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A NOTE ON SOME RATES OF CONVERGENCE IN FIRST-PASSAGE PERCOLATION[1]

BY KENNETH S. ALEXANDER

University of Southern California

A variation is given of the van den Berg–Kesten inequality on the probability of disjoint occurrence of events enabling it to apply to random variables, rather than just to events, associated with various subsets of an index set. This is used to establish superadditivity of a certain family of generating functions associated with first-passage percolation. This leads to improved estimates for the rates of convergence of the expected values of certain passage times.

1. Introduction. One of the leading groups of open questions in the subject of first-passage percolation involves the rate of convergence to the time constant for various normalized passage times. Such questions have two parts: How spread out is the normalized passage time around its expected value, and how close is the expected value to its limiting value (the time constant)? Recently Kesten (1993) was able to dramatically improve on the best previous results, which can be found in Kesten (1986). In this work we will show how Kesten's results on the spread of certain normalized passage times around their expected values, together with a new superadditivity relation, can be used to improve Kesten's rates of convergence of the expected normalized passage times to the time constant. For a_{0n} the passage time from the origin to $(n, 0, \dots, 0)$ and for μ the time constant, we will show that

$$n\mu \le Ea_{0n} \le n\mu + O(n^{1/2}\log n).$$

Our superadditivity relation is derived from a variant of the van den Berg–Kesten inequality on the disjoint occurrence of events, which may be of independent interest. Specifically, if S is a finite index set, Ω_b is a finite set of nonnegative integers for each $b \in S$, $\Omega = \prod_{b\in S}\Omega_b$, $\omega \in \Omega$ is a configuration on S, $C \subset \Omega$ is an event and $A \subset S$, we say that C *occurs on* A *in* ω if every ω' which agrees with ω on A is in C. In this way one obtains a two-valued random variable

$$f_C(A,\omega) := \begin{cases} 0, & \text{if } C \text{ occurs on } A \text{ in } \omega, \\ \infty, & \text{otherwise,} \end{cases} \tag{1.1}$$

which associates a value to each set A and configuration ω. One can define the

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event of disjoint occurrence, for $C, D \subset \Omega$:

$$C \,\square\, D := \{\omega \in \Omega\colon C \text{ occurs on } A \text{ and } D \text{ occurs on } B \text{ for some } A, B \subset S \text{ with } A \cap B = \phi\}.$$

The van den Berg–Kesten inequality (1985) is the statement that for increasing events C, D, under any product measure on Ω,

$$P(C \,\square\, D) \leq P(C)P(D). \tag{1.2}$$

This can be reformulated as follows. Let ω and ω' be independent configurations under some product measure on Ω, let C and D be increasing events, and define random variables

$$X := \min\{f_C(A, \omega) + f_D(B, \omega)\colon A, B \subset S \text{ with } A \cap B = \phi\},$$

$$X^* := \min\{f_C(A, \omega) + f_D(B, \omega')\colon A, B \subset S \text{ with } A \cap B = \phi\},$$

$$Y := \min\{f_C(A, \omega)\colon A \subset S\},$$

$$Z := \min\{f_D(B, \omega')\colon B \subset S\}.$$

Inequality (1.2) then states that

$$P(X = 0) \leq P(Y = Z = 0). \tag{1.3}$$

Since $P(Y = Z = 0) = P(Y + Z = 0)$, this is equivalent to

$$X \text{ is stochastically larger than } Y + Z. \tag{1.4}$$

The proof in van den Berg and Fiebig (1987) actually shows the stronger fact that

$$X \text{ is stochastically larger than } X^*; \tag{1.5}$$

clearly $X^* \geq Y + Z$.

Inequality (1.2) is extremely useful in studying percolation; usually S is a set of bonds or sites and $\Omega_b = \{0, 1\}$ corresponding to {vacant, occupied}. The formulations (1.4) and (1.5) extend naturally to first-passage percolation, where one deals implicitly with functions $f(A, \omega)$ which may take arbitrary values from $\mathbb{R}$, in contrast to the indicator function in (1.1). For example, if A is a lattice path, $f(A, \omega)$ could be the passage time along A. Our extension of (1.5) will require only minimal modification of the proof in van den Berg and Fiebig (1987) of a different extension of (1.2).

