The Annals of Applied Probability
1992, Vol. 2, No. 4, 846–856

MATCHING RANDOM SAMPLES IN MANY DIMENSIONS[1]

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Consider any norm N on $\mathbb{R}^d$, $d \geq 3$, and independent uniformly distributed points $X_1, \dots, X_n, \dots; Y_1, \dots, Y_n, \dots$ in $[0,1]^d$. Consider the random variable $M_n = \inf \sum_{i \leq n} N(X_i - Y_{\sigma(i)})$, where the infimum is taken over all permutations σ of $\{1, \dots, n\}$. We show that for some universal constant K, we have

$$\limsup_{n \to \infty} M_n n^{-1+1/d} \leq r_N \left(1 + K \frac{\log d}{d}\right) \quad \text{a.s.,}$$

where r_N is the radius of the ball for N of volume 1.

1. Introduction. Consider a norm N on $\mathbb{R}^d$. Consider independent identically distributed points $X_1, \dots, X_n, \dots; Y_1, \dots, Y_n, \dots$ in $[0,1]^d$, and denote by M_n^N the average length of the edge in an optimal matching between $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, that is,

$$M_n^N = \inf_{\sigma \in S} \sum_{i \leq n} N(X_i - Y_{\sigma(i)}),$$

where the infimum is taken over the set S of all permutations σ of $\{1, \dots, n\}$. Ajtai, Komlos and Tusnady [1] proved that, when $d = 2$, with high probability, M_n^N is of order $(n \log n)^{1/2}$ (a truly remarkable result). When $d \geq 3$, the nature of the result changes, and M_n is of order $n^{1-1/d}$ (a fact of considerably less depth). This means that, with high probability, we have $c_{d,N} n^{1-1/d} \leq M_n^N \leq C_{d,N} n^{1-1/d}$, for two constants $c_{d,N}, C_{d,N}$ dependent on d, N. We are interested here in the behavior of these constants as $d \to \infty$. The method of [1] relies on a partitioning scheme that partitions $[0,1]^d$ into parallelepipeds. This method is extremely ill adapted to the study of $C_{d,N}, c_{d,N}$, in particular in the case where N is the Euclidean ball, as parallelepipeds are far from balls (and the higher the dimension, the more so). In the present paper, we introduce a different (and very elementary) method that is better adapted to the problem (but fails to give the correct result when $d = 2$). We obtain the following result.

Received February 1991; revised November 1991.
[1]This research was supported in part by an NSF grant.
AMS 1980 *subject classifications*. 60C05, 05C70.
Key words and phrases. Matchings, transportation cost, empirical measure.

THEOREM 1. *Consider the number r_N such that the ball for N of radius r_N has volume* 1. *Then*

$$r_N\left(1 - \frac{1}{d+1}\right) \le \liminf_{n\to\infty} \frac{M_n^N}{n^{1-1/d}} \le \limsup_{n\to\infty} \frac{M_n^N}{n^{1-1/d}}$$

$$\le r_N\left(1 + K\frac{\log d}{d}\right) \quad a.e.,$$

where K is universal.

We close this section with a discussion of some open problems. It is routine to prove (using "subadditivity arguments") that $\liminf n^{-1+1/d} M_n^N$ exists a.e. Denote by $\alpha_d(N)$ this limit.

PROBLEM 1. In the case where N is the Euclidean ball in $\mathbb{R}^d$, what is the rate of convergence of $|r_N^{-1}\alpha_d(N) - 1|$ to 0 as $d \to \infty$? Our result implies that this quantity is $O(\log d/d)$.

Another natural question of interest is as follows. For $p \ge 1$, define

$$M_n^{N,p} = \inf_{\sigma\in S} \sum_{i\le n} N(X_i - Y_{\sigma(i)})^p.$$

PROBLEM 2. Is it true that, given $p > 1$, we have

$$\limsup_{n\to\infty} \frac{M_n^{N,p}}{n^{1-p/d}} \le r_N^p(1 + o(d^{-1}))?$$

The function of d^{-1} that is implicit in the notation $o(d^{-1})$ might depend on p. We conjecture, however, that, provided $p \le \varphi(d)$, where φ is a certain function such that $\lim_{d\to\infty}\varphi(d) = \infty$ [possibly, $\varphi(d) = \alpha d$], we can take a function independent of p.

