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THE EQUIVALENCE OF THE COX PROCESS WITH SQUARED RADIAL ORNSTEIN–UHLENBECK INTENSITY AND THE DEATH PROCESS IN A SIMPLE POPULATION MODEL

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Two kinds of stationary point process are considered. One is generated by the sequence of death times in a simple immigration, birth and death process; the other is the Cox process with intensity given by the square of the radial Ornstein–Uhlenbeck process. By comparison of the coincidence densities, we show that the two classes of processes are equivalent. An explicit expression is given for the coincidence density of arbitrary order.

1. Introduction. In statistical applications the objective in analyzing a Cox process [Cox (1955)] is to draw inferences about the unobserved intensity process from the observed pattern of points. A simplifying assumption is that the intensity is a Markov process. Smith (1984), for example, has proposed a method for dealing with the case in which the intensity is a finite state Markov chain. In the continuous case it would be convenient to have available a flexible parametric family of diffusions to model the intensity process. Minimally, three parameters are required: two to control the scale and shape of the equilibrium intensity and one to control the smoothness of the trajectories. The final choice of model will also be influenced by considerations of mathematical tractability and the ease with which the point process can be simulated. A possible candidate is the d-dimensional squared radial Ornstein–Uhlenbeck process.

The purpose of this paper is to show that if a Cox process has an intensity that is the square of a stationary radial Ornstein–Uhlenbeck process, then it is equivalent to the process of death times in a simple stationary immigration, birth and death process; a process that is easy to simulate. The result is surprising because it provides a theoretical link between two of the most basic stochastic processes. The equivalence of the point processes in the case $d = 2$ was first noted by Srinivasan (1988). However the result is not well known. Our method of proof is simpler than that of Srinivasan and we believe that the generalization is new.

The plan of the paper is as follows. In Sections 2 and 3 we collect basic results about the simple immigration, birth and death process and the radial Ornstein–Uhlenbeck process. In Section 4 we prove some lemmas relating

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the factorial moments and ordinary moments of the two processes. Sections 5 and 6 contain statements of the main results and their proofs.

2. The immigration, birth and death process. Let $\{Y(t)\}$ be a stationary Markov population process with constant rates of immigration, birth and death, given by ν, λ and μ, respectively. In other words, if $Y(t) = n$, then the intensity of transition to $n+1$ is $\nu + n\lambda$ and the intensity of the transition to $n-1$ is $n\mu$ for $n \geq 0$. See, for example, Moran (1968). For convenience, we define the following parameters:

$$(1) \qquad \alpha = \frac{\mu - \lambda}{2}, \qquad \beta = \frac{\lambda}{\mu - \lambda}, \qquad \delta = \frac{\nu}{\lambda}.$$

It is well known that $\{Y(t)\}$ has an equilibrium distribution provided $\lambda < \mu$ and we shall insist on this condition throughout. Furthermore, the equilibrium distribution is negative binomial and is given by

$$(2) \qquad \operatorname{Prob}(Y(0) = j) = \left(1 - \frac{\lambda}{\mu}\right)^{\delta} \frac{\Gamma(\delta + j)}{\Gamma(\delta) j!} \left(\frac{\lambda}{\mu}\right)^{j}, \qquad j = 0, 1, 2, \dots,$$

with associated probability generating function

$$(3) \qquad G_0(z) = E(z^{Y(0)}) = (1 - \beta(z-1))^{-\delta}.$$

Given that $Y(0) = y$, the distribution of $Y(t)$ can be represented as

$$(4) \qquad Y(t) = \sum_{i=1}^{y} W_i(t) + N(t),$$

where $W_i(t)$ is the number of survivors in the family that originated with the ith member of the population at time $t = 0$, and $N(t)$ is the contribution to the population size arising from new immigrations in $(0, t)$. Because $N(t), W_1(t), W_2(t), \dots$ are independent and $W_1(t), W_2(t), \dots$ are identically distributed, it follows that the probability generating function (p.g.f.) of $Y(t)$ is