2. Definitions and statement of results. Let us first define the passage times of interest, with notation mostly as in Kesten (1986) and Smythe and Wieman (1978). Let there be a bond between each nearest-neighbor pair of sites in $\mathbb{Z}^d$; let $\mathscr{B}$ denote the set of all such bonds. Attached to each bond is a random passage time $\omega_b \geq 0$; the passage times $\{\omega_b\colon b \in \mathscr{B}\}$ are iid with d.f. F. Let ω denote the configuration $\{\omega_b\colon b \in \mathscr{B}\}$. For a given lattice path γ, the *passage time* $T(\gamma)$ is the sum of the passage times of the bonds comprising γ. (When we say "lattice path" we always implicitly mean a self-avoiding one.) If

A and B are subsets of, or points of $\mathbb{Z}^d$, and C is a subset of $\mathbb{R}^d$, then

$$T(A,B) := \inf\{T(\gamma): \gamma \text{ a lattice path from } A \text{ to } B\}$$

and, letting γ_0 denote γ with its endpoints deleted,

$$T(A,B;C) := \inf\{T(\gamma): \gamma \text{ a lattice path from } A \text{ to } B,\ \gamma_0 \subset C\}.$$

We denote points of $\mathbb{Z}^d$ as (n, j) with $n \in \mathbb{Z}$ and $j \in \mathbb{Z}^{d-1}$, and similarly for $\mathbb{R}^d$. Define hyperplanes

$$H_n := \{(n,j): j \in \mathbb{Z}^{d-1}\}$$

and slabs

$$Q_{mn} := \{(x,y) \in \mathbb{R}^d: m < x < n\}.$$

The passage times of interest are

$$a_{mn} := T((m,0),(n,0)),$$
$$b_{mn} := T((m,0), H_n).$$

Note $b_{mn} \le a_{mn}$. To ensure finite means for the passage times [see Kesten (1986)] we assume throughout that letting $b(1), \dots, b(2d)$ denote the bonds with an endpoint at 0,

$$E \min(\omega_{b(1)}, \dots, \omega_{b(2d)}) < \infty.$$

Now [see Smythe and Wierman (1978) for definitions and proofs]

(2.1) $$\{a_{mn}, 0 \le m \le n\} \text{ is a subadditive process}$$

and $\{Ea_{0n}, n \ge 0\}$ is subadditive, that is,

(2.2) $$Ea_{0,n+m} \le Ea_{0m} + Ea_{0n}.$$

These relations are useful in establishing [see Smythe and Wierman (1978)] that there is a time constant $\mu = \mu(F) \ge 0$ such that

$$\theta_{0n}/n \to \mu \quad \text{a.s. and in } L^1 \text{ for } \theta = a \text{ or } b.$$

For $\theta = a$ this is an example of Kingman's (1968) theorem, by (2.1). We will use the convention throughout that θ stands for either one of the passage times a or b. Kesten (1986) showed that

$$\mu(F) = 0 \quad \text{if and only if } F(0) \ge p_c(\mathbb{Z}^d),$$

where $p_c(\mathbb{Z}^d)$ denotes the critical probability for Bernoulli bond percolation on $\mathbb{Z}^d$. It follows by standard subadditivity arguments from (2.2) that

(2.3) $$Ea_{0n} \ge n\mu.$$

In fact, Kesten (1986) proved that if

(2.4) $$F(0) < p_c(\mathbb{Z}^d)$$

and

(2.5) $$\int e^{\lambda x}\, dF(x) < \infty \quad \text{for some } \lambda > 0,$$

then for some constants $C_i(F, d)$,

$$n\mu \le Ea_{0n} \le n\mu + O\big(n^{5/6}(\log n)^{1/3}\big), \tag{2.6}$$

$$Eb_{0n} \le n\mu + O\big(n^{5/6}(\log n)^{1/3}\big), \tag{2.7}$$

$$P\big[|\theta_{0n} - E\theta_{0n}| \ge xn^{1/2}\big] \le C_1 e^{-C_2 x} \quad \text{for } x \le C_3 n, \text{ for } \theta = a \text{ or } b. \tag{2.8}$$