2. Simple facts. We fix the norm N on $\mathbb{R}^d$. We say that a function f on $\mathbb{R}^d$ is *Lipschitz* if $|f(x) - f(y)| \le N(x - y)$ for all $x, y \in \mathbb{R}^d$. We denote by $\mathscr{L}$ the class of Lipschitz functions that are 0 at the origin. We will proceed (in a standard fashion; e.g., [5]) through duality.

LEMMA 1.

$$M_n = \sup_{f\in\mathscr{L}} \left| \sum_{i\le n} (f(X_i) - f(Y_i)) \right|.$$

PROOF. Since for $f \in \mathscr{L}$, we have $|f(X_i) - f(Y_i)| \le N(X_i - Y_j)$, we certainly have

$$\left|\sum_{i \le n} (f(X_i) - f(Y_i))\right| \le M_n.$$

The converse is less obvious. It relies on the fact that

$$M_n = \sup\left(\sum_{i \le n} u_i - \sum_{i \le n} v_i\right),$$

where the supremum is taken over all sequences $(u_i)_{i \le n}, (v_j)_{j \le n}$ for which $u_i \le \min_{j \le n}\{v_j + N(X_i - Y_j)\}$. (This is a simple consequence of the duality principle in linear programming; see [2].) For any such sequences $(u_i)_{i \le n}, (v_j)_{j \le n}$, consider the function

$$g(x) = \min_{j \le n}\{v_j + N(x - Y_j)\}.$$

Then $g(Y_j) \le v_j, g(X_i) \ge u_i$, so that

$$\sum_{i \le n} g(X_i) - \sum_{i \le n} g(Y_i) \ge \sum_{i \le n} u_i - \sum_{i \le n} v_i.$$

Also, it is simple to see from the definition that g is Lipschitz, so that $f(x) = g(x) - g(0) \in \mathscr{L}$. This completes the proof. □

We set $D = \sup\{N(x - y); x, y \in [0, 1]^d\}$. Thus $|f(x)| \le D$ for $f \in \mathscr{L}$.

LEMMA 2.

$$P(|M_n - E(M_n)| \ge t) \le \exp\left(-\frac{t^2}{8nD^2}\right).$$

PROOF. This is an immediate consequence of the martingale difference method as, for example, in [4]. □

A consequence of this statement is that, in the statement of Theorem 1, it suffices to replace M_n by $E(M_n)$.

We will assume in the sequel that $r_N = 1$. This is no loss of generality, as is seen by replacing N by N/r_N. This means that the ball of N of radius 1 has volume 1.

We prove the lower bound for M_n, which is certainly well known, and is based on the observation that

$$N(X_i - Y_{\sigma(i)}) \ge \min_{j \le n} N(X_i - Y_j).$$

Thus, conditioning on $X_1, \ldots, X_n$, we get that

$$E(M_n) \ge n \min_{x \in [0,1]^2} E\left(\min_{j \le n} N(x - Y_j)\right). \tag{1}$$

We denote by $B(x, t)$ the ball for N centered at x of radius t. Thus

$$B(x,t) = \{y \in \mathbb{R}^d;\ N(x-y) \le t\}.$$

The d-dimensional volume of a set A will be denoted by $|A|$. Since $r_N = 1$, we have $|B(x,t)| = t^d$, so that

$$\left|B(x,t) \cap [0,1]^d\right| \le t^d$$

and

$$P\Big(\min_{j \le n} N(x - Y_j) \ge t\Big) \ge (1-t^d)^n.$$

Thus

$$E \min_{j\le n} N(x-Y_j) \ge \int_0^1 (1-t^d)^n \, dt.$$

By (1) and change of variable $t = n^{-1/d}u$, we get

$$E(M_n) \ge n^{1-1/d} \int_0^{n^{1/d}} (1-u^d/n)^n \, du.$$

Thus, by Fatou's lemma,

$$\liminf_{n\to\infty} \frac{E(M_n)}{n^{1-1/d}} \ge \int_0^\infty \exp(-u^d)\, du \ge \int_0^1 (1-u^d)\, du = 1 - \frac{1}{d+1}.$$

3. The approach. We consider a parameter $\eta > 1$ to be adjusted later on. We set $r = \eta n^{-1/d}$, so that $|B(x,r)| \doteq \eta^d/n$. We set

$$u(i,j) = \begin{cases} 1, & \text{if } N(X_i - Y_j) \le r, \\ 0, & \text{otherwise.} \end{cases}$$

We set

$$b(x) = \left|B(x,r) \cap [0,1]^d\right|.$$

Note that $b(x) \le \eta^d/n$.

