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OPTIMAL CONTROL AND REPLACEMENT WITH STATE-DEPENDENT FAILURE RATE: AN INVARIANT MEASURE APPROACH[1]

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Stochastic control problems in which the payoff depends on the running maximum of a diffusion process are considered. Such processes provide appealing models for physical processes that evolve in a *continuous* and *increasing* manner and fail at a random time. The controller must make two decisions: first, she must choose how fast to work (this decision determines the rate of profit as well as the proximity of failure), and second, she must decide when to replace a deteriorated system with a new one. Preventive replacement becomes an important option when the cost for replacement after a failure is larger than the cost of a preventive replacement. Single-cycle and long-term average criteria are used to evaluate the control and replacement decisions.

We model the process via a martingale problem formulation. This enables the long-term average control problem to be rephrased as an LP over the invariant measures of the process. We identify the invariant measures corresponding to each control and replacement decision and determine the optimal solution using an iterative scheme.

1. Introduction. This paper is concerned with the following optimal control problem: A process increases randomly in time until it fails, at which time the process is renewed and the cycle repeats. Failure is also random and its rate depends on the level of the process. A profit is earned while the process is running, but a cost is incurred when the process is renewed. The controller has influence on the system in two ways. First, she decides the drift rate of the process; higher drift rates improve the profit rate but also increase the rate of failure. Second, she can decide to renew the process before failure at a cost that is less than the cost incurred at failure. Her task is to obtain the maximum long-term profit rate.

Intimately related to this long-term average control problem is a single-cycle control problem in which the controller seeks to maximize the total profit obtained during the first cycle. In the companion paper Heinricher and Stockbridge (1993), the *single-cycle* problem is studied in depth using dynamic programming, and the relationship between the single-cycle and long-term average problems is briefly discussed. An important aspect of the paper is that failure occurs according to a *nonstandard* state-dependent rate and

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only when the process increases. This localizes failure to a subregion of the state-space.

This paper concentrates on the *long-term average* problem using an invariant measure approach. The process is modelled via a constrained martingale problem developed by Kurtz (1990); the nonstandard failure mechanism is easily represented by a constraining operator on the subregion in which failure may occur. The problem is then reformulated as a linear program over the space of invariant measures of the controlled process. The key to this approach is the characterization and identification of the invariant measures through the basic adjoint relation (7). The paper also demonstrates how the single-cycle problem can be solved from knowledge of the invariant measures.

Invariant measure methods have some advantages over dynamic programming. Modelling the dynamics of the process via a martingale problem allows milder conditions on the coefficients of the problem in order to establish existence of solutions. It is also not necessary to require the control processes to be renewed along with the state process. In comparing this approach with the standard method of passing to the limit with discounted problems, the present approach has the advantage that the long-term average problem can be studied directly in terms of the invariant measures of the process. One present disadvantage is that only *existence* of the optimal controls is determined; they are not obtained directly in feedback form as with dynamic programming. (This difficulty will be overcome in the near future in joint work with T. Kurtz.)

This linear programming approach has been studied extensively in discrete settings, for example, Manne (1960), Derman (1962), Wolfe and Dantzig (1962), Denardo (1970) and Pittel (1971), and has been extended to continuous time with general state and control spaces by Stockbridge (1990) and Heinricher and Stockbridge (1992). A similar convex programming approach has recently been used on finite-horizon problems by Fleming and Vermes (1989); the measures involved are not the invariant measures of the process.

The paper is organized as follows. Section 2 formulates the model and the objective functions considered throughout the remainder of the paper. Section 3 then reformulates the long-term average criterion as an LP over the invariant measures of the process and in Section 4, these measures are identified, parametrized by the control and replacement policies. Section 5 shows how the single-cycle problem can be solved using the invariant measures. This is used in Section 6 to obtain the solution to the long-term average problem. One limitation to the solution in Section 6 is that it is given in terms of the optimal long-term average value λ^*. This limitation is overcome in Section 7 using a λ-maximization technique [extending a method in Taylor (1975) and Aven and Bergman (1986)] and an illustrative example is presented.

2. Formulation of the model. We begin by defining the notation used throughout the paper. For this purpose, S represents various subsets of $\mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{Z}^2$ (for several values of d) that will be identified when used. $\hat{C}(S)$

denotes the space of continuous functions on S vanishing at infinity, $C_c(S)$ denotes the space of continuous functions on S having compact support, $C^{2,1}(S)$ denotes the space of functions that are twice continuously differentiable in the first argument and continuously differentiable in the second argument (here, $S \subset \mathbb{R}^2$), $D_S[0,\infty)$ denotes the space of functions from $[0,\infty)$ into S which are right continuous and have left limits, $\mathscr{P}(S)$ denotes the space of probability measures on S, $\mathscr{M}(S)$ denotes the space of finite measures on S and $\mathscr{B}(S)$ denotes the collection of Borel sets of S. For measures μ, μ_n on S, $\mu_n \Rightarrow \mu$ means $\int f d\mu_n \to \int f d\mu$ for all bounded, continuous functions f.

2.1. *Dynamics of the system.* The dynamics of the process are formulated as solutions of Kurtz's constrained martingale problem [Kurtz (1990)] which identifies the behaviour of the process in different regions of the plane. We define the process for all times $t \geq 0$ in order to study the long-term average reward; the single-cycle reward is obtained by considering the process up to the time of the first jump (renewal).

In this application, we choose a level Δ at which a *planned* replacement will automatically occur once the process reaches it but *unplanned replacement* due to failure may occur at any level $y < \Delta$. So define $E_0 = \{(x, y)$): $-\infty \leq x < y,\ 0 < y < \Delta\}$ as the region in which the running max is constant while the diffusion operates (there is no possibility of failure), $E_1 = \{(x, y)$: $x = y,\ 0 \leq y < \Delta\}$ as the region in which the diffusion operates, the running max increases and failure occurs according to a state-dependent rate $k(y)$, and $E_2 = \{(x, y)$: $x = \Delta,\ y = \Delta\}$ as the region in which the process instantaneously jumps to the origin (a preventive replacement occurs). Let $E = E_0 \cup E_1 \cup E_2$. In addition, we append to the process components (n and r below) which count the number of failures and the number of preventive replacements; let $N = \{0, 1, 2, \dots\}$.

It is important to observe that we are restricting the model to require failure *only* when the diffusion is at its running maximum. This is a nonstandard failure mechanism which nevertheless makes sense; failure should be a result of increased "wear."

Let $\mathscr{D}_1 = \{\phi \in \hat{C}(E \times N^2))$: $\phi(\cdot,\cdot,n,r) \in C^{2,1}(E)\ \forall\ n, r\}$ and define the following operators on $\mathscr{D}_1$ in each region:

$$\begin{aligned}
A\phi(x,y,n,r,u) &= f(x,y,u)\phi_x(x,y,n,r) \\
&\quad + \tfrac{1}{2}\sigma^2(x,y,u)\phi_{xx}(x,y,n,r) \qquad \forall\,(x,y,n,r) \in E_0 \times N^2, \\
B_1\phi(x,y,n,r) &= \phi_y(x,y,n,r) \\
&\quad + k(y)[\phi(0,0,n+1,r) - \phi(x,y,n,r)] \\
&\qquad\qquad\qquad\qquad \forall\,(x,y,n,r) \in E_1 \times N^2, \\
B_2\phi(x,y,n,r) &= \phi(0,0,n,r+1) \\
&\quad - \phi(x,y,n,r) \qquad \forall\,(x,y,n,r) \in E_2 \times N^2.
\end{aligned}$$

We assume the controls take values in a compact set $U \subset \mathbb{R}$ and that the coefficients satisfy the following conditions:

CONDITION 1. f and σ are bounded, continuous functions on $E \times U$.

CONDITION 2. There exists a constant α such that

$$0 < \alpha \leq f(x, y, u), \sigma(x, y, u) \quad \forall\, (x, y, u) \in E \times U.$$

CONDITION 3. k is nonnegative, bounded, continuous with $\int_0^\infty k(y)\,dy = +\infty$.