[Inequality (2.6) is actually only stated, but a similar fact is proved.] Note the order-$n^{1/2}$ bound on the spread around the expectation is much smaller than the bound on the difference $E\theta_{0n} - n\mu$. Simulations [see Kesten (1993) for references] suggest that the spread around the expectation may actually be of order $n^{1/3}$. Here we will improve the upper bounds in (2.6) and (2.7) as follows.

THEOREM 2.1. *Under the hypotheses* (2.4) *and* (2.5),

$$E\theta_{0n} \le n\mu + O\big(n^{1/2}\log n\big) \quad \textit{for } \theta = a \textit{ or } b. \tag{2.9}$$

The proof will involve the generating functions

$$g_n(\beta) := -\log\Bigg(\sum_{j\in\mathbb{Z}^{d-1}} Ee^{-\beta T((0,0),(n,j))}\Bigg), \qquad \beta > 0. \tag{2.10}$$

Heuristically one expects $g_n(\beta)$ to behave like $-\log Ee^{-\beta b_{0n}}$; in fact

$$g_n(\beta) \le -\log Ee^{-\beta b_{0n}} \le \beta Eb_{0n}. \tag{2.11}$$

It is easily checked that for fixed β, $g_n(\beta)$ is positive for sufficiently large n; see the proof of Lemma 3.1. The key property of $g_n(\beta)$ is the following, which should be contrasted with (2.2) and which will follow from an extension of (1.5).

PROPOSITION 2.2. *$g_n(\beta)$ is superadditive, that is,*

$$g_{n+m}(\beta) \ge g_n(\beta) + g_m(\beta) \quad \textit{for all } m, n \ge 0 \textit{ and all } \beta > 0. \tag{2.12}$$

Consequently, for some constants $\nu_\beta \le \mu$,

$$\lim_n g_n(\beta)/n = \sup_n g_n(\beta)/n = \beta\nu_\beta \quad \textit{for each } \beta > 0. \tag{2.13}$$

The most important consequence of Proposition 2.2 is that $g_n(\beta)/\beta \le n\mu$, in contrast to (2.3); this is valid even if we choose β depending on n. We now have a deterministic quantity $g_n(\beta)/\beta$ closely related to b_{0n} which [cf. (2.6)] is on the opposite site of $n\mu$ from the expected value of at least one of the passage times of interest. Roughly speaking, this and (2.8) prevent b_{0n} from straying too far above $n\mu$ with any significant probability, much as (2.3) and (2.8) prevent a_{0n} from significant straying below $n\mu$.

Since $P[T((0,0),(n,0)) = 0] \ge F(0)^n$, we have $\nu_\beta \le -\log F(0)/\beta$. Thus ν_β can be strictly less than μ.

A subset C of a partially ordered set Ω is called *increasing* if $x \in C$ and $x \leq y$ imply $y \in C$, and *decreasing* if its complement is increasing. A probability measure Q on Ω is said to *have positive correlations* if $Q(C \cap D) \geq Q(C)Q(D)$ for all increasing C, D. Every probability measure on a totally ordered set has positive correlations, and Harris (1960) proved that every product measure on $\{0, 1\}^n$ has positive correlations.

We turn now to the extension of (1.5). The particular special case we need, (3.2), is contained in (4.13) of Kesten (1986), but to bring out the key ideas we present here an abstract formulation. The method in Kesten (1986) extends readily to handle this formulation, but we will give a short direct proof along the lines of van der Berg and Fiebig (1987).

Let $\mathscr{F}(S)$ denote the collection of all finite subsets of a countable set S.