$$(5) \qquad G_t(z; y) = E(z^{Y(t)} | Y(0) = y) = P_W^y(z) P_N(z),$$

where $P_W(z)$ and $P_N(z)$ are the p.g.f.s of W and N, respectively. Both P_W and P_N are well known [Moran (1968)]. With our notation they can be rewritten as

$$(6) \qquad \begin{aligned} P_W(z) &= 1 + \rho^2 (z-1) m(z-1), \\ P_N(z) &= [m(z-1)]^{\delta}, \end{aligned}$$

where

$$(7) \qquad m(u) = (1 - \beta(1 - \rho^2) u)^{-1}$$

and $\rho = \exp(-\alpha t)$.

3. The radial Ornstein–Uhlenbeck process. Let $\{R\{t\}\}$ denote a stationary radial Ornstein–Uhlenbeck process represented by the stochastic differential equation

$$dR(t) = \left[(\delta - 1/2)R^{-1}(t) - R(t)\right] dt + dB(t), \tag{8}$$

where $\delta > 0$ is given by (1) and B is Brownian motion. When 2δ is an integer, $R(t)$ can be thought of as the distance from the origin of a Brownian particle moving in 2δ dimensions, subject to a restoring force directly proportional to the distance. In equilibrium the squared distance has a χ^2 distribution with 2δ degrees of freedom. The transition density of $\{R(t)\}$ [i.e., the density at $R(t) = r$ given that $R(0) = s$] is given by

$$\begin{aligned} p(t,s,r) &= 2r^{2\delta-1}e^{-r^2}\frac{1}{1-e^{-2t}}\exp\left[\frac{-(s^2+r^2)e^{-2t}}{1-e^{-2t}}\right] \\ &\quad\times\left(rse^{-t}\right)^{1-\delta}I_{\delta-1}\left(\frac{2rse^{-t}}{1-e^{-2t}}\right), \end{aligned} \tag{9}$$

where I is the modified Bessel function [Karlin and Taylor (1981), page 334]. We will be interested in the process $\{V(t)\}$ defined by

$$V(t) = \frac{\beta}{2}R^2(\alpha t), \quad \text{for } -\infty < t < \infty, \tag{10}$$

which is also a diffusion and therefore a Markov process. The parameters α and β are related to μ, λ and ν by (1). From (9), using properties of the Bessel function [Abramovitz and Stegun (1965)], the conditional moment generating function (m.g.f.) of $V(t)$ given $V(0) = y$ can be evaluated as

$$M_t(u;y) = E\left(e^{uV(t)}|V(0) = y\right) = \exp\left(y\rho^2 um(u)\right)m^{\delta}(u). \tag{11}$$

Note that $m^{\delta}(u)$ is the m.g.f. of a gamma distribution with scale parameter $\lambda(1-\rho^2)/(\mu-\lambda)$ and shape parameter δ. The m.g.f. of the equilibrium distribution is obtained by letting $t \to \infty$ or equivalently $\rho \to 0$ because $\rho = \exp(-\alpha t)$. Denoting the equilibrium m.g.f. by $M_0(u)$ and recalling that we have assumed that the process $\{V(t)\}$ is stationary, we have

$$M_0(u) = E(e^{uV(0)}) = (1-\beta u)^{-\delta}, \tag{12}$$

which is the m.g.f. of a gamma distribution with scale parameter β and shape δ.

The conditional m.g.f. can also be written in the form

$$M_t(u;y) = \sum_{j=0}^{\infty}\frac{e^{-\kappa}\kappa^j}{j!}m^{\delta+j}(u), \tag{13}$$

where $\kappa = \rho^2 y/((1-\rho^2)\beta)$; that is, the conditional distribution is a Poisson mixture of gamma distributions. This provides a straightforward way of simulating the process $\{V(t)\}$ once a discrete set of times has been chosen.

