1$. It is clear from the equation that this is a travelling wave solution with velocity 1. Microscopically, configurations are translated one unit to the right each step.
2. Left density $\rho > 1$, right density $\lambda < 1$. In this case the macroscopic equation gives us two travelling waves moving in opposite directions. At time t the solution is undefined in the region $-t < r < t$. The microscopic solution also gives these two travelling waves, but in the region in between, the solution is $u = 1$.
3. Left density $\rho > 1$, right density $\lambda > 1$. In this case we get a travelling wave solution with velocity -1 microscopically and macroscopically.
4. Monotone initial conditions with more than one step. In this case both macroscopically and microscopically the system behaves the same.

7. Relations with particle systems in $\{0,1\}^{\mathbb{Z}}$. In this section we show that the BLCA is isomorphic to two simple exclusion-type automata in $\{0,1\}^{\mathbb{Z}}$. We start by defining the models.

The asymmetric simple exclusion cellular automata. For a given configuration $\xi \in \{0,1\}^{\mathbb{Z}}$, define $B_1\xi$ as the configuration

$$B_1\xi(2z+1) = 1\{\xi(2z) + \xi(2z+1) \geq 1\},$$

$$B_1\xi(2z) = \xi(2z) + \xi(2z+1) - 1\{\xi(2z) + \xi(2z+1) \geq 1\},$$

and $B_2\xi$ as the configuration

$$B_2\xi(2z) = 1\{\xi(2z-1) + \xi(2z) \geq 1\},$$
$$B_2\xi(2z-1) = \xi(2z-1) + \xi(2z) - 1\{\xi(2z-1) + \xi(2z) \geq 1\}.$$

Now define the asymmetric simple exclusion automata

$$\xi_{2t} = B_1\xi_{2t-1} \quad \text{and} \quad \xi_{2t+1} = B_2\xi_{2t}. \tag{7.1}$$

In words, at even times, all particles occupying even sites that can jump to the right (i.e., those whose successive odd site is empty), jump to the right. At odd times, the particles occupying the odd sites do the same.

We prove that this is isomorphic to a subsystem of the BLCA when the advection is applied before the collision rule. As observed before, the BLCA consists of two independent subsystems: $\{\eta(x,s,t)\colon x+t \text{ odd}\}$ and $\{\eta(x,s,t)\colon x+t \text{ even}\}$. Consider the subsystem $\{\eta(x,s,t)\colon x+t \text{ odd}\}$ and define the configuration $\xi = \xi(\eta)$ by

$$\xi(2x+1) = \begin{cases} \eta(2x+1,-1,t), & \text{for } t \text{ even},\\ \eta(2x,1,t), & \text{for } t \text{ odd} \end{cases}$$

and

$$\xi(2x) = \begin{cases} \eta(2x-1,1,t), & \text{for } t \text{ even},\\ \eta(2x,-1,t), & \text{for } t \text{ odd}. \end{cases}$$

We have the following result, the proof of which is straightforward (see Figure 2).

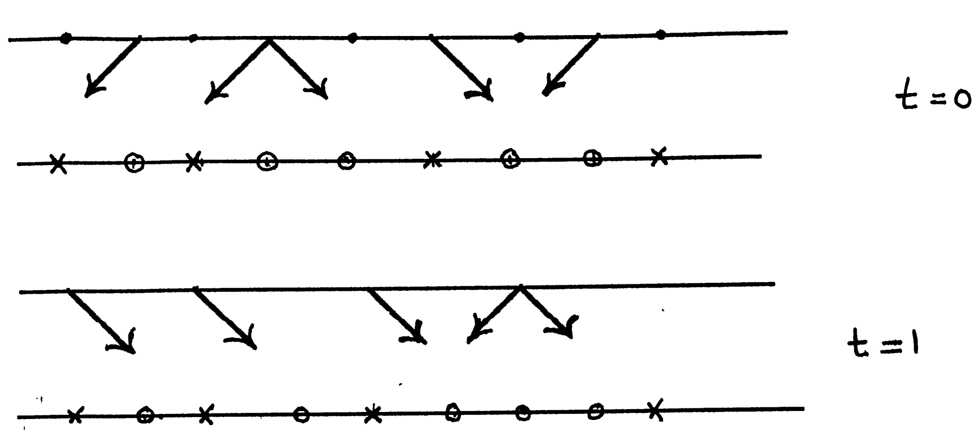

FIG. 2. ○, *particles in exclusion automaton*; ×, *vacant sites in exclusion automaton*; ·, *vacant sites in* BLCA; ↘ , *particle with velocity* 1 *in* BLCA; ↙ , *particle with velocity* −1 *in* BLCA.

LEMMA 1. *The transformation ξ defines an isomorphism between the subsystem $\{\eta(x,s,t)\colon x+t \text{ odd}\}$ and ξ_t, $t \in \mathbb{Z}$, such that ξ_t is the asymmetric simple exclusion cellular automata, with distribution described by* (7.1).

The automaton 184. This model has been classified by Wolfram (1983), and studied by Krug and Spohn (1988), who related it to an interface problem and studied convergence to equilibrium. Let $B\gamma$ be the configuration defined by

$$B\gamma(z) = \begin{cases} 1, & \text{if } \gamma(z-1) = 1 \text{ and } \gamma(z) = 0, \\ 0, & \text{if } \gamma(z) = 1 \text{ and } \gamma(z+1) = 0, \\ \gamma(z), & \text{otherwise.} \end{cases}$$

In words, $B\gamma$ is the configuration obtained when all particles of γ allowed to jump one unit to the right do it. Define the automaton by $\gamma_t = B\gamma_{t-1}$. Assume now that at time 0, all even sites are empty. In this case this is isomorphic to ξ_t with the same initial configuration. On the other hand, if all even sites are occupied, it is also isomorphic to ξ_t. Nevertheless, for other configurations this system is not isomorphic, presenting a richer structure. But for those types of initial conditions the results proved for the BLCA hold.

Acknowledgments. We would like to thank Z. Cheng, J. L. Lebowitz and E. R. Speer for discussions on the BLCA which led to this work. We are grateful to Errico Presutti for his valuable comments. K. R. would like to thank the Instituto de Matemática e Estatística da Universidade de São Paulo for its hospitality and would like to acknowledge the support provided by the Grant for Research and Creative projects at SUNY, New Paltz. Work partially supported by FAPESP and CNPq. We thank the referees for their comments.

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