THEOREM 2.3. *Let S be a countable set, let Ω_b be a partially ordered set for each $b \in S$, with σ-algebra induced by the ordering, and let Q_b be a probability measure on Ω_b which has positive correlations. Let $\Omega := \prod_{b \in S} \Omega_b$ be the configuration space, and let $Q := \prod_{b \in S} Q_b$. Let f, g: $\mathscr{F}(S) \times \Omega \to \mathbb{R} \cup \{\infty\}$ with $f(A, \cdot\,)$ and $g(B, \cdot\,)$ increasing for every $A, B \in \mathscr{F}(S)$. Assume $f(A, \omega)$ and $g(B, \omega)$ are determined by the restrictions of ω to A and to B, respectively. Let $\mathscr{S} \subset \mathscr{F}(S) \times \mathscr{F}(S)$ with $A \cap B = \phi$ for each $(A, B) \in \mathscr{S}$, let ω and ω' be two independent configurations with distribution Q, and define*

$$X := \inf\{ f(A, \omega) + g(B, \omega) \colon (A, B) \in \mathscr{S}\},$$
$$X^* := \inf\{ f(A, \omega) + g(B, \omega') \colon (A, B) \in \mathscr{S}\}.$$

Then X is stochastically larger than X^*.

For our purposes we only need the case $\Omega_b = \mathbb{R}$ for all b; we have given a general formulation to emphasize that only the positive-correlations property of every probability measure on $\mathbb{R}$ is used.

One can replace $\mathbb{R} \cup \{\infty\}$ with $\mathbb{R} \cup \{-\infty\}$ in Theorem 2.3; it is only necessary to avoid indeterminate expressions $\infty + (-\infty)$ in the definition of X and X^*. One can also replace "increasing" with "decreasing" and extend to more than two functions in an obvious way.

3. Proofs.

To motivate the formulation of Theorem 2.3 we will begin with the proof of Proposition 2.2 from it.

PROOF OF PROPOSITION 2.2. For $m, n \geq 0$ and $j, k \in \mathbb{Z}^{d-1}$, let $\tau(m, n, j, k)$ denote the shortest passage time among all paths from $(0, 0)$ to $(m + n, k)$ which first meet H_m at (m, j). Then

$$Ee^{-\beta T((0,0),(m+n,k))} \leq \sum_{j \in \mathbb{Z}^{d-1}} Ee^{-\beta\tau(m,n,j,k)}. \tag{3.1}$$

Let $\Phi(m, j)$ denote the set of all lattice paths from $(0, 0)$ to (m, j) which first meet H_m at (m, j), and let $\Psi((m, j), (m + n, k))$ denote the set of all lattice paths from (m, j) to $(m + n, k)$. For $A \in \mathscr{F}(S)$ and ω any configuration, define

$$f(A,\omega) := \begin{cases} T(A), & \text{if } A \in \Phi(m,j), \\ \infty, & \text{otherwise} \end{cases}$$

and

$$g(A,\omega) := \begin{cases} T(A), & \text{if } A \in \Psi((m,j),(m+n,k)), \\ \infty, & \text{otherwise.} \end{cases}$$

Now $\inf\{f(A,\cdot): A \in \mathscr{F}(S)\}$ is stochastically larger than $T((0,0),(m,j))$, and $\inf\{g(B,\cdot): B \in \mathscr{F}(S)\}$ has the distribution of $T((0,0),(n,k-j))$, so Theorem 2.3 [or (4.13) of Kesten (1986)] says that

$$\begin{aligned} &\tau(m,n,j,k) \text{ is stochastically larger than} \\ &\qquad T((0,0),(m,j)) + T((0,0),(n,k-j))', \end{aligned} \tag{3.2}$$

where the prime denotes that this is a copy of $T((0,0),(n,k-j))$ which is independent of $T((0,0),(m,j))$. Therefore using (3.1) and (3.2),

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^{d-1}} Ee^{-\beta T((0,0),(m+n,k))} \\ &\quad \le \sum_{k\in\mathbb{Z}^{d-1}} \sum_{j\in\mathbb{Z}^{d-1}} Ee^{-\beta T((0,0),(m,j))} Ee^{-\beta T((0,0),(n,k-j))} \\ &\quad = \Bigg(\sum_{j\in\mathbb{Z}^{d-1}} Ee^{-\beta T((0,0),(m,j))}\Bigg)\Bigg(\sum_{j\in\mathbb{Z}^{d-1}} Ee^{-\beta T((0,0),(n,j))}\Bigg), \end{aligned}$$

which becomes (2.12) when negative logs are taken. Equation (2.13) follows from standard subadditivity arguments. The inequality $\nu_\beta \le \mu$ follows from (2.11). □