DEFINITION 2.1. An $E \times N^2 \times U$-valued process $\{(x_t, y_t, n_t, r_t, u_t)\}_{t \geq 0}$ taking values in $D_{E \times N^2 \times U}[0, \infty)$ is a solution of the (controlled) constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$ if there exist a filtration $\{\mathscr{F}_t\}$ and nondecreasing processes λ_1 and λ_2 such that:

(a) $\{(x_t, y_t, n_t, r_t, u_t)\}_{t \geq 0}$ is $\{\mathscr{F}_t\}$-progressively measurable.
(b) For $i = 1, 2$, $\int_0^t \chi_{\overline{E}_i}(x_s, y_s))\,d\lambda_i(s) = \lambda_i(t)$.
(c) λ_2 is a counting process.
(d) For each $\phi \in \mathscr{D}_1$,

$$\begin{aligned} &\phi(x_t, y_t, n_t, r_t) - \int_0^t A\phi(x_s, y_s, n_s, r_s, u_s)\,ds \\ &\qquad - \int_0^t B_1\phi(x_{s-}, y_{s-}, n_{s-}, r_{s-})\,d\lambda_1(s) \\ &\qquad - \int_0^t B_2\phi(x_{s-}, y_{s-}, n_{s-}, r_{s-})\,d\lambda_2(s) \end{aligned} \tag{1}$$

is an $\{\mathscr{F}_t\}$-martingale.

A U-valued process $\{u_t\}_{t \geq 0}$ is said to be *admissible* if there exists a solution of the constrained martingale problem in which $\{u_t\}_{t \geq 0}$ is the control component.

REMARK 2.2. Existence of a solution to the constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$ is established by Kurtz [(1990), Section 2] when the control process is constant, $u_t \equiv u$, $t \geq 0$. Theorem 3.1 provides existence of solutions for a larger collection of control processes.

REMARK 2.3. Choosing $\phi(x, y, n, r) = n$ and approximating by elements of $\mathscr{D}_1$, it follows from (1) that $n_t - \int_0^t k(y_s)\,d\lambda_1(s)$ is a local martingale. Similarly, choosing $\phi(x, y, n; r) = r$ implies that $r_t - \lambda_2(t)$ is a local martingale.

2.2. *Decision criteria.* In this paper, we consider two decision criteria: the long-term average reward obtained subject to replacement costs; and a single-cycle criterion in which the controller seeks to maximize the total expected reward until the process fails or a preventive replacement occurs (excluding the cost for replacement).

2.2.1. LONG-TERM AVERAGE CRITERION. Let h be bounded and continuous on $E \times U$ and let R_1 denote the cost for *replacement after failure* and R_2 denote the cost for a *preventive replacement*. We assume $0 \le R_2 \le R_1$. The long-term objective is to maximize

$$\liminf_{t\to\infty} \frac{1}{t} E\left[\int_0^t h(x_s, y_s, u_s)\, ds - R_1 n_t - R_2 r_t\right] \tag{2}$$

over the admissible controls $\{u_t\}_{t\ge 0}$ and preventive replacement levels Δ. In light of Remark (2.3), it is clear that the objective can be written as

$$\liminf_{t\to\infty} \frac{1}{t} E\left[\int_0^t h(x_s, y_s, u_s)\, ds - R_1 \int_0^t k(y_s)\, d\lambda_1(s) - R_2 \lambda_2(t)\right]. \tag{3}$$

The benefit of this representation of the objective function is that the counting processes $\{n_t\}_{t\ge 0}$ and $\{r_t\}_{t\ge 0}$ of the number of replacements do not enter the computation of the value; the only components of the process involved are $\{(x_t, y_t, u_t)\}_{t\ge 0}$ and the "local time processes" λ_1 and λ_2.

2.2.2. SINGLE-CYCLE CRITERION. Define

$$\zeta \wedge \tau(\Delta) = \inf\{t > 0: (x_t, y_t) = (0,0)\}. \tag{4}$$

Since the first renewal of the process can result from a failure or a preventive decision (the process reaches level Δ before it fails), $\zeta \wedge \tau(\Delta)$ represents the minimum of the first time of failure and the time the process reaches level Δ. Let h be as above. The single-cycle objective is to maximize

$$E\left[\int_0^{\zeta\wedge\tau(\Delta)} h(x_t, y_t, u_t)\, dt\right] \tag{5}$$

over the collection of admissible control processes $\{u_t\}_{t\ge 0}$ and replacement levels Δ.

2.3. *Dynamics revisited.* Since both the long-term average value and the single-cycle value of a solution $\{(x_t, y_t, n_t, r_t, u_t)\}_{t\ge 0}$ of the constrained martingale problem can be obtained from the $\{(x_t, y_t, u_t)\}_{t\ge 0}$ process and the "local time" processes λ_1 and λ_2, it is advantageous to represent the dynamics of the process so that the processes $\{n_t\}_{t\ge 0}$ and $\{r_t\}_{t\ge 0}$ are not involved. To do this, restrict the domain $\mathscr{D}_1$ to the subset $\mathscr{D} = \hat{C}^{2,1}(E)$; that is, to those elements of $\mathscr{D}_1$ which are functions of *only* x and y. Then the operators take the form:

$$A\phi(x,y,u) = f(x,y,u)\phi_x(x,y) + \tfrac{1}{2}\sigma^2(x,y,u)\phi_{xx}(x,y) \quad \forall (x,y) \in E_0,$$
$$B_1\phi(x,y) = \phi_y(x,y) + k(y)[\phi(0,0) - \phi(x,y)] \quad \forall (x,y) \in E_1,$$
$$B_2\phi(x,y) = \phi(0,0) - \phi(x,y) \quad \forall (x,y) \in E_2.$$

Throughout the remainder of this paper, we make this restriction on the domain and refer to the process $\{(x_t, y_t, u_t)\}_{t \geq 0}$ *as a solution of the constrained martingale problem.* [In fact, it is possible to start by defining the martingale problem on the domain $\mathscr{D}$ and extend it to $\mathscr{D}_1$. We have adopted the present approach since it provides an intuitive explanation of the objective function (3).]

It will be necessary to compactify the space of control processes in order to characterize the invariant measures for the solutions of the constrained martingale problem and to identify optimal solutions. We therefore define a *relaxed* solution to the constrained martingale problem as follows:

DEFINITION 2.4. An $E \times \mathscr{P}(U)$-valued process $\{(x_t, y_t, \pi_t)\}_{t \geq 0}$ taking values in $D_{E \times \mathscr{P}(U)}[0,\infty)$ is a relaxed solution of the (controlled) constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$ if there exist a filtration $\{\mathscr{F}_t\}$ and nondecreasing processes λ_1 and λ_2 such that:

(a) $\{(x_t, y_t, \pi_t)\}_{t \geq 0}$ is $\{\mathscr{F}_t\}$-progressively measurable.
(b) For $i = 1, 2$, $\int_0^t \chi_{\overline{E}_i}(x_s, y_s))\, d\lambda_i(s) = \lambda_i(t)$.
(c) λ_2 is a counting process on E_2.
(d) For each $\phi \in \mathscr{D}$,

$$\phi(x_t, y_t) - \int_0^t \int_U A\phi(x_s, y_s, u)\pi_s(du)\, ds - \int_0^t B_1\phi(x_{s-}, y_{s-})\, d\lambda_1(s)$$
$$- \int_0^t B_2\phi(x_{s-}, y_{s-})\, d\lambda_2(s)$$

is an $\{\mathscr{F}_t\}$-martingale.

3. LP reformulation of the long-term average criterion. This section shows how to compute the long-term average value in terms of the invariant measures of the process, and identifies the basic adjoint relationship which characterizes these invariant measures. We *fix* the replacement level $\Delta < \infty$ throughout this section.

3.1. *Characterization of invariant measures.* We are interested in the limits of occupation measures of the controlled process as $t \to \infty$ so we restrict attention to a collection of times which are bounded away from 0.

Let $\{(x_t, y_t, u_t)\}_{t \geq 0}$ be a solution of the constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$. Define the occupation measures $\{\mu_t^0\}_{t \geq 1}$ on $E \times U$ by

$$\mu_t^0(\Gamma) = \frac{1}{t} E\left[\int_0^t \chi_\Gamma(x_s, y_s, u_s)\right] ds \qquad \forall\, \Gamma \in \mathscr{B}(E \times U).$$

Note that for each t, μ_t^0 is a probability measure. The collection of measures $\{\mu_t^0\}_{t \geq 1}$ is tight since $E \times U$ is compact and is, therefore, relatively compact. Thus there exist weak limits, as $t \to \infty$, which are probability measures on $E \times U$.

