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METASTABILITY IN THE GREENBERG-HASTINGS MODEL

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The Greenberg–Hastings model (GHM) is a family of multitype cellular automata that emulate excitable media, exhibiting the nucleation and spiral formation characteristic of such complex systems. In this paper we study the asymptotic frequency of nucleation in GHM dynamics on $\mathbb{Z}^2$ as the number of types, or *colors*, becomes large. Starting from uniform product measure over κ colors, and assuming that the excitation threshold θ is not too large, the box size L_κ needed for formation of a spiral core is shown to grow exponentially: $L_\kappa \approx \exp\{C_\kappa\}$ as $\kappa \to \infty$. By exploiting connections with percolation theory, we find that $C = 0.23 \pm 0.06$ in the nearest neighbor, $\theta = 1$ case. In contrast, GHM rules obey power law nucleation scaling when started from a suitable nonuniform product measure over the κ colors. This effect is driven by critical percolation. Finally, we present some analogous results for a random GHM, an interacting Markovian system closely related to the epidemic with regrowth of Durrett and Neuhauser.

1. Introduction. In the context of spatial dynamical systems, *metastability* refers to the displacement of one apparent steady state by another, often abruptly and after a prolonged period of time. Such "punctuated equilibria" are typically caused by the appearance of structures with superior self-organizational ability, due either to exceptionally large dynamic fluctuations or propagation across great distances from rare nucleating pockets in the initial state. Metastability effects are especially prevalent in the variety of complex systems known as excitable media, outstanding examples of which are neural signal transmission, atrial fibrillation and the Belousov–Zhabotinski (BZ) chemical reaction.

The *Greenberg–Hastings model* (GHM) γ_t is a very simple cellular automaton that emulates an excitable medium. At each time t, $\gamma_t \in \{0, \dots, \kappa - 1\}^{\mathbb{Z}^d}$. This means that for each $x \in \mathbb{Z}^d$, $\gamma_t(x)$ has one of κ possible types $0, \dots, \kappa - 1$, which appear as different colors in computer simulations, so we refer to them as *colors*. Recall that a cellular automaton (CA) (see Toffoli and Margolus [23]) is a dynamic configuration on a lattice of sites that updates in parallel according to a local homogeneous deterministic rule. Modeling of excitable media by iterated maps goes back to Wiener and Rosenblueth [25]

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in the mid 1940s. Several researchers contributed to the formalization of their idea in terms of CA dynamics; then about 15 years ago Greenberg and Hastings isolated the essential phenomenological features in their simple three-color nearest-neighbor rule [14]. Ever since, especially with the advent of microcomputers, many scientists across the spectrum of applied science have generalized GHM to incorporate more detailed aspects of the BZ reaction and other excitable media. The papers of Winfree, Winfree and Seifert [26] and Gerhardt, Schuster and Tyson [11] exemplify this work.

To strike a balance between physical verisimilitude and mathematical tractability, as in [9], we consider the GHM to comprise a family of CA rules indexed by three parameters: a finite neighborhood $\mathscr{N}$ of the origin, a threshold number θ of sites needed for excitation and the number κ of available colors. For given $\mathscr{N}$, θ and κ, taking the neighborhood of x to be the translate of $\mathscr{N}$, $\mathscr{N}_x = x + \mathscr{N}$, the update algorithm is as follows:

$$(1.1)\quad \gamma_{t+1}(x) = \begin{cases} (\gamma_t(x) + 1) \bmod \kappa, & \text{if } \gamma_t(x) \geq 1 \textit{ or} \\ & \#\{y \in \mathscr{N}_x \colon \gamma_t(y) = 1\} \geq \theta, \\ \gamma_t(x), & \text{otherwise} \end{cases}$$

(# denotes cardinality). In words, if a site x has color k at time t, the color may become $(k + 1) \bmod \kappa$ at time $t + 1$. If $k \geq 1$ this change is automatic, that is, it occurs inevitably, whereas if $k = 0$ the change is by contact: it happens only if at least θ sites within $\mathscr{N}_x$ have color 1.

One can define the GHM on a subset A of $\mathbb{Z}^d$ as well, assuming *free boundary*: For $x \in A$ we take $\mathscr{N}_x = \{y\colon y \in A \cap (x + \mathscr{N})\}$. One such subset is the box $B_L = \{z\colon \|z\|_\infty \leq L\}$, in which case *periodic boundary* is also natural: opposite boundaries are identified so the neighbor structure wraps around at the edges. The choice of boundary condition will play no significant rule in our analysis. Throughout this paper we consider planar GHM dynamics ($d = 2$), either on B_L or on all of $\mathbb{Z}^2$; some of our ideas generalize easily to higher dimensions but others do not. We will start from random initial states γ_0. Specifically, γ_0 will be μ-distributed, where μ is a translation invariant product measure.

What happens when GHM begins in a homogeneous disordered initial state? Figure 1 shows six gray-scale "still frames" of one representative evolution on a 240×240 box: the rule with $\mathscr{N} = B_3$, $\theta = 6$, $\kappa = 8$, initial densities $P_\mu(\gamma_0 = 0) = 2/9$, $P_\mu(\gamma_0 = k) = 1/9$ for $1 \leq k \leq 7$ and periodic boundary. Type 0 ("rested," or "excitable") is the lightest shade of gray; type 1 ("excited") is black and types $2, \dots, 7$ are intermediate shades from darker to lighter. Evidently by time 10 the overwhelming majority of sites have color 0. Note that the all 0's configuration is a trap from which no further excitation is possible. However, four isolated pockets of 1's have formed stable "pacemakers" that generate wave activity. By time 20 these nucleating centers give rise to remarkably coherent spiral pairs, sometimes called "ram's horns." Each spiral pair generates concentric rings that spread out until, by time 40, they annihilate upon collision with one another. In this manner, by time 80