4. Conditional moments. First recall that if $h(z)$ is the p.g.f. of an nonnegative integer-valued random variable X and if $h(1+u)$ is finite for all u in some interval about zero, then the kth factorial moment of X is given by

$$E(X^{(k)}) = \left.\frac{d^k h(z)}{dz^k}\right|_{z=1} = \left.\frac{d^k h(1+u)}{du^k}\right|_{u=0}. \tag{14}$$

Equivalently, $E(X^{(k)})$ is the coefficient of $u^k/k!$ in the expansion of $h(1+u)$.

We now compare the moments of the stationary processes $\{Y(t)\}$ and $\{V(t)\}$.

LEMMA 1. *The factorial moments of $Y(0)$ are equal to the ordinary moments of $V(0)$.*

PROOF. From (3) and (12) we have

$$G_0(1+u) = M_0(u). \tag{15}$$

Taking derivatives of kth order and evaluating at $u = 0$, the result follows because on the left-hand side we have the kth factorial moment of $Y(0)$ using (14), and on the right-hand side we have the kth ordinary moment of $V(0)$ by the definition of a m.g.f. □

Let $p(y) = a_0 + a_1 y + \cdots + a_k y^k$ be an arbitrary polynomial in y and define $\tilde{p}(y)$ to be

$$\tilde{p}(y) = a_0 + a_1 y^{(1)} + \cdots + a_k y^{(k)}. \tag{16}$$

We will refer to $\tilde{p}$ as the factorial polynomial associated with p.

LEMMA 2. *For each $k = 1, 2, \ldots$ the conditional expectation $E(Y^{(k)}(t)|Y(0) = y)$, considered as a function of y, is the factorial polynomial associated with $E(V^k(t)|V(0) = y)$.*

PROOF. First note that both $E(Y^{(k)}(t)|Y(0) = y)$ and $E(V^k(t)|V(0) = y)$ are polynomials in y of order k. For example,

$$\begin{aligned} E(Y^{(k)}(t)|Y(0) = y) &= \left.\frac{d^k}{du^k} G_t(1+u; y)\right|_{u=0} \\ &= \left.\frac{d^k}{du^k}(1+\rho^2 u m(u))^y m(u)^\delta\right|_{u=0}, \end{aligned} \tag{17}$$

which means that each successive differentiation introduces a higher power of y. The resulting expression is a polynomial in y because $1 + \rho^2 u m(u) = 1$ when $u = 0$. We can collect factorial powers together to obtain the expression

$$E(Y^{(k)}(t)|Y(0) = y) = a_{k0} + a_{k1} y^{(1)} + \cdots + a_{kk} y^{(k)} = \tilde{p}_k(y), \tag{18}$$

where $\tilde{p}_k(y)$ is a factorial polynomial. In a similar fashion, the kth conditional moment of $V(t)$ given $V(0) = y$ can be written as

$$E\big(V^k(t)|V(0) = y\big) = b_{k0} + b_{k1}y + \cdots + b_{kk}y^k. \tag{19}$$

To prove the lemma we must show that $a_{kj} = b_{kj}$ for all k and for all $j \le k$.

Using (18) the conditional p.g.f. can be written as

$$G_t(1 + u; y) = \sum_{k=0}^{\infty} \frac{u^k}{k!}\tilde{p}_k(y) \tag{20}$$

for $|u|$ sufficiently small.