In a similar manner, define the occupation measures $\{\mu_t^1\}_{t \geq 1}$ and $\{\mu_t^2\}_{t \geq 1}$ on $\overline{E}_1$ and E_2, respectively, by

$$\mu_t^1(H) = \frac{1}{t} E\left[\int_0^t \chi_H(x_{s-}, y_{s-})\, d\lambda_1(s)\right] \qquad \forall\, H \in \mathscr{B}(\overline{E}_1),$$

$$\mu_t^2(\{(\Delta, \Delta)\}) = \frac{1}{t} E\left[\int_0^t \chi_{\{(\Delta,\Delta)\}}(x_{s-}, y_{s-})\, d\lambda_2(s)\right].$$

Since $\lambda_2(t)$ is a counting process for the number of visits to (Δ, Δ), μ_t^2 has mass of size $E[\lambda_2(t)]/t$. These masses have a finite upper bound, say L_1, since the drift f and the diffusion σ are bounded above and hence the expected passage time from $(0,0)$ to (Δ, Δ) is bounded away from 0. Therefore, there exist weak limits of $\{\mu_t^2\}_{t \geq 1}$ as $t \to \infty$.

We now show the existence of weak limits of $\{\mu_t^1\}_{t \geq 1}$. Define

$$(6) \quad \theta(x) = \begin{cases} 0, & x \leq -2, \\ 1 + 10(x+1)^3 + 15(x+1)^4 + 6(x+1)^5, & -2 \leq x \leq -1, \\ 1, & -1 \leq x \end{cases}$$

and $\psi(y) = \int_0^y \exp\{\int_z^y k(v)\, dv\}\, dz$. Let $\phi(x, y) = \theta(x)\psi(y)$, $(x, y) \in E$, and note that $\phi \in \mathscr{D}$. Also note that $0 \leq \phi(x, y) \leq \phi(\Delta, \Delta) < \infty$ for $(x, y) \in E$, and $\phi(x, y) = \psi(y)$ for $x \geq -1$; in particular, $\phi(y, y) = \psi(y)$ for all $y \in [0, \Delta]$. It now follows that on E,

$$B_1\phi(y, y) = 1,$$
$$B_2\phi(\Delta, \Delta) = -\phi(\Delta, \Delta)$$

and

$$A\phi(x, y, u) = \begin{cases} 0, \quad x \leq -2,\ x \geq -1, \\ \psi(y)\big[\big(30(x+1) + 90(x+1)^2 + 60(x+1)^3\big)\sigma^2(x, y, u) \\ \quad + \big(30(x+1)^2 + 60(x+1)^3 + 30(x+1)^4\big) f(x, y, u)\big], \\ \hfill -2 \leq x \leq -1 \end{cases}$$

and that $A\phi \geq -3L_2\psi(\Delta)$ on E, where $\sigma^2 \leq L_2$. Thus using this ϕ and the fact that $\{(x_t, y_t, u_t)\}_{t \geq 0}$ is a solution of the constrained martingale problem, we have

$$E[\phi(x_0, y_0)] = E\Big[\phi(x_t, y_t) - \int_0^t A\phi(x_s, y_s, u_s)\,ds$$
$$- \int_0^t B_1\phi(x_{s-}, y_{s-})\,d\lambda_1(s) - \int_0^t B_2\phi(x_{s-}, y_{s-})\,d\lambda_2(s)\Big].$$

The above observations imply

$$\begin{aligned} E[\lambda_1(t)] &= E[\phi(x_t, y_t)] - E[\phi(x_0, y_0)] \\ &\quad - E\Big[\int_0^t A\phi(x_s, y_s, u_s)\,ds\Big] + \phi(\Delta, \Delta)E[\lambda_2(t)] \\ &\le \phi(\Delta, \Delta)(1 + 3L_2 t + E[\lambda_2(t)]). \end{aligned}$$

It then follows that, for $t \ge 1$, $\mu_t^1(\overline{E}_1) \le \phi(\Delta, \Delta)[1 + 3L_2 + L_1]$ and that $\{\mu_t^1\}_{t \ge 1}$ is relatively compact.

Let $\{t_k\}$ be a sequence, $t_k \to \infty$, such that there exist measures μ^0, μ^1 and μ^2 with $\mu_{t_k}^0 \Rightarrow \mu^0$, $\mu_{t_k}^1 \Rightarrow \mu^1$, and $\mu_{t_k}^2 \Rightarrow \mu^2$ as $k \to \infty$. Then since $\{(x_t, y_t, u_t)\}_{t \ge 0}$ is a solution of the constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$, for each $\phi \in \mathscr{D}$,

$$\begin{aligned} 0 &= \frac{1}{t_k}E\Big[\phi(x_{t_k}, y_{t_k}) - \phi(x_0, y_0) - \int_0^{t_k} A\phi(x_s, y_s, u_s)\,ds \\ &\qquad - \int_0^{t_k} B_1\phi(x_s, y_s)\,d\lambda_1(s) - \int_0^{t_k} B_2\phi(x_s, y_s)\,d\lambda_2(s)\Big] \\ &= \frac{1}{t_k}E[\phi(x_{t_k}, y_{t_k}) - \phi(x_0, y_0)] - \int A\phi(x, y, u)\,d\mu_{t_k}^0 \\ &\qquad - \int B_1\phi(x, y)\,d\mu_{t_k}^1 - \int B_2\phi(x, y)\,d\mu_{t_k}^2. \end{aligned}$$

Letting $k \to \infty$, we obtain

$$\int A\phi\,d\mu^0 + \int B_1\phi\,d\mu^1 + \int B_2\phi\,d\mu^2 = 0 \qquad \forall\,\phi \in \mathscr{D}. \tag{7}$$

The relation (7) is a necessary condition which the invariant distributions of solutions must satisfy. The following theorem of Kurtz shows that (7) is sufficient as well.

THEOREM 3.1. *Suppose $\mu^0 \in \mathscr{P}(E \times U)$, $\mu^1 \in \mathscr{M}(\overline{E}_1)$ and $\mu^2 \in \mathscr{M}(E_2)$ satisfy (7). Then there exists a stationary relaxed solution $\{(x_t, y_t, \pi_t)\}_{t \ge 0}$ of the constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$ satisfying*

$$E[\chi_{\Gamma_1}(x_t, y_t)\pi_t(\Gamma_2)] = \mu^0(\Gamma_1 \times \Gamma_2)$$

for all $\Gamma_1 \in \mathscr{B}(E)$ and $\Gamma_2 \in \mathscr{B}(U)$.

PROOF. The proof is essentially an application of Theorem 4.1 of Kurtz (1991). □

3.2. *Long-term average value computation.* Again let $\{(x_t, y_t, u_t)\}_{t \geq 0}$ be a solution of the constrained martingale problem for $(A, E_0, B_1, E_1, B_2, E_2)$ and let $\{t_k\}$ be a sequence, $t_k \to \infty$, such that

$$\lim_{k \to \infty} \frac{1}{t_k} E\left[\int_0^{t_k} h(x_s, y_s, u_s)\, ds - R_1 \int_0^{t_k} k(y_s)\, d\lambda_1(s) - R_2 \lambda_2(t_k)\right]$$
$$= \liminf_{t \to \infty} \frac{1}{t} E\left[\int_0^{t} h(x_s, y_s, u_s)\, ds - R_1 \int_0^{t} k(y_s)\, d\lambda_1(s) - R_2 \lambda_2(t)\right].$$

For some subsequence of $\{t_k\}$ (without loss of generality, say the entire sequence), there exist measures μ^0, μ^1 and μ^2 with $\mu^0_{t_k} \Rightarrow \mu^0$, $\mu^1_{t_k} \Rightarrow \mu^1$ and $\mu^2_{t_k} \Rightarrow \mu^2$ as $k \to \infty$. Then the long-term average value (3) for the process $\{(x_t, y_t, u_t)\}_{t \geq 0}$ equals

$$\int_{E \times U} h(x, y, u)\, d\mu^0(x, y, u) - R_1 \int_{\overline{E}_1} k(y)\, d\mu^1(y) - R_2\, \mu^2(\Delta, \Delta).$$

3.3. *LP reformulation*: *Long-term average problem.* Combining the characterization of the invariant measures with the computation of the long-term average value, we obtain the following LP problem:

Maximize

$$\int_{E \times U} h(x, y, u)\, d\mu^0(x, y, u) - R_1 \int_{\overline{E}_1} k(y)\, d\mu^1(y) - R_2\, \mu^2(\Delta, \Delta)$$

subject to

$$\int_{E \times U} A\phi(x, y, u)\, d\mu^0(x, y, u)$$
$$+ \int_{\overline{E}_1} B_1 \phi(y, y)\, d\mu^1(y) + \int_{E_2} B_2 \phi(y, y)\, d\mu^2(y) = 0 \qquad \forall\, \phi \in \mathscr{D}$$

and

$$\mu^0 \in \mathscr{P}(E \times U), \qquad \mu^1 \in \mathscr{M}(\overline{E}_1), \qquad \mu^2 \in \mathscr{M}(E_2).$$

4. Identification of the invariant measures μ^0, μ^1 and μ^2. The previous section shows that the invariant measures for the constrained martingale problem are characterized by the basic adjoint relation (7). It is obvious that different choices of controls and replacement levels will produce different invariant measures. In this section, we identify the measures μ^0, μ^1 and μ^2 (parametrized by the control and replacement level) which satisfy (7).