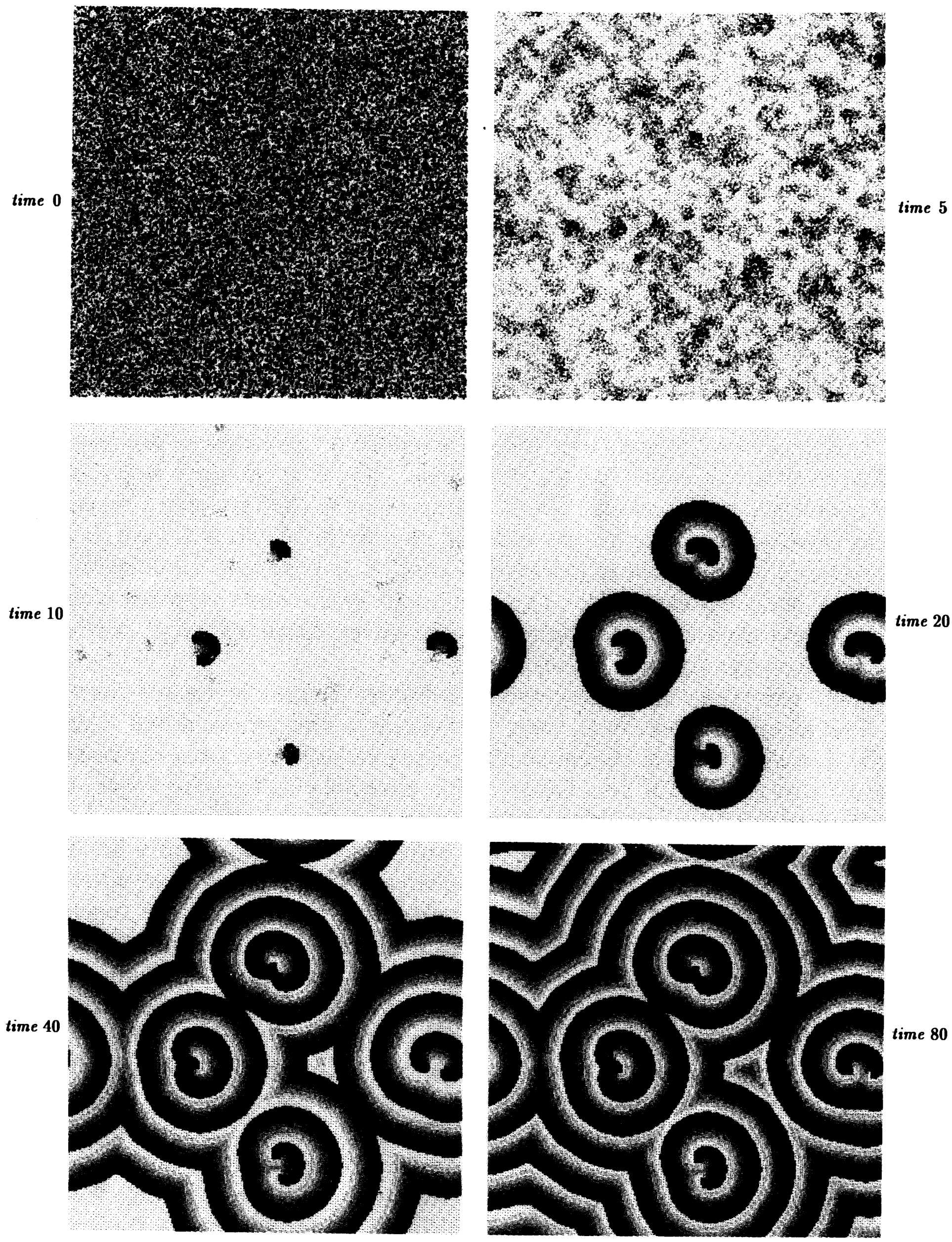


FIG. 1. *Nucleation of ram's horns in a range 3 Greenberg–Hastings model.*

the entire space is covered with a locally periodic steady state of oscillating waves. Nucleation of spirals, spiral pairs and/or target patterns from noisy starting configurations is characteristic of many real-world excitable media, although the nucleation mechanism is poorly understood in most instances. Our paper [9] contains color computer graphics that illustrate additional varieties of complex self-organization, as well as an extensive bibliography of research on excitable media. The even more exotic dynamics of closely related *cyclic cellular automaton* (CCA) models are also described and illustrated. Figure 1 was created using EXCITE! [10], $\geq$ 386-compatible freeware that facilitates GHM and CCA experimentation; the program is available by request.

The simplest choice for γ_0 is *primordial soup*: the random configuration that paints each site independently with a color chosen uniformly from the available palette $\{0, 1, \dots, \kappa - 1\}$. Let us reserve the notation π for this distribution, either on B_L or $\mathbb{Z}^2$ according to context. Our rather whimsical terminology is motivated by some emerging questions of *artificial life*: How are iterates of local logical rules able to transform randomness into organized structures that can move about, reproduce, compete, compute and carry out other evolutionary functions? See Waldrop's report [24] or Levy's recent book [20] for popular accounts of this exciting new field. We do not wish to impart any profound significance to primordial soup π. It is simply a mathematically expedient starting state that (by the monkey-at-the-typewriter principle) has every possible finite configuration, however extraordinary, represented somewhere on the infinite lattice.

If one starts the rule that generated Figure 1 from π rather than the μ with double weight on color 0, then γ_t invariably dies out on a 240×240 box, that is, fixates in all 0's. (Because any color except 0 advances automatically, the only way the GHM can fixate is for all colors to become 0 eventually.) Here "invariably" means the probability of forming a spiral core is so small that thousands of EXCITE! trials are unlikely to yield a single instance of nucleation. To expect to obtain ram's horns starting from π, one must either decrease the threshold θ to 5 or increase the box size L far beyond the capability of any currently feasible graphics array. In the infinite GHM, that is, on all of $\mathbb{Z}^2$, π will definitely generate ram's horns. Indeed, somewhere in the infinite initial soup there is an exact copy of the Figure 1 configuration at time $t = 80$. The ram's horns there are examples of *stable periodic objects* (spos) [9]: configurations $\gamma(\Lambda)$ on a finite $\Lambda \subset \mathbb{Z}^2$ with the property that the color at every site x within Λ sees at least θ of its successor color within $(x + \mathscr{N}) \cap \Lambda$. A moment's thought reveals that the color at every site of an spo advances every time under GHM dynamics, irrespective of the state of the system off Λ. In general, if a GHM rule admits spos, then it cannot die out on $\mathbb{Z}^2$ starting from π. See Durrett and Griffeath [4] for results on the existence of spos for GHM dynamics.

Our central problem here is the following: If a GHM with specific parameters nucleates on $\mathbb{Z}^2$, then how large an L is needed so that the finite system on B_L survives, that is, does not die out? In the formulation of our results we typically fix $\mathscr{N}$ and θ, letting $\kappa \to \infty$. A natural interpretation of GHM dynamics calls states $2, \dots, \kappa - 1$ *refractory*, $\kappa - 2$ updates being the deterministic recovery time for a cell after it has been excited before it is again ready to fire. In this context, we are studying asymptotic system behavior as the refractory period becomes large. As it turns out, roughly speaking, the size of the finite system needed to support nucleation grows exponentially with the number of colors κ. Alternatively, this exponential scaling can be interpreted as the length scale that separates nucleating pacemakers in the infinite system when κ is large. From the point of view of a local observer at the origin $\mathbf{0}$, with overwhelming probability γ_t appears to die out (by time $t = 5\kappa$, say). However, after a period of time that is exponential in κ, $\mathbf{0}$ is suddenly "overrun" by waves of activity that started far, far away. The apparently stable all 0's state is displaced by self-organized local periodicity.

This paper presents a series of theorems that give precise meaning to some aspects of the caricature of the previous paragraphs. Before turning to the results, let us digress briefly to mention some methodological aspects of research on excitable cellular automata that motivate our work. First, there is an intriguing interplay between computer visualization and deductive reasoning. The theory that we develop here was motivated in large part by innumerable experiments with the CAM-6 cellular automaton machine [23], EXCITE! and other programs. However, in the study of subtle phenomena such as self-organization and metastability, one quickly learns that a video display does not always offer an accurate window onto the behavior of an infinite interacting system, or even a very large finite one. There is nothing like a traditional theorem and proof to dispel paranoia about "monsters" out there somewhere that might fundamentally alter system behavior.