LEMMA 3.

$$\eta^d E(M_n) \le \eta^d rn + 2G(n) + G_1(n),$$

where

$$G(n) = E \sup_{f\in\mathscr{L}} \left| \sum_{i\le n} f(X_i) \sum_{j\le n} \big(u(i,j) - b(X_i)\big)\right|,$$

$$G_1(n) = E \sup_{f\in\mathscr{L}} \left| \sum_{i\le n} f(X_i)\big(\eta^d - nb(X_i)\big) - \sum_{j\le n} f(Y_j)\big(\eta^d - nb(Y_j)\big)\right|.$$

PROOF. We write for $f \in \mathscr{L}$,

$$\eta^d \Big| \sum_{i \le n} f(X_i) - \sum_{j \le n} f(Y_j) \Big|$$

$$\le \Big| \sum_{i \le n} f(X_i) \Big(\sum_{j \le n} u(i,j) \Big) - \sum_{j \le n} f(Y_j) \Big(\sum_{i \le n} u(i,j) \Big) \Big|$$

$$+ \Big| \sum_{i \le n} f(X_i) \sum_{j \le n} \Big(u(i,j) - \frac{\eta^d}{n} \Big) - \sum_{j \le n} f(Y_j) \sum_{i \le n} \Big(u(i,j) - \frac{\eta^d}{n} \Big) \Big|$$

$$:= A_1(f) + A_2(f).$$

We have

$$A_1(f) \le \sum_{i,j \le n} u(i,j) \big| f(X_i) - f(Y_j) \big| \le r \sum_{i,j \le n} u(i,j),$$

since f is Lipschitz and $u(i,j) = 0$ unless $N(X_i - Y_j) \le r$. Thus

$$E \sup_{f \in \mathscr{L}} A_1(f) \le r \sum_{i,j \le n} Eu(i,j).$$

Now

$$E(u(i,j)|X_i) = b(X_i) \le \frac{\eta^d}{n},$$

so that $E(u(i,j)) \le \eta^d/n$, and this implies that

$$E \sup_{f \in \mathscr{L}} A_1(f) \le r\eta^d n.$$

Now we write

$$A_2(f) \le \Big| \sum_{i \le n} f(X_i) \sum_{j \le n} \big(u(i,j) - b(X_i) \big) \Big| + \Big| \sum_{j \le n} f(Y_j) \sum_{i \le n} \big(u(i,j) - b(Y_j) \big) \Big|$$

$$+ \Big| \sum_{i \le n} f(X_i) \big(nb(X_i) - \eta^d \big) - \sum_{j \le n} f(Y_j) \big(nb(Y_j) - \eta^d \big) \Big|.$$

Taking the supremum over f and then expectation yields the result. □

Next we will derive upper bounds on $G_1(n)$ and $G(n)$ and show that with a proper choice of η these terms are low-order terms. This will lead to Theorem 1.

LEMMA 4.

$$\lim_{n \to \infty} \frac{G_1(n)}{n^{1-1/d}} = 0.$$

PROOF. For a measurable function f on $[0,1]^d$, we set

$$\Delta(f,n) = \left| \sum_{i \le n} f(X_i)\big(\eta^d - nb(X_i)\big) - \sum_{j \le n} f(Y_j)\big(\eta^d - nb(Y_j)\big) \right|.$$

Thus

$$\Delta(f,n) \le \|f\|_\infty \left(\sum_{i \le n} \left|\eta^d - nb(X_i)\right| + \sum_{j \le n} \left|\eta^d - nb(Y_j)\right| \right).$$