Again, let Δ be a fixed replacement level, $0 < \Delta < \infty$, and μ^0, μ^1 and μ^2 satisfy (7). Define $\mu^0_E \in \mathscr{P}(E)$ to be the state marginal of μ^0,

(8a) $$\mu^0_E(G) = \mu^0(G \times U) \qquad \forall\, G \in \mathscr{B}(E)$$

and $\nu: E \times \mathscr{B}(U) \to [0,1]$ to be the regular conditional distribution of u given x and y; that is, ν satisfies

$$\text{(8b)} \quad \mu^0(\Gamma) = \int_E \int_U \chi_\Gamma(x,y,u)\nu(x,y,du)\mu_E^0(dx,dy) \qquad \forall\, \Gamma \in \mathscr{B}(E \times U).$$

Note that $\nu(x,y,\cdot) \in \mathscr{P}(U)$ for μ_E^0-almost every (x,y). Intuitively, ν acts as a *relaxed control* by determining the distribution on the control space U corresponding to each state (x,y). (This intuitive statement is not rigorously established by Theorem 3.1. However, it will be proven in forthcoming work with Kurtz.)

The measures μ^0, μ^1 and μ^2 will be determined in terms of the conditional distribution ν on the control space U and the replacement level Δ.

To determine the invariant measures, we begin formally. Suppose that μ_E^0 has a density $p(x,y)$ and μ^1 has a density $\hat{p}(y)$. Define the "averaged" coefficients

$$\tilde{f}(x,y) = \int_U f(x,y,u)\nu(x,y,du) \text{ and } \tilde{\sigma}^2(x,y) = \int_U \sigma^2(x,y,u)\nu(x,y,du).$$

Then for each $\phi \in \mathscr{D}$, (7) can be written as

$$\begin{aligned}
0 &= \int_0^\Delta \int_{-\infty}^y \int_U \left[f(x,y,u)\phi_x(x,y) + \frac{1}{2}\sigma^2(x,y,u)\phi_{xx}(x,y) \right] \\
&\quad \times \nu(x,y,du)p(x,y)\,dx\,dy \\
&\quad + \int_0^\Delta [\phi_y(y,y) + k(y)(\phi(0,0) - \phi(y,y))]\hat{p}(y)\,dy \\
&\quad + [\phi(0,0) - \phi(\Delta,\Delta)]\mu^2(\Delta,\Delta) \\
&= \int_0^\Delta \int_{-\infty}^y \left[\tilde{f}(x,y)\phi_x(x,y) + \frac{1}{2}\hat{\sigma}^2(x,y)\phi_{xx}(x,y) \right] p(x,y)\,dx\,dy \\
&\quad + \int_0^\Delta [\phi_y(y,y) + k(y)(\phi(0,0) - \phi(y,y))]\hat{p}(y)\,dy \\
&\quad + [\phi(0,0) - \phi(\Delta,\Delta)]\mu^2(\Delta,\Delta).
\end{aligned}$$

Integrating by parts shows that this expression

$$\begin{aligned}
&= \int_0^\Delta \int_{-\infty}^y \left[\tilde{f}(x,y)p(x,y) - \frac{1}{2}\frac{\partial}{\partial x}(\tilde{\sigma}^2(x,y)p(x,y)) \right] \phi_x(x,y)\,dx\,dy \\
&\quad + \int_0^\Delta \frac{1}{2}\tilde{\sigma}^2(y,y)p(y,y)\phi_x(y,y)\,dy \\
&\quad + \int_0^\Delta [\phi_y(y,y) + k(y)(\phi(0,0) - \phi(y,y))]\hat{p}(y)\,dy \\
&\quad + [\phi(0,0) - \phi(\Delta,\Delta)]\mu^2(\Delta,\Delta).
\end{aligned}$$

Adding and subtracting $\phi_y(y, y)$ in the appropriate place gives

$$= \int_0^\Delta \int_{-\infty}^y \left[\tilde{f}(x,y)p(x,y) - \frac{1}{2}\frac{\partial}{\partial x}(\tilde{\sigma}^2(x,y)p(x,y)) \right] \phi_x(x,y)\,dx\,dy$$
$$+ \int_0^\Delta \frac{1}{2}\tilde{\sigma}^2(y,y)p(y,y)[\phi_x(y,y) + \phi_y(y,y)]\,dy$$
$$+ \int_0^\Delta \left[\hat{p}(y) - \frac{1}{2}\hat{\sigma}^2(y,y)p(y,y) \right] \phi_y(y,y)\,dy$$
$$+ \int_0^\Delta k(y)[\phi(0,0) - \phi(y,y)]\hat{p}(y)\,dy$$
$$+ [\phi(0,0) - \phi(\Delta,\Delta)]\,\mu^2(\Delta,\Delta)$$

and using the fact that $\phi_x(y, y) + \phi_y(y, y) = (d/dy)\phi(y, y)$, another integration by parts leads to

$$= \int_0^\Delta \int_{-\infty}^y \left[\tilde{f}(x,y)p(x,y) - \frac{1}{2}\frac{\partial}{\partial x}(\tilde{\sigma}^2(x,y)p(x,y)) \right] \phi_x(x,y)\,dx\,dy$$
$$+ \int_0^\Delta \left[\hat{p}(y) - \frac{1}{2}\hat{\sigma}^2(y,y)p(y,y) \right] \phi_y(y,y)\,dy$$
$$- \int_0^\Delta \left[\frac{1}{2}\frac{d}{dy}(\tilde{\sigma}^2(y,y)p(y,y)) + k(y)\hat{p}(y) \right] \phi(y,y)\,dy$$
$$+ \frac{1}{2}\tilde{\sigma}^2(\Delta,\Delta)p(\Delta,\Delta)\phi(\Delta,\Delta) - \frac{1}{2}\tilde{\sigma}^2(0,0)p(0,0)\phi(0,0)$$
$$+ \int_0^\Delta k(y)\hat{p}(y)\,dy\,\phi(0,0) + [\phi(0,0) - \phi(\Delta,\Delta)]\,\mu^2(\Delta,\Delta).$$

Considering the first three items, the densities p and $\hat{p}$ should satisfy

$$\tilde{f}(x,y)p(x,y) - \frac{1}{2}\frac{\partial}{\partial x}(\tilde{\sigma}^2(x,y)p(x,y)) = 0, \tag{9}$$

$$\hat{p}(y) - \tfrac{1}{2}\hat{\sigma}^2(y,y)p(y,y) = 0, \tag{10}$$

$$k(y)\hat{p}(y) + \frac{1}{2}\frac{d}{dy}(\tilde{\sigma}^2(y,y)p(y,y)) = 0. \tag{11}$$

Observe that the p and $\hat{p}$ which satisfy these conditions will also clean up the extra terms since

$$\int_0^\Delta k(y)\hat{p}(y)\,dy = -\int_0^\Delta \frac{1}{2}\frac{d}{dy}(\hat{\sigma}^2(y,y)p(y,y))\,dy$$
$$= \frac{1}{2}\tilde{\sigma}^2(0,0)p(0,0) - \frac{1}{2}\tilde{\sigma}^2(\Delta,\Delta)p(\Delta,\Delta).$$