Moreover, our asymptotic scaling results offer some insight into the mechanism of GHM nucleation. As the reader will soon see, connectivity questions play a key role and several of our arguments depend on recent, remarkable advances in percolation theory. Ideally we would hope that probabilistic analysis of nucleation and self-organization might prove illuminating to applied researchers in the area of excitable media. As mathematicians, we began studying cellular automata to gain a better understanding of some closely related interacting particle systems [1, 7]. The last section of this paper illustrates the utility of such connections. By focusing on excitable CA rules for the past several years, we have become devoted to these models in their own right. They are beautiful examples of complex deterministic dynamics started from random initial states, a kind of stochastic process that offers promising new directions for applied probability.

The organization of this paper is as follows. We begin in the next section by establishing a general metastability theorem for the $(\mathscr{N}, \theta, \kappa)$ Greenberg–Hastings model (1.1). If θ is too large in comparison to $\#\mathscr{N}$, then nucleation is impossible and γ_t dies out. The critical value $\theta_c(\mathscr{N}, \kappa)$ for death is not known, but as a warm-up for the main theme of nucleation scaling we show that if $\mathscr{N}$ is symmetric ($-\mathscr{N} = \mathscr{N}$) and $\theta > \#\mathscr{N}/2$, then the system dies out by a fixed time T regardless of the initial state γ_0, on either the infinite lattice or a large finite one. This is a more general sufficient condition for a stronger kind of GHM death, with a simpler proof, than claim (2) of Section 4 in [9].

THEOREM 1. *Let γ_t be the GHM on $\mathbb{Z}^2$ or B_L (L large), with symmetric neighbor set $\mathcal{N}$, threshold θ, and κ colors. Assume that $\theta > \#\mathcal{N}/2$. Then γ_t dies out globally, that is, there is a deterministic time $T = T(\mathcal{N}, \theta)$, independent of the initial configuration γ_0, such that $\gamma_{T+\kappa} \equiv 0$.*

If θ is sufficiently small in comparison with $\#\mathcal{N}$, then (as in Figure 1) stable periodic objects exist and nucleation is to be expected on $\mathbb{Z}^2$. Again, just how small θ needs to be is a difficult question (cf. [4]), but for box neighborhoods $\mathcal{N} = B_\rho$ and $\theta \le \rho^2/4$, it is quite simple to construct spos for *any* $\kappa \ge 3$. In this context, as $\kappa \to \infty$, and starting from primordial soup π, we will show that the GHM on B_L dies out with overwhelming probability if $L < e^{c\theta\kappa}$, but survives with overwhelming probability if $L > e^{C\rho^2\kappa}$. Here and throughout the paper, c, C and other customary letters denote positive finite constants that may change from line to line or page to page according to the context.

THEOREM 2. *Let γ_t be the GHM on B_L with neighbor set $\mathcal{N}$, threshold θ and κ colors, started from uniform product measure π.*

(a) *For each $\alpha \in (0, \frac{1}{2})$ there is a constant $c = c_\alpha$ such that if $\alpha\#\mathcal{N} \le \theta \le \#\mathcal{N}$, $5 \le \alpha\kappa$ and $L < e^{c\theta\kappa}$, then $P_\pi(\text{dies out}) \ge 1 - e^{c\theta\kappa}$.*

(b) *Assume that $\mathcal{N} = B_\rho$ and $\theta \le \rho^2/4$. There is a constant C such that if $L \ge e^{C\rho^2\kappa}$, then $P_\pi(\text{survives}) \to 1$ as $\rho \to \infty$ or $\kappa \to \infty$.*

Beginning in Section 3 and for the remainder of the paper, we specialize to the Greenberg–Hastings model with $\mathcal{N} = \{z\colon \|z\|_1 = 1\}$ and $\theta = 1$. Let us call this nearest-neighbor threshold-1 case the *basic* GHM. The reasons for specializing are largely technical. Various dynamical invariants such as *open bonds*, the *index* of a loop and *defects*, notions that were originally formulated for the basic CCA in [8], apply equally well to the basic GHM. It follows that the basic GHM on $\mathbb{Z}^2$, started from π, is uniformly locally periodic with probability 1, that is, every site eventually advances around the color wheel at every time step (see [8]). Moreover, by exploiting these invariants one can carry the nucleation analysis to a deeper level. Part (a) of our next result is merely a special case of the previous theorem. However, in part (b) our description of the infinite system as observed from the origin more fully captures the metastability effect: $\mathbf{0}$ has color 0 by time κ, retains that color until time $e^{c\kappa}$ and then is repeatedly excited by a 1 after time $e^{C\kappa}$. [Presumably the time intervals in (b) can be replaced by $[e^{C\kappa} + (i-1)\kappa,\ e^{C\kappa} + i\kappa]$, that is, the origin becomes periodic by a time that is exponential in κ, but we do not know how to prove this.]

THEOREM 3. *Let γ_t be the basic κ-color GHM started from uniform product measure π. There are constants $0 < c \le C$ such that the following are true*:

(a) *Consider γ_t on B_L. Then as $\kappa \to \infty$,*

$$P_\pi(\text{dies out}) \to 1 \quad \text{if } L \le e^{c\kappa}$$
$$\to 0 \quad \text{if } L \ge e^{C\kappa}.$$

(b) *Consider γ_t on $\mathbb{Z}^2$. Then as $\kappa \to \infty$ the probability of the following event converges to* 1:

$$\gamma_t(\mathbf{0}) = 0 \quad \text{for } \kappa - 1 \le t \le e^{c_\kappa},$$

and then

$$\gamma_t(\mathbf{0}) = 1 \quad \text{for at least one } t \in \left[(i-1)e^{C\kappa}, ie^{C\kappa}\right];\ i = 1, 2, \dots .$$

Define c^* (resp. C^*) to be the supremum of all c's (resp. the infimum of all C's) for which the conclusion in Theorem 3(a) holds. It is quite plausible that $c^* = C^*$, but proving this equality seems beyond the reach of our methods. Instead, we settle for reasonably good rigorous bounds on c^* and C^*.

THEOREM 4. $0.177 < c^* \le C^* < 0.288$.

Although these numbers themselves do not convey much, our ability to estimate this closely the scaling rate for a complex effect in an interacting system entails some understanding of the underlying mechanism. The derivation of our upper bound is particularly illuminating, involving dynamic defect formation and an interesting comparison with a nonhomogeneous oriented percolation problem.