Now note that

$$\sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} y^{(k)} = \theta^k, \tag{21}$$

so that

$$\sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} \tilde{p}_k(y) = p_k(\theta), \tag{22}$$

where p_k is the ordinary polynomial associated with $\tilde{p}_k$. Using (20) and (22),

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} G_t(1 + u; y) &= \sum_{k=0}^{\infty} \frac{u^k}{k!} p_k(\theta) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{u^k\theta^j}{k!} a_{kj}, \end{aligned} \tag{23}$$

and substituting the explicit form for $G_t(1 + u; y)$ from (5) we have

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} G_t(1 + u; y) &= \sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} \big(1 + \rho^2 u m(u)\big)^y m^{\delta}(u) \\ &= e^{\theta\rho^2 u m(u)} m^{\delta}(u) \\ &= M_t(u, \theta). \end{aligned} \tag{24}$$

Comparison of (23) and (24) shows that a_{kj} is the coefficient of $u^k\theta^j/k!$ in $M_t(u, \theta)$. However, from (19) the coefficient of $u^k y^j/k!$ in $M_t(u, y)$ is b_{kj} so that $a_{kj} = b_{kj}$, which proves the lemma. □

LEMMA 3. *For each $k = 1, 2, \ldots$ the conditional expectation $E(Y^{(k)}(t)|Y(0) = y)$ considered as a function of δ is a polynomial in δ of order k.*

PROOF. The proof follows from (17) because each derivative introduces an extra power of δ. □

LEMMA 4. *For each $k = 1, 2, \ldots$ the conditional expectation $E(Y^{(k)}(t)|Y(0) = y)$ is differentiable with respect to t so that*

$$E\big(Y^{(k)}(t)|Y(\tau) = y\big) = E\big(Y^{(k)}(t)|Y(0) = y\big) + o(\tau) \quad \text{as } \tau \to 0. \tag{25}$$

PROOF. Again this follows directly from (17) because the joint distribution of $(Y(t), Y(\tau))$ is the same as that of $(Y(t-\tau), Y(0))$ by stationarity and $\rho = \exp(-\alpha t)$ is a differentiable function of t. □

5. Equivalence of the two point processes. Our treatment of point processes will follow that of Daley and Vere-Jones (1988), who provide a comprehensive theoretical background. We are interested in point processes that are both orderly and stationary. The kth order coincidence density of such a process is defined to be

$$h_k(t_1, \dots, t_k) = \lim_{\tau \to 0+} \tau^{-k} E\big[I(t_1, t_1 + \tau) \cdots I(t_k, t_k + \tau)\big] \tag{26}$$

for arbitrary $t_1 < \cdots < t_k$, where $I(t, t+\tau) = 1$ if there is exactly one point in the interval $(t, t+\tau)$ and $I(t, t+\tau) = 0$ otherwise.

Coincidence densities, or higher order intensities as they are sometimes known, were studied extensively by Macchi (1975). It is known that the coincidence densities characterize the point process [Daley and Vere-Jones (1988), Section 5.4]; that is, if two orderly point processes have the same coincidence densities of all orders, then they are stochastically equivalent. We will make use of this fact to demonstrate our main result.

THEOREM 1. *The point process associated with the times of deaths in the population process* $\{Y(t)\}$ *is stochastically equivalent to the Cox process with intensity process* $\{\mu V(t)\}$.

PROOF. We start by considering the Cox process with intensity $\Lambda(t)$ defined by $\Lambda(t) = \mu V(t)$ for all t. Conditional on the intensity, the point process is an inhomogeneous Poisson process and the kth order coincidence density is simply the product of the intensities $\Lambda(t_1) \cdots \Lambda(t_k)$. Unconditionally, this gives

$$h_k(t_1, \dots, t_k) = E\big(\Lambda(t_1) \cdots \Lambda(t_k)\big), \tag{27}$$

and in terms of $V(t)$,

$$h_k(t_1, \dots, t_k) = \mu^k E(V_1 \cdots V_k), \tag{28}$$

where we use the notation $V_i = V(t_i)$, $i = 1, \dots, k$, for convenience.