Thus these formal calculations show that the left-hand side of (7)

$$= \int_0^\Delta \int_{-\infty}^y \left[\tilde{f}(x,y)p(x,y) - \frac{1}{2}\frac{\partial}{\partial x}(\tilde{\sigma}^2(x,y)p(x,y)) \right] \phi_x(x,y)\,dx\,dy$$

$$+ \int_0^\Delta \left[\hat{p}(y) - \frac{1}{2}\tilde{\sigma}^2(y,y)p(y,y) \right] \phi_y(y,y)\,dy$$

$$- \int_0^\Delta \left[\frac{1}{2}\frac{d}{dy}(\tilde{\sigma}^2(y,y)p(y,y)) + k(y)\hat{p}(y) \right] \phi(y,y)\,dy$$

$$+ \left(\mu^2(\Delta,\Delta) - \frac{1}{2}\tilde{\sigma}^2(\Delta,\Delta)p(\Delta,\Delta) \right) [\phi(0,0) - \phi(\Delta,\Delta)].$$

The p and $\hat{p}$ which satisfy (9), (10) and (11) have the form

$$p(x,y) = \frac{2C}{\tilde{\sigma}^2(x,y)} \exp\left\{ -\int_0^y k(v)\,dv \right\} \exp\left\{ -\int_x^y \frac{2\tilde{f}(t,y)}{\tilde{\sigma}^2(t,y)}\,dt \right\} \quad \forall\, x \le y,\ 0 \le y \le \Delta \tag{12}$$

and

$$\hat{p}(y) = C \exp\left\{ -\int_0^y k(v)\,dv \right\} \qquad \forall\, 0 \le y \le \Delta. \tag{13}$$

Fix $C = 1$ in (12) and (13) and let K be the normalizing constant for the probability measure μ_E^0:

$$K = \left[\int_0^\Delta \int_{-\infty}^y \frac{2}{\tilde{\sigma}^2(x,y)} \exp\left\{ -\int_0^y k(v)\,dv \right\} \times \exp\left\{ -\int_x^y \frac{2\tilde{f}(t,y)}{\tilde{\sigma}^2(t,y)}\,dt \right\} dx\,dy \right]^{-1}. \tag{14}$$

Note that this means *the solution p is an unnormalized density for* μ_E^0. With μ^2 defined to have point mass at (Δ, Δ) of size

$$\tfrac{1}{2}\tilde{\sigma}^2(\Delta,\Delta)p(\Delta,\Delta) = K\hat{p}(\Delta) = K \exp\left\{ -\int_0^\Delta k(v)\,dv \right\}, \tag{15}$$

the basic adjoint relation (7) is satisfied for every $\phi \in \mathscr{D}$.

Observe that the integration by parts computations are valid with these choices of p and $\hat{p}$. Thus we have identified the invariant measures given by densities. The next theorem proves that *all* the invariant measures have densities of this form.

Observe also that the dependence on the conditional distribution enters through the averaged coefficients $\tilde{f}$ and $\tilde{\sigma}^2$. In addition, note that $\hat{p}$ and μ^2 only depend on ν through the normalizing constant K. This shows that the failure mechanism is not affected by the control and *allows the optimization procedure to separate the control and replacement decisions.*

THEOREM 4.1. *Let μ^0, μ^1 nd μ^2 satisfy* (7) *and μ_E^0 and ν be defined by* (8a) *and* (8b), *respectively. Then μ_E^0 has density p given by* (12) *and μ^1 has density $\hat{p}$ given by* (13).

PROOF. Using the conditional distribution ν provided by μ^0, and p and $\hat{p}$ given by (12) and (13), respectively, *define* the measures $\tilde{\mu}_E^0 \in \mathscr{P}(E)$ and $\tilde{\mu}^1 \in \mathscr{M}(E_1)$ to have (unnormalized) densities p and $\hat{p}$, respectively. Define the measure $\tilde{\mu}^0 \in \mathscr{P}(E \times U)$ by

$$\tilde{\mu}^0(\Gamma) = K\int_E\int_U \chi_\Gamma(x,y,u)\nu(x,y,du)p(x,y)\,dx\,dy \qquad \forall\, \Gamma \in \mathscr{B}(E \times U).$$

Also define the measure $\tilde{\mu}^2$ to have point mass given by (15); denote this positive constant by $\tilde{M}$. Then (μ^0, μ^1, μ^2) satisfies the basic adjoint relation (7) by assumption and $(\tilde{\mu}^0, \tilde{\mu}^1, \tilde{\mu}^2)$ satisfies (7) by construction.

Let $\tilde{A}\phi(x,y) = \tilde{f}(x,y)\phi_x(x,y) + (1/2)\tilde{\sigma}^2(x,y)\phi_{xx}(x,y)$ and let M denote the mass of μ^2. M is positive since $\Delta < \infty$. Then, for every $\phi \in \mathscr{D}$, (7) implies

$$\begin{aligned}
&\int_E \tilde{A}\phi(x,y)\,d\mu_E^0(x,y) + \int_{E_1} B_1\phi(y,y)\,d\mu^1(y)\\
&\quad = -M[\phi(0,0) - \phi(\Delta,\Delta)]\\
&\quad = -\frac{M}{\tilde{M}}\tilde{M}[\phi(0,0) - \phi(\Delta,\Delta)]\\
&\quad = \frac{M}{\tilde{M}}\left[\int_E \tilde{A}\phi(x,y)\,d\tilde{\mu}_E^0(x,y) + \int_{E_1} B_1\phi(y,y)\,d\tilde{\mu}^1(y)\right].
\end{aligned}\tag{16}$$

Fix $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $a_1 \le a_2 \le b_1 \le b_2$; that is, such that $G = [a_1, a_2] \times [b_1, b_2] \subset E$. Define I_G to be the indicator of the set G. The idea is to solve the system

$$\begin{aligned}
\tilde{A}\phi(x,y) &= I_G(x,y) &\qquad& \forall\,(x,y) \in E,\\
B_1\phi(y,y) &= 0 && \forall\, 0 \le y \le \Delta,
\end{aligned}$$

for ϕ^* and use this in (16) to show that the measures μ_E^0 and $\tilde{\mu}_E^0$ agree on the set G and hence are the same since a_1, a_2, b_1, b_2 are arbitrary. The difficulty is that $\phi^* \notin \mathscr{D}$ so it is necessary to approximate.

Let $f_n, \sigma_n, I_n \in C_c^{2,1}(E)$ be chosen such that as $n \to \infty$, $f_n \to \tilde{f}$, $\sigma_n \to \tilde{\sigma}^2$ and $I_n \to I_G$ and such that for each n, the support of σ_n properly contains the supports of f_n and I_n, and $\sigma_n \ge \tilde{\sigma}^2$ on the supports of f_n and I_n. Recalling the definition of θ in (6), define

$$\xi_n(x,y) = \theta\left(\frac{x-y}{n}\right)$$

and note that $\xi_n \in \mathscr{D}$ and $\xi_n(x, y) \to 1$ as $n \to \infty$ for all $(x, y) \in E$. Define ψ_n by

$$\psi_n(x, y) = \int_0^x \int_0^z \frac{2}{\sigma_n(w, y)} I_n(w, y) \exp\left\{-\int_w^z \frac{2 f_n(t, y)}{\sigma_n(t, y)}\, dt\right\} dw\, dz \qquad \forall (x, y) \in E,$$

where $I_n(x, y)/\sigma_n(x, y)$ and $f_n(x, y)/\sigma_n(x, y)$ are defined to be 0 when both numerator and denominator are 0. Define

$$C_n(y) = \int_0^y \left[k(z)\psi_n(z, z) - \frac{\partial}{\partial y}\psi_n(z, z)\right] \exp\left\{\int_z^y k(v)\, dv\right\} dz \qquad \forall\, 0 \le y \le \Delta$$

and finally define $\phi_n \in \mathscr{D}$ by

$$\phi_n(x, y) = \xi_n(x, y)[\psi_n(x, y) + C_n(y)] \qquad \forall (x, y) \in E. \tag{17}$$