Section 3 concludes with a brief discussion of our CAM-based numerical estimation of the ostensible common value $c^* = C^*$. The data supports a best guess of

$$c^* = C^* \approx 0.223 \pm 0.01.$$

In Section 4 we turn to the issue of κ-color GHM nucleation starting from product measures other than primordial soup. Under π, an enormous finite volume is needed to support spiral formation once κ is large, so it is natural to ask whether some other homogeneous random medium is more conducive to nucleation, and if so, which medium is optimal. We feel that such questions are fundamental to the subject of CA self-organization, and that probability theory will play a crucial role in resolving them. Let μ denote a product density with $P_\mu(\gamma_0(x) = k) = \mu_k$, $0 \le k \le \kappa - 1$, on $\mathbb{Z}^2$ or B_L according to context. We first show that γ_t is almost surely locally periodic on $\mathbb{Z}^2$ if and only if μ attaches positive density to 0, 1 and at least one additional color. Then we indicate that the nucleation scaling is still exponential in κ for the great preponderance of initial product measures μ. However, certain special choices of the densities (μ_k) can lead to power-law scaling. The main result of Section 4 (Theorem 7) asserts that if P_μ is concentrated on $\{0,1,2\}$, with

$$\mu_0 \ge p_c - \frac{1}{\kappa^3}, \qquad \mu_2 \ge 1 - p_c - \frac{1}{\kappa^3} \quad \text{and} \quad \mu_1 \ge \frac{1}{\kappa^\lambda} \quad \text{for some } \lambda > 0,$$

where $p_c \approx 0.59$ is the critical value for site percolation on $\mathbb{Z}^2$, then for a suitable $\nu > 0$, the GHM on B_{κ^ν} satisfies

$$P_\mu(\gamma_t \text{ survives}) \to 1 \quad \text{as } \kappa \to \infty.$$

We illustrate this effect by growing a 170 color spiral pair on a 400×400 box.

Last, in Section 5 we turn to a variant of the GHM with random dynamics. The random Greenberg–Hastings model (RGHM) $\tilde{\gamma}_t$ is a nearest neighbor continuous-time Markovian interaction on $\{0,\dots,\kappa-1\}^{\mathbb{Z}^2}$. Besides κ, there is an additional parameter β that determines the system's generator. Let $\tilde{\gamma}_t(x) = k$. If $k \ge 1$, then κ changes to $(k+1) \bmod \kappa$ at exponential rate 1, whereas if $k = 0$ a change to 1 occurs at rate $\beta \cdot \#\{y \in \mathscr{N}_x\colon \tilde{\gamma}_t(x) = 1\}$. This process is closely related to the epidemic with regrowth of Durrett and Neuhauser [5]. We are able to apply results from that paper and the earlier work of Cox and Durrett [2] to obtain counterparts to our previous GHM results for the RGHM on exponential boxes $B_{e^{c\kappa}}$ with κ large. Detailed discussion and formulation of these theorems is deferred to Section 5. The point we want to stress is that many of the Markov processes commonly known as interacting particle systems, in particular those that self-organize by means of wave propagation, can profitably be viewed as perturbations of deterministic CA dynamics started from random initial states.

2. A metastability result for Greenberg–Hastings models with general threshold and range. In this section, we consider Greenberg–Hastings models with neighbor set $\mathscr{N}$, threshold θ and κ colors. We prove Theorems 1 and 2 of the Introduction. Our first theorem establishes global death of either finite or infinite systems, started from any initial configuration, when $\mathscr{N}$ is symmetric and $\theta > \#\mathscr{N}/2$.

PROOF OF THEOREM 1. Abbreviate $n = \#\mathscr{N}$ and let $\rho = \max\{\|z\|_\infty\colon z \in \mathscr{N}\}$ be the range of $\mathscr{N}$. For the purpose of this argument, a (directed) *edge* at time t connects a 1 at some $z \in \mathbb{Z}^2$ to a 0 in $z + \mathscr{N}$. Fix a large t and note that if $\gamma_t(x) = 1$, then there must have been at least θ edges to x at time $t-1$, hence at least θ 1's within range ρ of x at that time. So by symmetry of $\mathscr{N}$, there can be at most $n - \theta$ edges from x at time t. More generally, if $x_1,\dots,x_{l_j}$ are distinct sites with color 1 at time $t-j$, then there are at least $l_j\theta$ edges to these sites at time $t-j-1$, and hence, because each 1 can contribute at most $n-\theta$ edges, there must be at least $[\theta/(n-\theta)]l_j$ 1's in $\bigcup_{i=1}^{l_j} \mathscr{N}_{x_i}$ at time $t-j-1$. Proceeding by induction on j, if $\gamma_t(x) = 1$, then at time $t-j$ there must be at least $\theta^j/(n-\theta)^{j-1}$ 1's within range $j\rho$ of x.

Because $\theta > n/2$, we have $\theta/(n-\theta) > 1$. Thus $\theta[\theta/(n-\theta)]^{j-1}$ grows exponentially, whereas the number of sites within $j\rho$ of x grows quadratically. Choose T so that $\theta[\theta/(n-\theta)]^T > (2\rho T + 1)^2$. There cannot possibly be enough 1's close enough to x in the initial configuration to generate a 1 at x at time T, so $\gamma_{T+\kappa} \equiv 0$ regardless of γ_0. □

Our second theorem deals with the GHM on a finite box $B_L \subset \mathbb{Z}^2$ starting from primordial soup π: $P_\pi(\gamma_0(x) = k) = 1/\kappa$ for $k = 0, \ldots, \kappa - 1$. For relatively small values of θ we show that with high probability, eventually in time, γ_t fixates at all 0's if L is too small but fluctuates if L is sufficiently large. The cutoff occurs for L that grow exponentially in the parameter κ.

PROOF OF THEOREM 2. (a) Suppose $\gamma_{\kappa-1}(x) = 1$. Such an x must have θ 2's in its neighborhood, each such 2 having θ 3's in its neighborhood, and so on. Hence, setting $x_{\kappa-2} = x$ at time $\kappa - 2$ there exists a sequence of sites x_j, $j = \kappa - 2, \kappa - 3, \ldots, 0$, such that x_j has at least θ neighboring sites of color $\kappa - j - 1$, one of which is x_{j-1} (this last for $j \geq 1$). If A_j denotes the sites that x_j "sees" with color $\kappa - j - 1$ at time $\kappa - 2$, then the A_j are clearly disjoint. Moreover, for $j \geq 1$ we can select $x_{j-1} \in A_j$ in a prescribed fashion, for example, the site in A_j that is closest to $\mathbf{0}$ with some convention in case of ties. Translated into a condition at time 0, this means that x_0 sees θ 1's on A_0, x_1 sees θ 0's on $A_1 \ni x_0$ and, for $j = 2, \ldots, \kappa - 2$, x_j sees θ sites in $I_j = \{0, \kappa - 1, \ldots, \kappa - j + 1\}$ on $A_j \ni x_{j-1}$.