We can evaluate the expectation by successive conditioning, making use of the fact that $V_1, \dots, V_k$ is a Markov sequence. Conditioning on $V_1, \dots, V_{k-1}$ and using the Markov property, we have

$$E\big(V_1 \cdots V_k | V_1, \dots, V_{k-1}\big) = V_1 \cdots V_{k-1} E\big(V_k | V_{k-1}\big). \tag{29}$$

The expectation $E(V_k | V_{k-1})$ is a polynomial in V_{k-1} of order 1 by (19). When multiplied by V_{k-1} we obtain a polynomial of order 2, which we denote by $q_2(V_{k-1})$. Note that here $\rho = \exp(-\alpha(t_k - t_{k-1}))$. Next, conditioning on $V_1, \dots, V_{k-2}$ the expectation becomes $V_1 \cdots V_{k-2} E[q_2(V_{k-1}) | V_{k-2}]$, which can be written as $V_1 \cdots V_{k-3} q_3(V_{k-2})$, where q_3 is a polynomial of order 3.

Continuing in this way, we have finally

$$E(V_1 \cdots V_k) = E(q_k(V_1)), \tag{30}$$

where q_k is a polynomial of order k.

Let us now consider the coincidence densities of the point process produced by the deaths in $\{Y(t)\}$. As before, we consider an arbitrary set of times $t_1 < \cdots < t_k$. We assume that τ is sufficiently small and that the intervals $(t_i, t_i + \tau)$ are disjoint. For convenience we write I_i instead of $I(t_i, t_i + \tau)$, for $i = 1, \dots, k$. Denoting the σ-field generated by $\{Y(s),\ s < t\}$ by $\mathscr{B}(t)$, we have

$$E(I_k | \mathscr{B}(t_k)) = E(I_k | Y(t_k)) = \tau\mu Y(t_k) + o(\tau), \tag{31}$$

where μ is the death intensity and we have used the Markov property of $\{Y(t)\}$. It follows that

$$E(I_1 \cdots I_k | \mathscr{B}(t_k)) = I_1 \cdots I_{k-1}[\tau\mu Y(t_k) + o(\tau)], \tag{32}$$

and taking conditional expectations given $\mathscr{B}(t_{k-1} + \tau)$,

$$\begin{aligned} &E(I_1 \cdots I_k | \mathscr{B}(t_{k-1} + \tau)) \\ &\quad = I_1 \cdots I_{k-1}\big[\tau\mu E\big[Y(t_k) | Y(t_{k-1} + \tau)\big] + o(\tau)\big]. \end{aligned} \tag{33}$$

To evaluate the conditional expectation given $\mathscr{B}(t_{k-1})$ we use

$$\begin{aligned} &E\big\{I_{k-1} E\big[Y(t_k) | Y(t_{k-1} + \tau)\big] | \mathscr{B}(t_{k-1})\big\} \\ &\quad = \operatorname{Prob}\big[I_{k-1} = 1 | Y(t_{k-1})\big] \\ &\qquad \times E\big[Y(t_k) | Y(t_{k-1} + \tau) = Y(t_{k-1}) - 1\big] + o(\tau) \\ &\quad = \tau\mu Y(t_{k-1}) E\big[Y(t_k) | Y(t_{k-1} + \tau) = Y(t_{k-1}) - 1\big] + o(\tau) \\ &\quad = \tau\mu \tilde{q}_2(Y(t_{k-1})) + o(\tau). \end{aligned} \tag{34}$$

Each equality in (34) introduces an additional term of $o(\tau)$. The first $o(\tau)$ term is introduced by the approximation

$$\operatorname{Prob}\big[I_{k-1} = 1 \cap Y(t_{k-1} + \tau) - Y(t_{k-1}) \neq -1 | Y(t_{k-1})\big] = o(\tau), \tag{35}$$

because the probability that more than one event occurs in $(t_{k-1}, t_{k-1} + \tau)$ is $o(\tau)$. The second arises from the approximation

$$\operatorname{Prob}\big[I_{k-1} = 1 | Y(t_{k-1})\big] = \tau\mu Y(t_{k-1}) + o(\tau) \tag{36}$$