Proof of Theorem 2.3. As mentioned above, this is quite similar to the "bond-splitting" proof in van den Berg and Fiebig (1987) which yields a different extension of (1.2). Because limits can be taken, we may assume S is finite.

Let ω and ω' be two independent configurations. For each $G \subset S$ define a new configuration ω^G by

$$\omega_b^G := \begin{cases} \omega_b, & \text{if } b \notin G, \\ \omega_b', & \text{if } b \in G. \end{cases}$$

Then define

$$X_G := \min\{f(A,\omega) + g(B,\omega^G) : (A,B) \in \mathscr{S}\}.$$

We wish to show that

$$X_\phi \text{ is stochastically larger than } X_S.$$

It is sufficient to show that for every $G \subset S$ and $a \notin G$,

$$X_G \text{ is stochastically larger than } X_{G\cup\{a\}}. \tag{3.3}$$

Fix $G \subset S$ and $a \notin G$, and for $s \in \Omega_a$ define a configuration $\omega^{G,a,s}$ by

$$\omega_b^{G,a,s} := \begin{cases} s, & \text{if } b = a, \\ \omega_b^G, & \text{if } b \neq a. \end{cases}$$

Let $\tilde{\omega}$ denote the restriction of ω to $S \setminus \{a\}$. For $s \in \Omega_a$ define

$$U(\tilde{\omega}, \tilde{\omega}', s) := \min\{f(A, \omega) + g(B, \omega^{G,a,s}) \colon (A, B) \in \mathscr{S}, b \in B\}$$

$$V(\tilde{\omega}, \tilde{\omega}', s) := \min\{f(A, \omega^{\phi,a,s}) + g(B, \omega^G) \colon (A, B) \in \mathscr{S}, b \in A\}$$

$$W(\tilde{\omega}, \tilde{\omega}') := \min\{f(A, \omega) + g(B, \omega^G) \colon (A, B) \in \mathscr{S}, b \notin A \cup B\}.$$

We may think of U as the best that can be achieved when g uses bond a and s is the value of bond a, V the similar best with f using bond a, and W the best with neither f nor g using bond a. Fix configurations $\tilde{\omega}$ and $\tilde{\omega}'$, and $x \in \mathbb{R}$. If $W(\tilde{\omega}, \tilde{\omega}') \leq x$, then

$$P\big[X_G > x | \tilde{\omega}, \tilde{\omega}'\big] = P\big[X_{G \cup \{a\}} > x | \tilde{\omega}, \tilde{\omega}'\big] = 0.$$

If $W(\tilde{\omega}, \tilde{\omega}') > x$, then

(3.4) $\quad X_G > x$ if and only if $U(\tilde{\omega}, \tilde{\omega}', \omega_a) > x$ and $V(\tilde{\omega}, \tilde{\omega}', \omega_a) > x$,

while

(3.5) $\quad X_{G \cup \{a\}} > x$ if and only if $U(\tilde{\omega}, \tilde{\omega}', \omega_a) > x$ and $V(\tilde{\omega}, \tilde{\omega}', \omega'_a) > x$.