Observe that

$$\begin{aligned}
\tilde{A}\phi_n(x, y) = {} & \xi_n(x, y) \frac{\tilde{\sigma}^2(x, y)}{\sigma_n(x, y)} I_n(x, y) \\
& + \xi_n(x, y)\left[\tilde{f}(x, y) - \frac{\tilde{\sigma}^2(x, y)}{\sigma_n(x, y)} f_n(x, y)\right] \frac{\partial \xi_n}{\partial x}(x, y) \\
& + \left[\frac{1}{2}\tilde{\sigma}^2(x, y) \frac{\partial^2 \xi_n}{\partial^2 x}(x, y) + \tilde{f}(x, y) \frac{\partial \xi_n}{\partial x}(x, y)\right] \\
& \times [\psi_n(x, y) + C_n(y)],
\end{aligned}$$

$$B_1\phi_n(y, y) = 0$$

and $\{\tilde{A}\phi_n\}$ is uniformly bounded and converges to I_G as $n \to \infty$. Thus using ϕ_n in (16) we obtain

$$\begin{aligned}
\mu_E^0(G) &= \lim_{n\to\infty} \int_E \tilde{A}\phi_n(x, y)\, d\mu_E^0(x, y) + \int_{E_1} B_1\phi_n(y, y)\, d\mu^1(y) \\
&= \lim_{n\to\infty} \frac{M}{\tilde{M}} \left[\int_E \tilde{A}\phi_n(x, y)\, d\tilde{\mu}_E^0(x, y) + \int_{E_1} B_1\phi_n(y, y)\, d\tilde{\mu}^1(y)\right] \\
&= \frac{M}{\tilde{M}} \tilde{\mu}_E^0(G).
\end{aligned}$$

Since this holds for every rectangle in E_0, $\mu_E^0 = (M/\tilde{M})\tilde{\mu}_E^0$ and since both μ_E^0 and $\tilde{\mu}_E^0$ are probability measures, it follows that $M = \tilde{M}$.

Similarly, to show that $\mu^1 = \tilde{\mu}^1$, approximate the solution of the system

$$\begin{aligned}
\tilde{A}\phi(x, y) &= 0 & \forall (x, y) \in E, \\
B_1\phi(y, y) &= I_{[a, b]}(y) & \forall\, 0 \le y \le \Delta.
\end{aligned}$$

□

5. Single-cycle criterion. We postpone further examination of the long-term average criterion to consider the single-cycle problem; its solution will play an integral part in the solution of the long-term average problem. In this section, we show how to determine the single-cycle value using the invariant measures.

Recall that $\zeta \wedge \tau(\Delta)$ is defined to be the first time of renewal and the objective is to maximize the total reward obtained up to time $\zeta \wedge \tau(\Delta)$. We begin by establishing an equivalence between this single-cycle problem and a related long-term average problem.

Let r^* denote the unknown optimal single-cycle value. Then for every control process $\{u_t\}_{t \geq 0}$ and replacement level $\Delta > 0$,

$$E\int_0^{\zeta \wedge \tau(\Delta)} h(x_t, y_t, u_t)\, dt \leq r^* \tag{18a}$$

and equality is achieved by an optimal decision policy $((u_t^*)_{t \geq 0}, \Delta^*)$. Rearranging the inequality (18a) and dividing by the positive value $E[\zeta \wedge \tau(\Delta)]$, it follows that

$$\frac{E\int_0^{\zeta \wedge \tau(\Delta)} h(x_t, y_t, u_t)\, dt - r^*}{E[\zeta \wedge \tau(\Delta)]} \leq 0 \tag{18b}$$

for every control and replacement level policy. The left-hand side of (18b) represents (via a renewal argument) the long-term average value of a process that is renewed whenever failure or preventive replacement occurs and for which the cost of renewal is r^*. Sections 3 and 4 show that this long-term average value can also be represented as

$$K_{\nu,\Delta} \int_0^{\Delta} \int_{-\infty}^{y} \int_U h(x, y, u) \nu(x, y, du) p_{\nu,\Delta}(x, y)\, dx\, dy - K_{\nu,\Delta} r^*, \tag{19}$$

where the dependences of the density $p(x, y)$ of (12) and the normalizing constant K of (14) on ν and Δ have been indicated with subscripts. Observe that inequality (18b) becomes

$$K_{\nu,\Delta} \int_0^{\Delta} \int_{-\infty}^{y} \int_U h(x, y, u) \nu(x, y, du) p_{\nu,\Delta}(x, y)\, dx\, dy - K_{\nu,\Delta} r^* \leq 0 \tag{20}$$

and, by inverting the previous rearrangement, (18a) takes the form

$$\int_0^{\Delta} \int_{-\infty}^{y} \int_U h(x, y, u) \nu(x, y, du) p_{\nu,\Delta}(x, y)\, dx\, dy \leq r^* \tag{21}$$

for every pair (ν, Δ), with equality achieved by an optimal pair. Corresponding to an optimal pair (ν^*, Δ^*) is a stationary relaxed control $(\pi_t^*)_{t \geq 0}$ and stationary process $(x_t^*, y_t^*)_{t \geq 0}$ which achieves the maximum long-term average value 0 and therefore, the optimal single-cycle value r^*. Thus (π_t^*, Δ^*) is optimal for the single-cycle problem as well, and it is sufficient to optimize over stationary solutions of the constrained martingale problem. The single-cycle problem is equivalent to the *unconstrained* optimization problem: Maximize

$$\int_0^{\Delta} \int_{-\infty}^{y} \int_U h(x, y, u) \nu(x, y, du) p_{\nu,\Delta}(x, y)\, dx\, dy$$

over conditional distributions ν and preventive replacement levels Δ.

6. Solution of the LP. In this section, we use the identification of the invariant measures (parametrized by ν and Δ) to also rewrite the LP problem for the long-term average value as an unconstrained optimization problem. This optimization problem has the advantage that the solution can be readily determined but has the disadvantage that it requires a priori knowledge of the long-term average value. The difficulty will be overcome in the next section.

Let λ^* denote the optimal long-term average value and let $\hat{p}_\Delta$ denote the density (13). Define

$$\Lambda(\nu,\Delta) = K_{\nu,\Delta}\int_E\int_U h(x,y,u)\nu(x,y,du)p_{\nu,\Delta}(x,y)\,dx\,dy - K_{\nu,\Delta}R_1\int_{\overline{E}_1} k(y)\hat{p}_\Delta(y)\,dy - K_{\nu,\Delta}R_2\hat{p}_\Delta(\Delta)$$

and observe that the second integral can be explicitly computed, yielding

$$\begin{aligned}\Lambda(\nu,\Delta) &= K_{\nu,\Delta}\int_E\int_U h(x,y,u)\nu(x,y,du)p_{\nu,\Delta}(x,y)\,dx\,dy\\ &\quad + K_{\nu,\Delta}(R_1-R_2)\exp\left\{-\int_0^\Delta k(v)\,dv\right\} - K_{\nu,\Delta}R_1.\end{aligned}\tag{22}$$

Note that Theorem 3.1 implies $\Lambda(\nu,\Delta)$ is the long-term average reward for some solution of the constrained martingale problem and thus for every conditional distribution ν and replacement level Δ,

$$\Lambda(\nu,\Delta) \le \lambda^* \tag{23}$$

and equality is obtained by an optimal pair (ν^*,Δ^*). Taking into account the value of $K_{\nu,\Delta}$ given by (14), the inequality (23) can be rewritten as

$$\begin{aligned}&\int_0^\Delta\int_{-\infty}^y\int_U [h(x,y,u)-\lambda^*]\nu(x,y,du)p_{\nu,\Delta}(x,y)\,dx\,dy\\ &\quad +(R_1-R_2)\exp\left\{-\int_0^\Phi k(v)\,dv\right\} \le R_1.\end{aligned}\tag{24}$$

REMARK 6.1. Observe that the first term corresponds to a single-cycle problem having reward rate $h-\lambda^*$.

REMARK 6.2. In this form, the conditional distribution ν only affects the first term, so ν^* of an optimal pair (ν^*,Δ^*) maximizes the value of the integral. As a result, ν^* must also maximize the value of the integral for any $\Delta < \Delta^*$; otherwise, replacing ν^* on $0 \le y \le \Delta$ by a conditional distribution that is optimal on $0 \le y \le \Delta$ will improve the value. In fact, there is nothing special about the value Δ^* in this argument. Thus, we can separate the maximization into two steps: maximizing the first term of (24) with respect to ν taking $\Delta = \infty$ (so that ν is defined everywhere), and then maximizing the entire expression in (24) over the replacement level Δ with the optimal ν from the first step.

7. λ-maximization. We address the issue of determining the optimal long-term average value λ^* by extending the "λ-minimization" technique for optimal stopping to our control and replacement problem. This method establishes a relation between λ^* and the failure cost R_1 and provides an iterative scheme to produce a sequence of values λ_n converging to λ^*. Note that this method is a *maximization* technique since the original problem is to maximize the long-term average reward.