and the third is introduced by applying Lemma 4 because $yE(Y(t_k) | Y(t_{k-1} + \tau) = y - 1) = yE(Y(t_k) | Y(t_{k-1}) = y - 1) + o(\tau)$. Finally, because $E[Y(t_k) | Y(t_{k-1}) = y]$ is the factorial polynomial associated with $E[V_k | V_{k-1} = y]$ by Lemma 2, it follows that $yE[Y(t_k) | Y(t_{k-1}) = y - 1]$ is the factorial polynomial associated with $yE[V_k | V_{k-1} = y]$ or $q_2(y)$ in our previous notation.

In other words, using (33) and (34),

$$E\big[I_1 \cdots I_k | \mathscr{B}(t_{k-1})\big] = \mu^2\tau^2 I_1 \cdots I_{k-2}\tilde{q}_2(Y(t_{k-1})) + o(\tau^2). \tag{37}$$

For the next two steps, we take conditional expectations with respect to $\mathscr{B}(t_{k-2}+\tau)$ and then $\mathscr{B}(t_{k-2})$. In the latter case we must consider $yE[\tilde{q}_2(Y(t_{k-1}))|Y(t_{k-2})=y-1]$. By Lemma 2, each of the factorial powers in the factorial polynomial $\tilde{q}_2$ has an expectation that can be expressed as a polynomial in factorial powers of $y-1$. Multiplying by y these become polynomials in factorial powers of y, and collecting terms, the whole expression becomes $\tilde{q}_3(y)$, the factorial polynomial associated with $q_3(y)$. Continuing in this way we have eventually

$$E(I_1 \cdots I_k) = E\big[E\big(I_1 \cdots I_k | Y(t_1)\big)\big] = \mu^k \tau^k E\big[\tilde{q}_k(Y(t_1))\big] + o(\tau^k), \tag{38}$$

so that the coincidence density of order k is $\mu^k E[\tilde{q}_k(Y(t_1))]$.

We now use Lemma 1 to compare the two coincidence densities (i.e., $\mu^k E[\tilde{q}_k(Y(t_1))]$ and $\mu^k E[q_k(V(t_1))]$). Because $\tilde{q}_k$ is a factorial polynomial and each of the factorial moments of $Y(t_1)$ is equal to the corresponding ordinary moments of $V(t_1)$ by Lemma 1, it follows that the two coincidence intensities are identical and the proof is complete. □

6. Explicit expression for coincidence density. To give an explicit expression for the kth order coincidence density we must introduce some graph-theoretic concepts. We consider an undirected graph with k vertices labelled $1, 2, \dots, k$. Suppose that every vertex in the graph is either isolated or of degree 2. In other words, the graph consists of connected components each of which is either an isolated vertex, a pair of vertices joined by two edges or a larger circuit containing three or more vertices. We denote the set of all such graphs by $\mathscr{G}$.

Let the set of pairs $S = \{(i,j),\ 1 \le i \le k,\ i \le j \le k\}$ be ordered by some convention, for example, $(1,1), \dots, (1,k), (2,2), \dots, (2,k), \dots, (k,k)$. Each graph $g \in \mathscr{G}$ can be described by $S(g)$, an associated vector whose elements are increasingly ordered members of S. The presence of the pair (i,i) signifies that vertex i is isolated. If (i,j) is present and $i \ne j$, then there is an edge joining vertex i to vertex j. Note that if the vertices $\{i,j\}$ form a connected component of size 2 in the graph, the pair (i,j) will be repeated, consecutively, in the vector. We use the notation $\prod_{s \in g}$ to indicate a product taken over each of the pairs s in the vector $S(g)$.