Now the subsets

$$J := \{s \in \Omega_a \colon U(\tilde{\omega}, \tilde{\omega}', s) > x\} \quad \text{and} \quad K := \{s \in \Omega_a \colon V(\tilde{\omega}, \tilde{\omega}', s) > x\}$$

are increasing subsets of Ω_a, and Q_a has positive correlations, so by (3.4) and (3.5),

$$P\big[X_G > x | \tilde{\omega}, \tilde{\omega}'\big] = Q_b(J \cap K) \geq Q_b(J) Q_b(K) = P\big[X_{G \cup \{a\}} > x | \tilde{\omega}, \tilde{\omega}'\big].$$

Since $\tilde{\omega}$, $\tilde{\omega}'$ and x are arbitrary, (3.3) follows. □

LEMMA 3.1. *$\nu_\beta \to \mu$ as $\beta \to 0$.*

PROOF. Fix $0 < \varepsilon < \mu$. Let us use C_i, $i \geq 4$, to denote various constants which may depend on ε, F and d. From Theorem 5.20 of Kesten (1986),

(3.6) $$P[b_{0k} \leq k(\mu - \varepsilon)] \leq C_4 e^{-C_5 k} \quad \text{for all } k \geq 1.$$

If $\beta\mu < C_5$, then

$$Ee^{-\beta b_{0k}} \leq C_4 e^{-C_5 k} + e^{-\beta k(\mu - \varepsilon)} \leq C_6 e^{-\beta k(\mu - \varepsilon)} \quad \text{for all } k \geq 1.$$

Therefore for n sufficiently large (depending on β, ε, F and d),

$$\sum_{0 \leq k \leq n} \sum_{j \colon \max|j_i| = k} Ee^{-\beta T((0,0),(n,j))} \leq C_7 n^{d-1} Ee^{-\beta b_{0n}} \leq e^{-\beta n(\mu - 2\varepsilon)}/2$$

and

$$\sum_{k>n}\ \sum_{j:\,\max|j_i|=k} Ee^{-\beta T((0,0),(n,j))} \le \sum_{k>n} C_8 k^{d-2} Ee^{-\beta b_{0k}} \le e^{-\beta n(\mu-2\varepsilon)}/2. \tag{3.7}$$

Thus

$$g_n(\beta) \ge \beta n(\mu - 2\varepsilon)$$

and the lemma follows from (2.13), since ε is arbitrary. □

PROOF OF THEOREM 2.1. Again we use C_i, $i \ge 8$, to denote various constants which may depend on F and d. Fix $n \ge 1$ and for C_2 as in (2.8), set $\beta_n := C_2 n^{-1/2}$. We want to show that the large values of j contribute negligibly to the sum in (2.10), which is $e^{-g_n(\beta_n)}$. In fact, provided n is sufficiently large as in (3.7), we obtain using (2.8):

$$\begin{aligned}\sum_{k>8n}\ \sum_{j:\,\max|j_i|=k} Ee^{-\beta_n T((0,0),(n,j))} &\le \sum_{k>8n} C_8 k^{d-2} Ee^{-\beta_n b_{0k}} \\ &\le \sum_{k>8n} C_8 k^{d-2}\big(e^{-\beta_n k\mu/2} + P[b_{0k} < k\mu/2]\big) \\ &\le \sum_{k>8n} C_9 k^{d-2} e^{-\beta_n k\mu/2} \\ &\le e^{-2\beta_n n\mu}.\end{aligned} \tag{3.8}$$

Also, from integration by parts and (2.8),

$$\begin{aligned} Ee^{-\beta_n b_{0n}} &\le \int_0^\infty \beta_n e^{-\beta_n x} P[b_{0n} \le x]\, dx \\ &\le e^{-\beta_n Eb_{0n}} + \int_0^{Eb_{0n}} \beta_n e^{-\beta_n x} C_1 \exp\big(-C_2 n^{-1/2}(Eb_{0n} - x)\big)\, dx \\ &= (1 + C_1\beta_n Eb_{0n})e^{-\beta_n Eb_{0n}} \\ &\le C_{10} n e^{-\beta_n Eb_{0n}}.\end{aligned} \tag{3.9}$$

Note that our choice of β_n makes the second integrand in (3.9) a constant. From (3.9),

$$\begin{aligned}\sum_{0\le k\le 8n}\ \sum_{j:\,\max|j_i|=k} Ee^{-\beta_n T((0,0),(n,j))} &\le C_{11} n^{d-1} Ee^{-\beta_n b_{0n}} \\ &\le C_{12} n^d e^{-\beta_n Eb_{0n}}.\end{aligned}$$