Set

$$\begin{aligned} V_n &= \left\{x \in [0,1]^d;\, b(x) \ne \eta^d/n\right\} \\ &= \left\{x \in [0,1]^d;\, B(x,r) \not\subset [0,1]^d\right\}. \end{aligned}$$

Since $\eta^d - nb(x) = 0$ unless $x \in V_n$, we have

$$\sup_{\|f\|_\infty \le a} \Delta(f,n) \le a\big(\operatorname{card}\{i \le n;\, X_i \in V_n\} + \operatorname{card}\{j \le n;\, Y_j \in V_n\}\big).$$

Thus

$$E \sup_{\|f\|_\infty \le a} \Delta(f,n) \le 2na|V_n|.$$

Now since $r = n^{-1/d}$, it is clear that for a constant C, depending on N, d, but not on n, we have $|V_n| \le Cn^{-1/d}$. Thus

$$E \sup_{\|f\|_\infty \le a} \Delta(f,n) \le 2aCn^{1-1/d}. \tag{2}$$

On the other hand, for a given bounded function f, we see that $E\Delta(f,n)^2 = C'n$, where C' is independent of n, so that $E\Delta(f,n) \le (C')^{1/2}\sqrt{n}$, and $\lim_{n\to\infty} E\Delta(f,n)/n^{1-1/d} = 0$ since $d > 2$. The result then follows from this observation, (2) and the fact that given any $a > 0$, $\mathscr{L}$ can be covered by finitely many sets of the type $\{g \in \mathscr{L};\, \|f - g\|_\infty \le a\}$. □

The hard part of the proof is the following statement.

LEMMA 5. *We have* $G(n) \le 14n^{1-1/d}\eta^{d/2+1}$.

If we combine with Lemma 3, and recall that $r = \eta n^{-1/d}$, we see that

$$E(M_n) \le n^{1-1/d}\left(\eta + \frac{10}{\eta^{d/2-1}}\right) + \eta^{-d}G_1(n).$$

Taking $\eta = 1 + K(\log d/d)$ for a constant K large enough gives $\eta^{d/2-1} \ge d$; combined with Lemma 4, this proves Theorem 1. It remains to prove Lemma 5.

4. Proof of Lemma 5. We set $W_i = \sum_{j \le n}(u(i,j) - b(X_i))$. The first idea in the proof of Lemma 5 is elementary.

LEMMA 6.

$$E|W_i| \le \eta^{d/2}.$$

PROOF. Observe that $E(u(i,j)|X_i) = b(X_i)$ so that

$$E(W_i^2|X_i) = nb(X_i)(1 - b(X_i)) \le \eta^d. \qquad \square$$

For a function g on $[0,1]^2$, we consider the random variable

$$\Gamma(g) = \left|\sum_{i \le n} g(X_i)W_i\right|.$$

Thus

$$G(n) = E\sup_{f \in \mathscr{L}} |\Gamma(f)|. \tag{3}$$

A second idea in the proof of Lemma 5 is that, for a given a set A, there is much cancellation among the variables W_i for$X_i \in A$. (Observe that these variables are not independent.) We denote by 1_A the indicator function of A.

LEMMA 7. *For a set $A \subset [0,1]^d$, we have*

$$E(\Gamma(1_A)^2) \le n|A|(\eta^{2d} + \eta^d) \le 2n|A|\eta^{2d}.$$

PROOF. For simplicity we set $v(i,j) = u(i,j) - b(X_i)$ if $X_i \in A$, $v(i,j) = 0$ otherwise. Thus $\Gamma(1_A) = |\sum_{i,j \le n} v(i,j)|$. Now

$$E(\Gamma(1_A)^2) = \sum_{i,i',j,j'} E(v(i,j)v(i',j')).$$

If $j \ne j'$, conditioning on $X_i, X_{i'}, Y_{j'}$, we see that $E(v(i,j)v(i',j')) = 0$. If $j = j'$ and $i \ne i'$, we see that