THEOREM 2. *The kth order coincidence density of the processes described in Section* 5 *is given by*

$$h_k(t_1, \dots, t_k) = (\mu\beta)^k \sum_{g \in \mathscr{G}} 2^a \delta^c \prod_{s \in g} \rho_s, \tag{39}$$

where $\rho_{ij} = \rho_{(i,j)} = \exp(-\alpha|t_i - t_j|)$, *c is the number of connected components in the graph g and a is the number of such components containing more than two vertices.*

PROOF. First of all, we will prove the theorem when 2δ is integer, by calculating the coincidence density for the Cox process. Writing $N = 2\delta$, the process $\{R^2(t)\}$ can be represented as

$$R^2(t) = \sum_{n=1}^{N} U_n^2(t), \quad \text{for all } t, \tag{40}$$

where $\{U_n(t)\}$ are independent real-valued stationary Ornstein–Uhlenbeck processes, each represented by the stochastic differential equation

$$dU(t) = -U(t)\,dt + dB(t). \tag{41}$$

The Ornstein–Uhlenbeck process is a Gaussian Markov process with zero mean and autocovariance $E[U(t)U(0)] = \exp(-|t|)$. When 2δ is integer this is the most straightforward way to simulate the process $\{R(t)\}$. For background, see Karlin and Taylor [(1981), Chapter 15].

To calculate the kth order coincidence density we must evaluate

$$\begin{aligned} h_k(t_1,\dots,t_k) &= E[\Lambda(t_1)\cdots\Lambda(t_k)] \\ &= \left(\frac{\mu\beta}{2}\right)^k E\left[R^2(\alpha t_1)\cdots R^2(\alpha t_k)\right]. \end{aligned} \tag{42}$$

Using (40),

$$\begin{aligned} E\left[R^2(\alpha t_1)\cdots R^2(\alpha t_k)\right] &= E\left[\prod_{j=1}^{k}\left(\sum_{n=1}^{N} U_{jn}^2\right)\right] \\ &= \sum_{(j_1,\dots,j_k)\in\{1,\dots,N\}^k} E\left[U_{1j_1}^2 \cdots U_{kj_k}^2\right], \end{aligned} \tag{43}$$

where the $U_{jn} = U_n(\alpha t_j)$ and the vectors $(U_{1n},\dots,U_{kn})$ for $n = 1,\dots,N$ are independent multivariate normals each with variance–covariance matrix $\{\rho_{ij}\}$ with $\rho_{ij} = \exp(-\alpha|t_i - t_j|)$. To evaluate the expectation we consider $H_U(\theta)$, the joint m.g.f. of the variables $\{U_{jn}\}$, where

$$\begin{aligned} H_U(\theta) &= E\left[\exp\left(\sum_{n=1}^{N}\sum_{j=1}^{k}\theta_{jn}U_{jn}\right)\right] \\ &= \exp\left(1/2\sum_{n=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{k}\theta_{in}\theta_{jn}\rho_{ij}\right), \end{aligned} \tag{44}$$

using the multivariate normality. For a typical term on the right-hand side of (43) we have

$$\begin{aligned} E\left[U_{1j_1}^2\cdots U_{kj_k}^2\right] &= \text{Coeff. of } \frac{\theta_{1j_1}^2\cdots\theta_{kj_k}^2}{2^k} \text{ in } H_U(\theta) \\ &= \text{Coeff. of } \theta_{1j_1}^2\cdots\theta_{kj_k}^2 \text{ in } \frac{1}{k!}\left(\sum_{n=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{k}\theta_{in}\theta_{jn}\rho_{ij}\right)^k. \end{aligned} \tag{45}$$

Now notice that any graph $g \in \mathscr{G}$ can be associated with a term of the form $\theta_{1j_1}^2\cdots\theta_{kj_k}^2$. We do this by indexing the components using the index set

$\{1,\ldots,N\}$ and then constructing the product of terms $\theta_{in}\theta_{jn}$ over all the pairs (i,j) in the vector $S(g)$. For example, the graph with four vertices, in which vertex 4 is isolated with index 2, and in which $\{1,2,3\}$ form a circuit with index 1, corresponds to the product $\theta_{11}^2\theta_{21}^2\theta_{31}^2\theta_{42}^2$. Conversely, any product of the form $\theta_{1j_1}^2 \cdots \theta_{kj_k}^2$ corresponds to a set of graphs in $\mathscr{G}$ with indexed components.