Fix j and a site x, and write $n = \#\mathscr{N}$. Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli random variables with $P(X_1 = 1) = j/\kappa$ ($j = \#I_j$). Applying standard large deviations estimation to the time 0 condition of the last paragraph, we see that

$$P_\pi\big(\text{at least } \theta \text{ sites of } \gamma_0(x + (\mathscr{N} \setminus \{0\})) \text{ have a color in } I_j\big)$$

$$\leq P(X_1 + X_2 + \cdots + X_n \geq \theta) \leq e^{-\lambda\theta}\big(E[e^{\lambda X_1}]\big)^n = e^{-\lambda\theta}\left(\frac{\kappa - j + e^\lambda j}{\kappa}\right)^n,$$

for each $\lambda > 0$. Assuming that $j \leq \theta\kappa/n$ and choosing $e^\lambda = \theta(\kappa/j - 1)/(n - \theta)$ to minimize this last expression, we obtain

$$\left(\frac{n-\theta}{\theta}\right)^\theta \left(\frac{n}{n-\theta}\right)^n \left(\frac{j}{\kappa - j}\right)^\theta \left(\frac{\kappa - j}{\kappa}\right)^n \leq e^\theta \left(\frac{n}{\theta}\right)^\theta \left(\frac{j}{\kappa - j}\right)^\theta.$$

Fix $\beta \in (0, \alpha)$ such that $\beta\kappa \geq 1$. Because $\alpha n \leq \theta$, it follows from the criterion of the previous paragraph, in terms of disjoint A_j, that

$$\begin{aligned} &P_\pi(\gamma_{\kappa-1}(x) = 1 \text{ for some } x \in B_L) \\ &\quad\leq (2L+1)^2 e^{\theta\beta\kappa}\left(\frac{n}{\theta}\right)^{\theta\beta\kappa}\left(\frac{1 \cdot \cdots \cdot \beta\kappa}{[(1-\beta)\kappa]^{\beta\kappa}}\right)^\theta \qquad (2.1)\\ &\quad\leq (2L+1)^2\left(\frac{en\beta}{\theta(1-\beta)}\right)^{\theta\beta\kappa}. \end{aligned}$$

Set $\beta = \alpha/4$ to get

$$P_\pi(\gamma_{\kappa-1}(x) = 1 \text{ for some } x \in B_L) \le (2L+1)^2 \left(\frac{e}{4-\alpha}\right)^{\theta\alpha\kappa/4}.$$

Because $\alpha \le \frac{1}{2}$, the preceding probability is majorized by $(2L+1)^2 e^{-\alpha\kappa\theta/16}$. Thus part (a) of the theorem is proved, with $c = \alpha/50$.

(b) We paint a suitable ring of blocks so that it inevitably creates an spo at about time κ. For any fixed κ one can exhibit a range ρ spo that occurs in the initial state with probability $\exp\{-c\rho^2 \kappa \log \kappa\}$ (cf. [9]). Hence it suffices to establish prevalent spo formation for large κ. Arrange 4κ $a \times a$ squares in a "square ring," and denote them by $Q_0, \dots, Q_{4\kappa-1}$, starting from the upper left corner of the ring and proceeding clockwise. One-quarter of the arrangement is shown in the array in the next paragraph. Take a to be the largest integer such that $a(\rho - a + 1) \ge \theta$, that is, $a = \lfloor (\rho + 1 + \sqrt{(\rho+1)^2 - 4\theta})/2 \rfloor$. (It is here that we need $\theta \le \rho^2/4$.) Then $(a+1)(\rho - a) \le \theta - 1$, and hence $a(\rho - a) \le \theta - 1$. Color the squares so that all of Q_0 has color 1, all of Q_1 has color 0 and all sites in Q_j, $j = 2, \dots, \kappa - 2$, have colors from the set $I_j = \{0, \kappa - 1, \dots, \kappa - j + 1\}$, whereas $Q_{\kappa-1}$ is colored entirely with 2's. Paint the remaining squares so that the colors of sites in Q_j satisfy the same requirements as colors in $Q_{j \bmod \kappa}$. The value of a is chosen so that, for example, Q_0 (entirely 1's) will infect every point in Q_1 (0's) but will not infect any point in Q_2. Hence 1's travel from full block to full block, reaching $Q_{\kappa-3}$ at time $t = \kappa - 3$, and leaving a "trail" of blocks colored $2, 3, \dots, \kappa - 2$. Also at this time, $Q_{\kappa-2}$ has all 0's and the 2's initially on $Q_{\kappa-1}$ turn to $\kappa - 1$'s.

If sites outside $\bigcup_j Q_j$ are removed, any of the specified colorings, therefore, produces an spo at time $\kappa - 3$. How can we ensure that sites from outside the ring do not interfere with this formation? In the following scheme, every letter stands for an $a \times a$ block:

$$\begin{array}{cccccccccc}
R_0 & R_0 & R_0 & R_0 & R_0 & R_0 & R_4 & R_5 & & R_{\kappa-4} \\
R_0 & R_0 & R_0 & R_0 & R_0 & R_0 & R_4 & R_5 & & R_{k-4} \\
R_0 & R_0 & Q_0 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & & Q_{\kappa-4} \\
R_0 & R_0 & Q_{4\kappa-1} & R_0 & R_0 & R_0 & R_4 & R_5 & \cdots & R_{\kappa-4} \\
R_0 & R_0 & Q_{4\kappa-2} & R_0 & R_0 & R_0 & R_4 & R_5 & & R_{\kappa-4} \\
R_0 & R_0 & Q_{4\kappa-3} & R_0 & R_0 & & & & &
\end{array}$$

Specifying conditions on the colors in $R_0, R_4, \dots, R_{\kappa-4}$ is enough: Other requirements are determined by 90° rotation symmetry. So set all colors in R_0 equal to 2, and require that R_j should not include any color from $\{1, 0, \kappa-1, \kappa-2, \dots, \kappa-j-1\}$, $j = 5, \dots, \kappa-3$. Let $\Lambda = \bigcup_j R_j \cup \bigcup_j Q_j$. It is not hard to check that if there is no 1 anywhere outside Λ, then an spo is created at time $\kappa - 3$. (Colors for the R's are chosen so that they cannot possibly be excited during this time period: they may contain 0's only after the 1's have already moved past.)

Hence we only need to ensure that no 1 from outside Λ will affect the dynamics in Λ. This is guaranteed if there is no 1 within l^∞ distance $\rho\kappa$ of Λ, an event with probability at least

$$\left(1 - \frac{1}{\kappa}\right)^{9\rho^2\kappa^2} \geq e^{-18\rho^2\kappa},$$

because $a \leq \rho$. On the other hand, our specification on Λ happens with probability at least

$$\left(\frac{1}{\kappa}\right)^{150a^2}\left(\frac{\kappa!}{\kappa^\kappa}\right)^{20a^2} \geq e^{-25\rho^2\kappa},$$

provided κ is large enough. Now choose $C = 25$ and subdivide B_L into squares of size $3\rho\kappa \times 3\rho\kappa$. The probability that the just described scenario occurs in at least one of these boxes is at least

$$1 - \left(1 - e^{-43\rho^2\kappa}\right)^{L^2/(3\rho\kappa)^2} \geq 1 - \exp\left\{-e^{7\rho^2\kappa}/(3\rho\kappa)^2\right\},$$

which clearly converges to 1 as $\rho\kappa \to \infty$. □

3. Metastability in the nearest neighbor GHM started from primordial soup. For the remainder of the paper, let us focus on the basic Greenberg–Hastings model γ_t with $\mathscr{N} = \{z\colon \|z\|_1 = 1\}$, $\theta = 1$ and $\kappa \geq 3$. Throughout this section the dynamics will start from uniform product measure π: $P_\pi(\gamma_0(x) = i) \equiv 1/\kappa$. We begin by proving Theorem 3. The overall strategy is the same as for Theorem 2, but dynamic invariants of the nearest-neighbor case, first identified in [8], enable us to carry out a somewhat more detailed analysis. A *clock* (cf. [8]) in γ_t is a loop $z_0, z_1, \dots, z_l = z_0$ in $\mathbb{Z}^2$ such that $\gamma_t(z_{i+1}) - \gamma_t(z_i) = 1 \bmod \kappa$ for $i = 0, 1, \dots, l-1$. Note that for the basic model a clock is a particularly simple variety of spo; in particular, clocks are unaffected by the external dynamics.