In the previous section, λ^* was assumed to be known and we optimized the left-hand side of (24). The fact that the value of the left-hand side is less than or equal to R_1 for every conditional distribution, replacement level pair (ν, Δ) and equal to R_1 for an optimal pair is observed but never used. This relation plays a central role in the identification of λ^* in this section.

We consider a family of objective functions of the form (24) in which the parameter λ replaces λ^*. Define

$$(25)\qquad J(\nu, \Delta; \lambda) = \int_E \int_U [h(x, y, u) - \lambda]\, \nu(x, y, du)\, p_{\nu, \Delta}(x, y)\, dx\, dy$$

and

$$(26)\qquad R(\Delta) = (R_1 - R_2) e^{-\int_0^\Delta k(v)\, dv}.$$

The following proposition relates the parameter λ to the optimal value λ^* through a comparison of $J(\nu, \Delta; \lambda) + R(\Delta)$ with R_1.

PROPOSITION 7.1.

(i) $J(\nu, \Delta; \lambda) + R(\Delta) \le R_1 \;\forall\, (\nu, \Delta)$ *implies* $\lambda \ge \lambda^*$.
(ii) $J(\nu, \Delta; \lambda) + R(\Delta) \ge R_1$ *for some* (ν, Δ) *implies* $\lambda \le \lambda^*$.

PROOF. Both parts rely on rewriting (as in the previous section) the inequalities

$$J(\nu, \Delta; \lambda) + R(\Delta) \le (\ge) R_1$$

in a form which exhibits the long-term average value corresponding to ν and Δ:

$$\Lambda(\nu, \Delta) \le (\ge) \lambda.$$

If the first inequality $(\le)$ holds for all (ν, Δ), $\lambda \ge \lambda^*$. If the second inequality $(\ge)$ holds for some (ν, Δ), $\lambda \le \lambda^*$. □

From this proposition it is clear that the long-term average problem can be solved using two steps. First, solve the family of maximization problems (parameterized by λ) having objective function $J(\nu, \Delta; \lambda) + R(\Delta)$. Then, determine the value of the parameter for which $J(\nu_\lambda, \Delta_\lambda; \lambda) + R(\Delta_\lambda) = R_1$, where $(\nu_\lambda, \Delta_\lambda)$ denotes an optimal pair for the λ-optimization problem. The following iteration method addresses the task of identification of the parameter value assuming that the optimization problem can be solved.

7.1. *Iteration scheme.* The iteration scheme begins with an arbitrary choice of conditional distribution ν_0, say one which puts full mass at a constant control, and an arbitrary replacement level Δ_0. Let $\lambda_0 = \Lambda(\nu_0, \Delta_0)$. The scheme first iterates on the conditional distribution to improve the value, then iterates on the replacement level and finally computes the value corresponding to the new conditional distribution and replacement level. The scheme then repeats, if necessary.

THE ITERATION ALGORITHM.

STEP 1 (Policy improvement step). Maximize $J(\nu, \infty; \lambda_n)$; a single-cycle problem with reward rate $h - \lambda_n$. Let ν_{n+1} denote a maximizer.

STEP 2 (Replacement level improvement step). Maximize $J(\nu_{n+1}, \Delta; \lambda_n) + R(\Delta)$. Let Δ_{n+1} denote a maximizer.

STEP 3 (Value computation). Compute $\lambda_{n+1} = \Lambda(\nu_{n+1}, \Delta_{n+1})$. If $\lambda_{n+1} = \lambda_n$, then $(\nu_{n+1}, \Delta_{n+1})$ is optimal. If $\lambda_{n+1} > \lambda_n$, then return to Step 1 with λ_{n+1}.

7.1.1. *Improvement of the values.* We show that the iterative scheme does indeed improve the long-term average values. To simplify notation, let $E_n = \{(x, y)): -\infty \le x \le y,\ 0 \le y \le \Delta_n\}$ and K_n denote the normalizing constant (14) corresponding to the pair (ν_n, Δ_n). We compare the values before and after improvement:

$$
\begin{aligned}
\lambda_{n+1} - \lambda_n &= K_{n+1} \int_{E_{n+1} \times U} h(x, y, u)\, \nu_{n+1}(y, du)\, p_{\nu_{n+1}}(x, y)\, dx\, dy \\
&\quad + K_{n+1} R(\Delta_{n+1}) - K_{n+1} R_1 - \lambda_n \\
&= K_{n+1} \Bigg[\int_{E_{n+1} \times U} [h(x, y, u) - \lambda_n]\, \nu_{n+1}(y, du)\, p_{\nu_{n+1}}(x, y)\, dx\, dy \\
&\qquad\qquad + R(\Delta_{n+1}) - R_1 \Bigg] \\
&\ge K_{n+1} \Bigg[\int_{E_n \times U} [h(x, y, u) - \lambda_n]\, \nu_{n+1}(y, du)\, p_{\nu_{n+1}}(x, y)\, dx\, dy \\
&\qquad\qquad + R(\Delta_n) - R_1 \Bigg] \\
&\ge K_{n+1} \Bigg[\int_{E_n \times U} [h(x, y, u) - \lambda_n]\, \nu_n(y, du)\, p_{\nu_n}(x, y)\, dx\, dy \\
&\qquad\qquad + R(\Delta_n) - R_1 \Bigg] \\
&= 0.
\end{aligned}
$$

Here, the inequalities follow from optimality of Δ_{n+1} for the optimization problem with $J(\nu_{n+1}, \Delta, \lambda_n) + R(\Delta)$, and optimality of ν_{n+1} for the optimization problem with $J(\nu, \Delta_n, \lambda_n)$ [cf. Remark (6.2)], and the final equality follows from the definition of λ_n.

7.1.2. CONVERGENCE OF VALUES. The iteration algorithm provides an increasing sequence of values $\{\lambda_n\}$ which are bounded above by the maximum value of h. Thus, there exists a limiting value λ_∞. We claim that $\lambda_\infty = \lambda^*$.

First, repeat the iteration algorithm once more using λ_∞ to determine ν_∞ and Δ_∞ which are optimal for the problem with parameter λ_∞. Allowing a slight abuse of notation which is nevertheless clear, let

$$\lambda_{\infty+1} = \Lambda(\nu_\infty, \Delta_\infty). \tag{27}$$

The algorithm improves values so $\lambda_{\infty+1} \geq \lambda_\infty$. Since (27) implies

$$J(\nu_\infty, \Delta_\infty; \lambda_{\infty+1}) + R(\Delta_\infty) = R_1$$

and $J(\nu_\infty, \Delta_\infty; \lambda) - R(\Delta_\infty))$ is a decreasing function of λ, it follows that $J(\nu_\infty, \Delta_\infty; \lambda_\infty) + R(\Delta_\infty) \geq R_1$.

We now demonstrate the opposite inequality.

$$\begin{aligned} J(\nu_\infty, \Delta_\infty; \lambda_\infty) &+ R(\Delta_\infty) \\ &= \int_{E_\infty \times U} [h(x, y, u) - \lambda_\infty] \nu_\infty(y, du) p_\infty(x, y)\, dx\, dy + R(\Delta_\infty) \\ &= \int_{E_\infty \times U} [h(x, y, u) - \lambda_n] \nu_\infty(y, du) p_\infty(x, y)\, dx\, dy + R(\Delta_\infty) \\ &\quad + \frac{\lambda_n - \lambda_\infty}{K_\infty}. \end{aligned}$$

By the optimality of $(\nu_{n+1}, \Delta_{n+1})$ for the λ_n-maximization problem, this quantity is

$$\begin{aligned} &\leq \int_{E_{n+1} \times U} [h(x, y, u) - \lambda_n] \nu_{n+1}(y, du) p_{n+1}(x, y)\, dx\, dy \\ &\quad + R(\Delta_{n+1}) + \frac{\lambda_n - \lambda_\infty}{K_\infty} \\ &= \int_{E_{n+1} \times U} [h(x, y, u) - \lambda_{n+1}] \nu_{n+1}(y, du) p_{n+1}(x, y)\, dx\, dy + R(\Delta_{n+1}) \\ &\quad + \frac{\lambda_n - \lambda_\infty}{K_\infty} + \frac{\lambda_{n+1} - \lambda_n}{K_{n+1}}. \end{aligned}$$

Since $\lambda_{n+1} = \Lambda(\nu_{n+1}, \Delta_{n+1})$ implies $J(\nu_{n+1}, \Delta_{n+1}; \lambda_{n+1}) + R(\Delta_{n+1}) = R_1$, the last expression becomes

$$= R_1 + \frac{\lambda_n - \lambda_\infty}{K_\infty} + \frac{\lambda_{n+1} - \lambda_n}{K_{n+1}}.$$

Letting $n \to \infty$, the result is established.