We now multiply out the k factors in the power term in (45) and sum as in (43), picking out all terms associated with a particular graph $g \in \mathscr{G}$. In this way, a product $A\prod_{s\in g}\rho_s$ is formed, where A depends on g. We now determine the multiplier A. We use the vector of pairs $S(g) = (s_1,\ldots,s_k)$ to decide which ρ term is chosen from each of the k factors, so that ρ associated with the pair s_i is selected from the ith factor, for $i = 1,\ldots,k$. If the graph has c components, each component can be indexed in N ways, which introduces a multiplier of N^c, and if the pair s does not correspond to an isolated vertex, then $2\rho_s$ will appear in each factor. This introduces a multiplier of 2^{k-v}, where v is the number of isolated vertices. However, the elements $S(g)$ can be allocated to the k factors of the power term in $k!$ ways, which would introduce a further multiplier of $k!$ except that components of size 2 have to be dealt with separately because their associated pair s appears twice in $S(g)$ and does not produce a distinguishable permutation. We must, therefore, divide by 2^p, where p is the number of components of size 2. The product $\prod_{s\in g}\rho_s$, therefore, appears with a multiplier of $k!2^{k-v-p}N^c$ or $k!2^{a+k}\delta^c$, where a is the number of components having three or more vertices. This proves the result in the case 2δ is integer. To complete the proof in the general case we use Lemma 3. Thus we know that the coincidence density is a polynomial in δ and we have shown that the result is true when δ is an arbitrary positive half-integer. It follows that the formula is valid for all positive δ.

As an illustration, the third order coincidence density is

$$(\mu\beta)^3\left(2\rho_{12}\rho_{23}\rho_{13}\delta + \rho_{12}^2\delta^2 + \rho_{13}^2\delta^2 + \rho_{23}^2\delta^2 + \delta^3\right) \tag{46}$$

because the only graphs on three vertices are the 3-circuit, a pair plus an isolated vertex and three isolated vertices. □

7. Remarks. The equivalence of the two point processes in the case $\delta = 1$ was first noticed by Srinivasan (1988). Srinivasan considered a population process with immigration, birth, death and emigration; the point process being the process of emigration. However, this formulation does not produce a wider class of processes because the emigration process can be thought of as a thinning of a death process with a larger death parameter. In the Cox process considered by Srinivasan the intensity is proportional to the squared modulus of a complex Gaussian Markov process. The squared modulus of a such an intensity process is stochastically equivalent to the process $\{(U_1^2(\alpha t) + U_2^2(\alpha t))/2\}$, where the U's are independent Ornstein–Uhlenbeck processes.

The coincidence densities for the Cox process in this special case are known [Daley and Vere-Jones (1988)] and have the simple form

$$h_k(t_1,\dots,t_k) = (\mu\beta)^k \operatorname{Perm}\{\rho_{ij}\}, \tag{47}$$

where Perm$\{\rho_{ij}\}$ is the permanent of the matrix $\{\rho_{ij}\}$.

The full equivalence of the two types of point process means that the parameters of the population process ν, λ and μ are mapped onto the three parameters of the squared radial Ornstein–Uhlenbeck process. The shape, scale and autoregression parameters δ, β and α are given in terms of ν, λ and μ in (1), and, conversely,

$$\nu = 2\alpha\beta\delta, \qquad \lambda = 2\alpha\beta \quad \text{and} \quad \mu = 2\alpha(1+\beta) \tag{48}$$

define the immigration, birth and death rates for the associated population model.

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