PROOF OF THEOREM 3. Excitation propagates along nearest-neighbor self-avoiding paths in the basic GHM. The number of such paths of length k is clearly at most $4 \cdot 3^{k-1}$, so a simpler version of the argument leading to (2.1) yields the estimate

$$P_\pi(\gamma_k(x) = 1 \text{ for some } x \in B_L) \leq (2L+1)^2\, 4 \cdot 3^k k!\, \kappa^{-k},$$

for $k \leq \kappa - 1$, on B_L. Choosing $k = \kappa/3$, the first claim of (a) follows for $c < \frac{1}{6}$. The second claim of (a) is essentially a special case of Theorem 2(b) (take $a = 1$ in that construction).

To verify the first claim of (b), observe that for κ large,

$$\begin{aligned}
&P_\pi(\gamma_t(\mathbf{0}) = 1 \text{ for some } t \in [\kappa - 1, e^{c\kappa}])\\
&\quad\leq P_\pi(\gamma_0(x) = 1 \text{ for some } x \in B_4)\\
&\qquad + P_\pi(\gamma_t \text{ on } B_\kappa \text{ has a 1 that survives for 3 time steps})\\
&\qquad + P_\pi(\gamma_t \text{ on } B_{e^{c\kappa}} \text{ has a 1 that survives for } \kappa/3 \text{ time steps}).
\end{aligned}$$

The first two probabilities on the right are $O(1/\kappa)$, and we have just seen that the last is exponentially small. Finally, we need to establish the second claim of (b). By the construction for the second part of (a), with probability tending to 1, there is a clock within a distance of $\mathbf{0}$ that is at most exponential in κ. Hence, to finish the proof, we now show that $\mathbf{0}$ is visited at least once during each block of time $[(i-1)e^{C\kappa}, ie^{C\kappa}]$, $i = 1, 2, \dots,$ for some C.

Suppose that x is connected to y via a shortest path $\wp$ of length k. Assume, moreover, that this path consists of all 0's, except that the color of y is 1 (at $t = 0$, say). Then x must have color 1 sometime during the time period $[0, k]$. This fact is easy to check by induction on k: If $k = 1$ it is obvious; otherwise, consider the closest 1 on the path at time $t = 1$ (which must be closer than k).

Now consider a shortest path $\wp$ from the site $x = \mathbf{0}$ to a clock at time t, assuming its length to be $k \le e^{C\kappa}$. Assume also that y is the farthest site from $\mathbf{0}$ on $\wp$ such that the part of $\wp$ between $\mathbf{0}$ and y consists of 0's. Let z be the next site beyond y on $\wp$. If no site on the path from $\mathbf{0}$ to y (including $\mathbf{0}$ and y) becomes 1 during the time interval $[t, t+\kappa]$, then at time $t + \kappa - 1$ the path from $\mathbf{0}$ to z must consist of 0's. Hence, according to the preceding paragraph, if $\mathbf{0}$ does not change for $(\kappa - 1)(k - 1) + \kappa$ time units, it must at time $(\kappa - 1)k$ be connected to the clock by a path of 0's. However, then it must become 1 during the next $\kappa + k$ time units (by the preceding paragraph once more). Hence, if k is very much larger than κ, $\mathbf{0}$ cannot stay unchanged for a period of $2k^2$ time units. After a suitable redefinition of C, the proof is complete. □

We now turn to the numerical bounds of Theorem 4 for the exponential rate of nucleation in the basic κ-color GHM as $\kappa \to \infty$. Our principal tool is an analysis of the following oriented percolation problem (see, e.g. [3]). Let ϕ: $(0, \infty) \to (0, 1]$ be a continuous function with the property that

$$\limsup_{p \to 0} \frac{-\log \phi(p)}{\psi(p)} < \infty,$$

where ψ is a positive nonincreasing function on $(0, \infty)$, integrable in a neighborhood of 0 and such that $p\psi(p) \to 0$ as $p \to 0$. This implies that for each finite $\alpha > 0$, there exist $\delta > 0$ and $K < \infty$ such that

$$-\log \phi(p) \le K\psi(p \wedge \delta) \quad \text{for } p \in (0, \alpha].$$

On the oriented percolation lattice $\mathscr{L} = \{(x, y) \in \mathbb{Z}^2\colon\ x, y \ge 0\}$ make site (x, y) *open* with probability $\phi((x+y)/\kappa)$, where κ is a fixed positive integer. We call this model *ϕ-oriented site percolation* (ϕ-OSP). The case $\phi(r) = r \wedge 1$ is called *linear oriented site percolation* (LOSP). Thus, in LOSP the probability that a site (x, y) is open increases linearly with its level $x + y$ until it reaches probability 1. Denote by $P(\kappa, k)$ the probability that $\mathbf{0}$ is connected to some site at level k in ϕ-OSP.

One can define the GHM on $\mathscr{L}$: Excitation is only transmitted along the directed edges of $\mathscr{L}$. Starting from a single 1 at $\mathbf{0}$, surrounded by all 0's, the probability that the excitation will persist up to time k is the same as the probability of an open path to level k in LOSP. Moreover, it is clear that if a 1 survives in the GHM on $\mathscr{L}$, then (under the natural coupling) it also survives in the basic GHM on $\mathbb{Z}^2$. These are the reasons that nonhomogeneous oriented percolation is of interest to us.

The abbreviation OSP will refer to standard (homogeneous) oriented site percolation (cf. [3]). For a fixed positive integer m and $p \in [0,1]$, let $Q(p,m)$ be the probability that OSP with density p of open sites, started from a single occupied site at $\mathbf{0}$, percolates up to level m. With p fixed, the sequence $-\log Q(p,m)$ is subadditive, and therefore $\lim_{m\to\infty}(1/m)(-\log Q(p,m))$ exists and is equal to $\inf_m(1/m)(-\log Q(p,m))$. We denote this limit by $\Phi(p)$. Also, let $p_c^+ = \inf\{p\colon Q(p,\infty) > 0\}$ be the critical density for OSP.