$$E(v(i,j)v(i',j')) = E\Big(E(v(i,j)|Y_j)^2\Big).$$

Thus

$$E(\Gamma^2(1_A)) = n^2E(v(1,1)^2) + n^2(n-1)E\Big(E(v(1,1)|Y_1)^2\Big).$$

For $s,t \ge 0$, we have $(s-t)^2 \le s^2 + t^2$, so that $v(1,1)^2 \le u(1,1) + b^2(X_1)$ and $v(1,1) = 0$ if $X_1 \notin A$. We observe that $u(1,1)^2 = u(1,1)$, so that

$$E(u(1,1)^2|X_1) = E(u(1,1)|X_1) = b(X_1).$$

Since $b(X_1) \le \eta^d/n$, we have shown that

$$E(v(1,1)^2) \le |A|\left(\frac{\eta^d}{n} + \frac{\eta^{2d}}{n^2}\right).$$

We have

$$E(v(1,1)|Y_1) = |A \cap B(Y_1, r)| - Ec,$$

where $c = b(X_1)$ for $X_1 \in A$ and $c = 0$ otherwise. Note that, since

$$E(v(1,1)) = 0,$$

we have

$$E|A \cap B(Y_1, r)| = Ec. \tag{4}$$

This implies first that

$$\begin{aligned} E\big(E(v(1,1)|Y_1)^2\big) &= E|A \cap B(Y_1, r)|^2 - (Ec)^2 \\ &\leq E|A \cap B(Y_1, r)|^2. \end{aligned}$$

Since $|A \cap B(Y_1, r)| \leq \eta^d/n$, we get, by (4),

$$\begin{aligned} E\big(E(v(1,1)|Y_1)^2\big) &\leq \frac{\eta^d}{n} E|A \cap B(Y_1, r)| \\ &\leq \frac{\eta^d}{n} Ec \leq |A| \frac{\eta^{2d}}{n^2}, \end{aligned}$$

since $c \leq \eta^d/n$ for $X_1 \in A$ and $c = 0$ otherwise. □

LEMMA 8.

$$E \sup_{\|g\|_\infty \leq b} \Gamma(g) \leq nb\eta^{d/2}.$$

PROOF. Write

$$\Gamma(g) \leq b \sum_{i \leq n} |W_i|$$

and use Lemma 6. □

For two functions h, g on $\mathbb{R}^d$, we recall that

$$h * g(x) = \int_{\mathbb{R}^d} h(x-t) g(t)\, dt. \tag{5}$$

In the cases we will consider the functions h, g will be continuous, and h will have a bounded support, so that $h * g$ will be well defined.

LEMMA 9.

$$E \sup_{\|g\|_\infty \leq b} \Gamma(h * g) \leq b \left(\int_{\mathbb{R}^d} E\Gamma^2(h^t)\, dt \right)^{1/2},$$

where $h^t(x) = h(x-t)$ *for* $x, t \in \mathbb{R}^d$.

PROOF. We have, using (5),

$$\begin{aligned}\Gamma(h*g) &= \left|\sum_{i\le n} W_i h*g(X_i)\right| = \left|\int_{\mathbb{R}^d}\sum_{i\le n} W_i g(t)h(X_i-t)\,dt\right| \\ &\le \int_{\mathbb{R}^d} g(t)\left|\sum_{i\le n} W_i h(X_i-t)\,dt\right| \\ &\le b\int_{\mathbb{R}^d}|\Gamma(h^t)|\,dt.\end{aligned}$$

Taking expectations, we get

$$E\sup_{\|g\|_\infty\le b}\Gamma(h*g)\le b\int_{\mathbb{R}^d}E|\Gamma(h^t)|\,dt.$$

Using Cauchy–Schwarz twice, we have

$$\begin{aligned}\int_{\mathbb{R}^d}E|\Gamma(h^t)|\,dt &\le \left(\int_{\mathbb{R}^d}(E|\Gamma(h^t)|)^2\,dt\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^d}E\Gamma^2(h^t)\,dt\right)^{1/2}.\end{aligned}$$

□

For $l\ge 1$, we consider the function h_l on $\mathbb{R}^d$ defined as follows. If $N(x)>2^l r$, then $h_l=0$, while if $N(x)\le 2^l r$, then $h_l=(2^l r)^{-d}$. Thus

$$\int_{\mathbb{R}^d}h_l(x)\,dx=1. \tag{6}$$

We now combine Lemmas 7 and 9.