7.2. *Simple model*: *The pure running max.* To illustrate the iterative scheme, we consider the pure running max model introduced in Heinricher and Stockbridge (1991) in which the reward rate h and diffusion coefficients f and σ are functions of *only* y and u, not of x. The maximization steps can be explicitly solved for this model.

We place the following additional assumptions on f, h and k:

(i) $h(y,u)$ is nonincreasing in y for fixed u.
(ii) $f(y,u)$ is nondecreasing in y for fixed u.
(iii) k is an increasing function.

These assumptions have a natural interpretation for applications in controlled wear. The first assumption says that the revenue rate decreases as the system wears. The second and third assumptions say that a worn system has higher wear and failure rates, respectively.

For Steps 1 and 2, let λ_n denote a generic long-term average value given by either initialization or Step 3 of the scheme.

7.2.1. STEP 1: MAXIMIZE $J(\nu,\infty;\lambda_n)$.

PROPOSITION 7.2. *For* $y \geq 0$, *let*

$$u_{n+1}(y) = \min\left\{v \in U: \frac{h(y,v) - \lambda_n}{f(y,v)} = \max_{u\in U} \frac{h(y,u) - \lambda_n}{f(y,u)}\right\}.$$

The conditional distribution ν_{n+1} *defined on* U *by* $\nu_{n+1}(x,y,\cdot) = \nu_{n+1}(y,\cdot) = \delta_{u_{n+1}(y)}(\cdot)$ *maximizes* $J(\nu,\infty;\lambda_n)$.

PROOF. For each conditional distribution ν, we have

$$\begin{aligned} J(\nu,\infty;\lambda_n) &= \int_0^\infty\int_{-\infty}^y\int_U \frac{[h(y,u)-\lambda_n]}{f(y,u)} f(y,u)\nu(x,y,du)p_\nu(x,y)\,dx\,dy \\ &\leq \int_0^\infty\int_{-\infty}^y \frac{[h(y,u_{n+1}(y))-\lambda_n]}{f(y,u_{n+1}(y))}\int_U f(y,u)\nu(x,y,du)p_\nu(x,y)dx dy. \end{aligned}$$

Using the fact that p_ν satisfies (9), we obtain

$$\begin{aligned} &= \int_0^\infty\int_{-\infty}^y \frac{[h(y,u_{n+1}(y))-\lambda_n]}{f(y,u_{n+1}(y))}\frac{1}{2}\frac{\partial}{\partial x}\big(\tilde{\sigma}^2(x,y)p_\nu(x,y)\big)\,dx\,dy \\ &= \int_0^\infty \frac{[h(y,u_{n+1}(y))-\lambda_n]}{f(y,u_{n+1}(y))}\frac{1}{2}\tilde{\sigma}^2(y,y)p_\nu(y,y)\,dy. \end{aligned}$$

Using (10) and (13), we get

$$= \int_0^\infty \frac{[h(y,u_{n+1}(y))-\lambda_n]}{f(y,u_{n+1}(y))} e^{-\int_0^y k(v)\,dv}\,dy. \tag{28}$$

The expression (12) for p simplifies when the conditional distribution is ν_{n+1}. It becomes

$$p_{n+1}(x, y) = \frac{2}{\sigma^2(u_{n+1}(y))} \exp\left\{\int_0^y k(v)\,dv + 2f(y, u_{n+1}(y))(x-y)/\sigma^2(u_{n+1}(y))\right\}$$

and direct computations with ν_{n+1} and p_{n+1} produce (28). □

7.2.2. STEP 2: MAXIMIZE $J(\nu_{n+1}, \Delta; \lambda_n) + R(\Delta)$. We rewrite $J(\nu_{n+1}, \Delta; \lambda_n + R(\Delta))$ once more to obtain a better form for optimization:

$$\begin{aligned} J(\nu_{n+1}, \Delta; \lambda_n) + R(\Delta) &= \int_0^\Delta \frac{h(y, u_{n+1}(y)) - \lambda_n}{f(y, u_{n+1}(y))} e^{-\int_0^y k(v)\,dv}\,dy \\ &\quad + (R_1 - R_2)e^{-\int_0^\Delta k(v)\,dv} \\ &= \int_0^\Delta \left[\frac{h(y, u_{n+1}(y)) - \lambda_n}{f(y, u_{n+1}(y))} - (R_1 - R_2)k(y)\right] \\ &\quad \times e^{-\int_0^y k(v)\,dv}\,dy + (R_1 - R_2). \end{aligned}$$

The assumptions on f and h imply $(h(y, u_{n+1}(y)) - \lambda_n)/f(y, u_{n+1}(y))$ is nonincreasing. To see this consider values y_1 and y_2 with $y_2 > y_1$. Then

$$\begin{aligned} \frac{h(y_2, u_{n+1}(y_2)) - \lambda_n}{f(y_2, u_{n+1}(y_2))} &\le \frac{h(y_1, u_{n+1}(y_2)) - \lambda_n}{f(y_1, u_{n+1}(y_2))} \\ &\le \frac{h(y_1, u_{n+1}(y_1)) - \lambda_n}{f(y_1, u_{n+1}(y_1))}. \end{aligned}$$

Since k is nondecreasing $(h(y, u_{n+1}(y)) - \lambda_n)/f(y, u_{n+1}(y)) - (R_2 - R_1)k(y)$ is nonincreasing and it is easily seen that the integral is maximized by choosing

$$\Delta_{n+1} = \inf\left\{y: \frac{h(y, u_{n+1}(y)) - \lambda_n}{f(y, u_{n+1}(y))} - (R_1 - R_2)k(y) \le 0\right\}. \tag{29}$$

7.2.3. SPECIFIC EXAMPLE. Let $h(y, u) \equiv h$, $f(y, u) = u \in [\alpha, \beta]$ with $\alpha > 0$, $\sigma(y, u) \equiv 1$ and

$$k(y) = \begin{cases} k_1, & \text{for } y \le N, \\ k_2, & \text{for } y > N. \end{cases}$$

Assume that α, β, k_1, k_2, N, R_1, and R_2 satisfy

$$\frac{k_2}{k_1}(1 - e^{-k_1 N}) > \frac{\beta R_1}{\alpha(R_1 - R_2)} > 1 + \frac{\beta}{\alpha}. \tag{30}$$

(The weak convergence arguments are still valid for this discontinuous k.)

Initialization: Let $\nu_0 = \delta_{\{\beta\}}$ and $\Delta_0 > N$. Then

$$\lambda_0 = h - \frac{\beta R_1(1 - e^{-k_1 N - k_2(\Delta_0 - N)}) + \beta R_2 e^{-k_1 N - k_2(\Delta_0 - N)})}{(1/k_1)(1 - e^{-k_1 N}) + (1/k_2)e^{-k_1 N}(1 - e^{-k_1 N - k_2(\Delta_0 - N)})}.$$

STEP 1. Since $h - \lambda_0 > 0$, the maximum is achieved by $\nu_1 \equiv \delta_\alpha$.

STEP 2. The condition $(h - \lambda_0)/\alpha - (R_1 - R_2)k(y) \le 0$ can be rewritten as

$$\frac{\beta R_1}{\alpha(R_1 - R_2)} - \frac{\beta}{\alpha} e^{-k_1 N - k_2(\Delta_0 - N)} \le \frac{k(y)}{k_1}(1 - e^{-k_1 N}) + \frac{k(y)}{k_2} e^{-k_1 N}(1 - e^{-k_1 N - k_2(\Delta_0 - N)})$$

and the assumptions (30) imply this inequality holds for $k(y) = k_2$ and fails for $k(y) = k_1$. Thus $\Delta_1 = N$.

STEP 3. The improved value is $\lambda_1 = h - \beta k_1 R_1 - \beta k_1 R_2 e^{-k_1 N}/(1 - e - k_1 N)$.

Second iteration: Again $h - \lambda_1 > 0$ so $\nu_2 \equiv \delta_\alpha$ and a similar computation to that above shows $\Delta_2 = N$. Thus an optimal solution has been reached.

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