Our immediate goal is to prove that for any $\alpha \in (0,\infty)$, there exists a $g(\alpha) > 0$ such that $P(\kappa, \alpha\kappa) = \exp\{-g(\alpha)\kappa + o(\kappa)\}$, and to identify $g(\alpha)$ *exactly* in terms of ϕ and Φ. This is done in the next proposition. First, though, we need a preliminary result about subcritical OSP. Let E_k denote the expectation operator corresponding to the probability measure $P_k = P(\cdot \mid \text{percolation to level } k)$.

LEMMA 1. *Consider OSP with density of open sites $p < p_c^+$. Let k be any positive integer. There exists an $M > 0$, independent of k, such that $E_k[\#\,\{\text{sites at level } k \text{ connected to } \mathbf{0}\}] \le M$.*

PROOF. Fix a large k and assume that $\wp$ is the highest path connecting $\mathbf{0}$ to level k. Note that this conditioning does not influence the probability that a site below $\wp$ is open. Because $p < p_c^+$ there is an $a > 0$ such that $Q(p,m) \le e^{-am}$ for large m (see [15]). Let y_0 be the second coordinate of the last site (the one at level k) of $\wp$. If there are at least m sites at level k connected to $\mathbf{0}$, given $\wp$, then there is an open connection between segments $\{(k - y_0, y);\ 0 \le y \le y_0 - m + 1\}$ and $\{(k - y, y);\ 0 \le y \le y_0 - m + 1\}$. Hence, for $y_0 \ge m$,

$$P_k(\#\,\{\text{sites at level } k \text{ connected to } \mathbf{0}\} \ge m)$$
$$\le \sum_{i=0}^{y_0 - m} e^{-a(y_0 - i)} \le e^{-am} \sum_{i=0}^{\infty} e^{-ai} =: e^{-am} \cdot \Sigma.$$

Because this probability is 0 for $y_0 < m$, we can take $M = \Sigma^2$. $\square$

The exact asymptotics for $P(\kappa, \alpha\kappa)$ are as follows.

PROPOSITION 1. *For each $\alpha > 0$,*

$$\lim_{\kappa\to\infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) = -\int_0^{\alpha} \Phi(\phi(p))\, dp. \tag{3.1}$$

REMARK. $\Phi(p) = 0$ for $p \geq p_c^+$, so one may just as well integrate over $[0, \alpha] \cap \{\phi < p_c^+\}$. In particular, the proposition implies that for LOSP,

$$\lim_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) = -\int_0^{\alpha \wedge p_c^+} \Phi(p)\, dp. \tag{3.2}$$

PROOF OF PROPOSITION 1. Fix a positive integer m, write $u_1 = \inf\{\phi(x): 1/\kappa \leq x \leq \alpha\}$ and define

$$I_i = \left[\frac{m(i-1)}{\kappa}, \frac{mi}{\kappa}\right], \qquad i \geq 1,$$

$$u_i = \inf\{\phi(x): x \in I_i\}, \qquad i \geq 2$$

and

$$U_i = \sup\{\phi(x): x \in I_i\}, \qquad i \geq 1.$$

We first prove the "$\geq$" half of (3.1). Conditioned on the event that **0** percolates to level $m(i-1)$, the probability that **0** percolates to level mi is at least $Q(u_i, m)$. It follows that $P(\kappa, k) \geq \prod_{i=1}^{k/m} Q(u_i, m)$. (The diligent reader should replace k/m by $\lfloor k/m \rfloor + 1$ here, and make similar adjustments in succeeding text.) Therefore,

$$\begin{aligned}
&\liminf_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) \\
&\geq \liminf_{\kappa \to \infty} \frac{1}{\kappa} \sum_{i=1}^{\alpha\kappa/m} \log Q(u_i, m) \\
&= \liminf_{\kappa \to \infty} \left(\frac{1}{\kappa} \log Q(u_1, m) + \frac{1}{m} \frac{m}{\kappa} \sum_{i=2}^{\alpha\kappa/m} \log Q(u_i, m) \right) \\
&= \frac{1}{m} \int_0^\alpha \log Q(\phi(p), m)\, dp.
\end{aligned} \tag{3.3}$$

To justify the last equality in (3.3), let

$$R_\kappa(p, m) = -\frac{1}{m} \log Q(u_i, m) \quad \text{if } p \in I_{i-1}, i \geq 2.$$

It is clear that $R_\kappa(p, m) \to -(1/m)\log Q(\phi(p), m)$ for each $p \in (0, \alpha]$ as $\kappa \to \infty$. Moreover, because

$$\frac{1}{m}(-\log Q(p, m)) \leq -\log Q(p, 1) = -\log(2p - p^2) \leq -\log p$$

for each m and p, it follows that $R_\kappa(p, m) \leq K\psi(p \wedge \delta)$. Now the fact that $-\log Q(u_1, m) \leq -m \log u_1 \leq mK\psi(1/\kappa)$ and bounded convergence yield (3.3). Sending $m \to \infty$ in the last expression of (3.3), bounded convergence also lets us conclude that

$$\liminf_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) \geq -\int_0^\alpha \Phi(\phi(p))\, dp.$$

Now we prove the " $\le$ " half of (3.1). Assume first that $\phi < p_c^+ - \varepsilon$ for some $\varepsilon > 0$. Denote by S_k the event that the ϕ-OSP survives up to level k. Then

$$
\begin{aligned}
P\big(S_{mi} | S_{m(i-1)}\big) & \\
\le \sum_{j=1}^{m(i-1)+1} & \Big[P\big(\#\{\text{sites at level } m(i-1) \text{ connected to } \mathbf{0}\} = j | S_{m(i-1)}\big) \\
& \qquad \times j\,Q(U_i, m)\Big] \\
= E\big[\#\{&\text{sites at level } m(i-1) \text{ connected to } \mathbf{0}\} | S_{m(i-1)}\big] Q(U_i, m).
\end{aligned}
$$

Using Lemma 1, we conclude that $P(\kappa, \alpha\kappa) \le \prod_{i=1}^{\alpha\kappa/m} MQ(U_i, m)$ and, therefore,

$$
\begin{aligned}
\limsup_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) &\le \frac{\alpha}{m} \log M + \limsup_{\kappa \to \infty} \frac{1}{\kappa} \sum_{i=1}^{\alpha\kappa/m} \log Q(U_i, m) \\
&= \frac{\alpha}{m} \log M + \frac{1}{m} \int_0^\alpha \log Q(\phi(p), m)\, dp
\end{aligned}
$$

for each m. Letting $m \to \infty$, we get

$$
\limsup_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) \le -\int_0^\alpha \Phi(\phi(p))\, dp.
$$

It follows that for each $\varepsilon > 0$,

$$
\limsup_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \alpha\kappa) \le -\int_0^\alpha \Phi(\phi(p)) 1_{\{\phi(p) < p_c^+ - \varepsilon\}}\, dp.
$$

Let $\varepsilon \to 0$ to finish the proof. □

Let us now derive the numerical bounds of Theorem 4 for the exponential rate of nucleation in the basic GHM as $\kappa \to \infty$. Let σ be the connectivity constant [17] for $\mathbb{Z}^2$ and let p_c be the critical value for two-dimensional *unoriented* site percolation [16]. We claim that the following lower and upper estimates hold:

$$
c^* \ge \frac{1}{2\sigma}, \tag{3.4}
$$

$$
C^* \le \int(\Phi) := \int_0^{p_c} \Phi(p)\, dp. \tag{3.5}
$$

To get (3.4), one merely refines the estimation for the first half of Theorem 3(a): Standard subadditivity arguments imply that the number of self-avoiding paths of length n grows like σ^n (at the exponential level), so the 3 in our previous argument can be replaced by σ, yielding a lower bound of $1/2\sigma$ instead of $1/6$. The derivation of (3.5) is more involved, relying on Proposition 1 in the LOSP case.