Combining with (3.8),

$$e^{-g_n(\beta_n)} \le e^{-2\beta_n n\mu} + C_{12} n^d e^{-\beta_n Eb_{0n}}. \tag{3.10}$$

If n is large, then from (2.13) and Lemma 3.1,

$$e^{-\beta_n n\mu} \le e^{-g_n(\beta_n)} \le 1/2,$$

so that (3.10) yields

$$e^{-g_n(\beta_n)} \le C_{13} n^d e^{-\beta_n Eb_{0n}}.$$

Using (2.13) and Lemma 3.1 again, then

$$n\mu\beta_n \geq g_n(\beta_n) \geq \beta_n Eb_{0n} - C_{14} \log n$$

and (2.9) follows for $\theta = b$.

Turning to $\theta = a$, from (2.8) we obtain for some C_{15},

$$\begin{aligned} 1/2 &\leq P\left[b_{0n} \leq Eb_{0n} + C_{15}n^{1/2}\right] \\ &\leq \sum_{j \in \mathbb{Z}^{d-1}} P\left[T((0,0),(n,j);Q_{-\infty,n}) \leq Eb_{0n} + C_{15}n^{1/2}\right]. \end{aligned} \tag{3.11}$$

For j with $\max|j_i| = k > 2n$ we have, using Theorem 5.20 of Kesten (1986) [cf. (3.6)]:

$$P\left[T((0,0),(n,j);Q_{-\infty,n}) \leq Eb_{0n} + C_{15}n^{1/2}\right] \leq P[b_{0k} \leq 3k\mu/4] \leq C_{16}e^{-C_{17}k}$$

so that for large n,

$$\sum_{j:\,\max|j_i| > 2n} P\left[T((0,0),(n,j);Q_{-\infty,n}) \leq Eb_{0n} + C_{15}n^{1/2}\right] \leq 1/4.$$

Hence the sum over the $(4n+1)^{d-1}$ values j with $\max|j_i| \leq 2n$ in (3.11) must exceed 1/4, which means there exists a j with

$$\left(4(4n+1)^{d-1}\right)^{-1} \leq P\left[T((0,0),(n,j);Q_{-\infty,n}) \leq Eb_{0n} + C_{15}n^{1/2}\right].$$

Since every path from $(0,0)$ to (n,j) in $Q_{-\infty,n}$ can be reflected through H_n to obtain a path from (n,j) to $(2n,0)$, this implies

$$\begin{aligned} &\left(4(4n+1)^{d-1}\right)^{-2} \\ &\quad \leq P\left[T((0,0),(2n,0) \leq 2\left(Eb_{0n} + C_{15}n^{1/2}\right)\right]. \end{aligned} \tag{3.12}$$

But for some C_{18}, by (2.8),

$$P\left[T((0,0),(2n,0) \leq Ea_{0,2n} - C_{18}n^{1/2} \log n\right] < \left(4(4n+1)^{d-1}\right)^{-2},$$

which with (3.12) yields

$$Ea_{0,2n} \leq 2Eb_{0n} + O(n^{1/2} \log n),$$

which proves (2.9) for $\theta = a$. □

It follows from this proof that if one could replace 1/2 with a smaller exponent in (2.8), the same exponent would be valid in (2.9).

REFERENCES

HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** 13–20.

KESTEN, H. (1986). Aspects of first-passage percolation. *Ecole d'Eté de Probabilités de Saint Flour XIV. Lecture Notes in Math.* **1180** 125–264. Springer, New York.

KESTEN, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* **3**.

KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.

SMYTHE, R. T. and WIERMAN, J. C. (1978). *First-Passage Percolation on the Square Lattice. Lecture Notes in Math.* **671**. Springer, New York.

VAN DEN BERG, J. and FIEBIG, U. (1987). On a combinatorial conjecture concerning disjoint occurrences of events. *Ann. Probab.* **15** 354–374.

VAN DEN BERG, J. and KESTEN, H. (1985). Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22** 556–569.

DEPARTMENT OF MATHEMATICS, DRB 155
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA 90089-1113