LEMMA 10.

$$E\sup_{\|g\|_\infty\le b}\Gamma(h_l*g)\le 2bn^{1/2}\eta^d(2^l r)^{-d/2}.$$

PROOF. Consider $t\in\mathbb{R}^d$, and

$$A_t=\left\{x\in[0,1]^d;\ h_l^t(x)\ne 0\right\}=\left\{x\in[0,1]^d;\ N(x-t)\le 2^l r\right\}.$$

Consider the indicator function 1_{A_t} of A_t. By Lemma 7, we have

$$E\left(\Gamma^2(1_{A_t})\right)\le 2n|A_t|\eta^{2d}.$$

Thus, setting $a_l=(2^l r)^{-d}$, we have

$$E\left(\Gamma\left(h_l^t\right)^2\right)\le 2a_l^2 n|A_t|\eta^{2d}$$

and

$$\int_{\mathbb{R}^d} E\left(\Gamma\left(h_l^t\right)^2\right) dt \le 2n\eta^{2d} a_l^2 \int_{\mathbb{R}^d} |A_t|\, dt$$
$$= 2n\eta^{2d} a_l^2 |B(0, 2^l r)|$$
$$= 2n\eta^{2d} a_l^2 a_l^{-1} = 2n\eta^{2d} a_l.$$

The result follows by Lemma 9. □

We now complete the proof of Lemma 5.

Consider a number q large enough that $2^q r \ge D$. For a function f on $\mathbb{R}$, we set

$$f_1 = f - f * h_1,$$
$$f_{q+1} = f * h_1 * \cdots * h_q.$$

For $2 \le l \le q$, we set $\xi_l = h_1 * \cdots * h_{l-1}$, and we set

$$f_l = (f - f * h_l) * \xi_l = (f - f * h_l) * \xi_{l-1} * h_{l-1}.$$

Thus $f = \sum_{l=1}^{q+1} f_l$. Consider the class

$$\mathscr{L}' = \{f\colon \mathbb{R}^d \to \mathbb{R};\ f \text{ Lipschitz},\ f(0) = 0, \|f\|_\infty \le D\}.$$

We have

$$E \sup_{f \in \mathscr{L}'} |\Gamma(f)| \le \sum_{l=1}^{q+1} E \sup_{f \in \mathscr{L}'} |\Gamma(f_l)|. \tag{7}$$

For $f \in \mathscr{L}'$, we have $\|f - f * h_l\|_\infty \le 2^l r$, since $\int h_l(x)\, dx = 1$ and $h_l(x) = 0$ for $N(x) \ge 2^l r$. It thus follows that if one sets $g_l = (f - f * h_l) * \xi_{l-1}$, then $\|g_l\|_\infty \le 2^l r$. Thus for $l \le q$, we have, by Lemma 10, and since $r = \eta n^{-1/d}$,

$$E \sup_{f \in \mathscr{L}'} |\Gamma(f_l)| \le 2^{l+1} n^{1/2} r \eta^d (2^{l-1} r)^{-d/2}$$
$$= \eta^{1+d/2} 2^{1+(l-1)(1-d/2)} n^{1-1/d}.$$

The same argument actually works for $l = q + 1$, since now $\|f\|_\infty \le D \le 2^q r$. Thus, from (7), we get, since $d \ge 3$,

$$E \sup_{f \in \mathscr{L}'} |\Gamma(f)| \le \eta^{1+d/2} n^{1-1/d} \left(\sum_{l \ge 0} 2^{2-l/2}\right)$$
$$\le 14 \eta^{1+d/2} n^{1-1/d}.$$

To finish, it suffices to show that

$$E \sup_{f \in \mathscr{L}'} |\Gamma(f)| = E \sup_{f \in \mathscr{L}} |\Gamma(f)|.$$

But this is the case since each function of $\mathscr{L}$ is the restriction to $[0, 1]^d$ of a function of $\mathscr{L}'$. □

Acknowledgment. This result was conjectured by W. Rhee, who found applications to the study of the Travelling Salesperson Problem in $\mathbb{R}^d$ [3].

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