Before proceeding to the proof of (3.5), let us show the concrete numerical estimates that (3.4) and (3.5) provide. If one computes the number of self-avoiding walks of length n, then the nth root of this number is an upper bound for σ. This is done for $n = 18$ in [17], giving $\sigma < 2.8165$, and hence $c > 0.177$. Actually, σ is believed to be about 2.7, which yields a nonrigorous lower bound of 0.185.

Subadditivity helps in applying (3.5) as well. It implies that

$$C^* < \frac{1}{2m}\int_0^{p_c}(-\log Q(p,m))\,dp.$$

For $m = 1$ this gives $C^* < p_c - \frac{1}{2}p_c \log p_c + \frac{1}{2}(2 - p_c)\log(2 - p_c) - \log 2$. The widely believed estimate $p_c \approx 0.5927$ yields a nonrigorous upper bound of 0.295. Replacing p_c by 1 does not hurt much, giving a rigorous bound of $1 - \log 2 < 0.307$. A larger m improves the estimate, but involves considerably more work. Numerical results for $p_c = 0.5927$ and $m = 2$, 3 and 4 give nonrigorous upper bounds for C^* of 0.289, 0.286 and 0.284, respectively. With $p_c = 1$, the same m's give 0.299, 0.294 and 0.291, whereas with $p_c = 0.6819$ (the best known rigorous upper bound, due to Zuev [27]) we get 0.295, 0.291 and 0.288. This last rigorous upper estimate is the one that appears in Theorem 4.

In light of the preceding discussion, Theorem 4 will be proved once we derive (3.5). Our argument relies on the notions of *open loop*, *index* and *defect* from [8]. A nearest-neighbor bond in $\mathbb{Z}^2$ is open at time t if the color increment across that bond in γ_t equals 0 or ± 1 (mod κ). An open loop at time t is a loop in $\mathbb{Z}^2$ that consists entirely of open bonds in γ_t. Straightforward case checking shows that open bonds, and hence open loops, cannot be destroyed by basic GHM dynamics. Now suppose that $l = (z_0, z_1, \dots, z_n = z_0)$ is an open loop. For $j = 1, \dots, n-1$, call $[j, j+1]$ $(n = 0)$ a clockwise (resp. counterclockwise) edge if z_j has color 0 (resp. 1) and z_{j+1} has color 1 (resp. 0). The index of l is the difference between the number of counterclockwise and clockwise edges. This index is also invariant under GHM dynamics, as can be seen by checking that edges are only created or annihilated in pairs of opposite variety. See [8] for more details and an illustration. (Although that paper deals with the basic CCA, all invariants identified there are shared by the basic GHM.) A *defect* is an open loop with nonzero index. Because any defect contains at least one site with color 1, index invariance implies that GHM dynamics cannot fixate once a defect is formed. We note in passing that CCA nucleation is much more complex; partial results in [12] indicate superexponential scaling.

Before proceeding to the proof of Theorem 4, let us state a simple lemma that plays an important role and will also be applied later in the paper. In essence it states that if, along a path, a "leading 1" sees 0's all the way to the edge of box B_L and if the boundary of B_L is suitably isolated from 1's, then later on some leading 1 (not necessarily the same) must also see all 0's on the path to the boundary. We omit the quick and easy proof by induction.

LEMMA 2. *Consider the GHM on B_L. For $0 \le t < u$, assume that at time t there is a self-avoiding path $z_0, z_1, \dots, z_k$ to the boundary of B_L such that $\gamma_t(z_0) = 1$ and $\gamma_t(z_i) = 0$ otherwise. Assume also that there are no additional 1's within l^1-distance $u - t$ of the boundary of B_L. Then there is a $j = j(u)$ such that $\gamma_u(z_j) = 1$ and $\gamma_u(z_i) = 0$ for $i > j$.*

PROOF OF THEOREM 4. Roughly speaking, our strategy is to exhibit a defect on $B_{4\kappa}$ by time κ with probability at least $\exp\{-\int(\Phi)\kappa\}$. Because defects cannot be destroyed and must include at least one excited site, this precludes death of γ_t. Then (3.5) is easy. A defect is certainly formed if a 1, starting at $\mathbf{0}$, survives on a path $\wp$ of size κ and then at time $\kappa - 1$ can be connected through 0's back to $\mathbf{0}$. However, survival until time $t_c = (p_c + \varepsilon)\kappa$ should suffice, for some $\varepsilon > 0$, because after t_c a 1 should have no trouble percolating back to its original position through sites that have already become 0's. As we shall see, Proposition 1 yields a lower bound on the probability of such a "good" percolating path $\wp$.

We will need a few facts from the theory of unoriented site percolation. The first is that $p_c > \frac{1}{2}$ (due to Higuchi [18]); hence we can choose $\delta > 0$ so that $\frac{1}{2} + \delta < p_c$. The second is that for $p > p_c$, $\mathbf{0}$ percolates with positive probability in the quarter space $Q = \{(x, y): x \ge 0, y \ge 0\}$ (a consequence of Russo's fundamental work [22]). The third is that for any $p < p_c$ there is an $a > 0$ such that $P(\mathbf{0}$ is connected to n sites$) \le e^{-an}$ for any $n \ge 1$ (the proof in [16] for bond percolation also applies to the site model). We will use this last fact for $p = \frac{1}{2} + \delta$, with $n = 4(\log \kappa / a)$.

Fix $\varepsilon > 0$ and let $k = \lfloor (p_c + \varepsilon)\kappa \rfloor$. Call a nearest-neighbor path $\wp$ in $\mathbb{Z}^2$ *fast* if it goes either up or to the right at each step. Fix a fast path $\wp$: $\mathbf{0} = z_0, z_1, \dots, z_k$ ($\|z_k\|_1 = k$). We now define several "good" events, each referring only to the configuration of γ_0 on $B_{5\kappa}$ (for the sake of clarity, we sometimes refer to colors 0 and 1 as κ and $\kappa + 1$, respectively):

$$G_1 = \{\gamma_0(z_i) \in \{\kappa + 1, \kappa, \kappa - 1, \dots, \kappa - i + 1\} \text{ and } \wp \text{ is the lowest fast path with this property}\},$$

$$G_2 = \{\gamma_0(x) \ne \kappa + 1 \text{ for any } x \in Q \setminus \{\mathbf{0}\} \text{ with } \|x\|_1 \le k \text{ which is at } \|\cdot\|_1\text{-distance at most } 2n \text{ from } \wp \text{ (this includes sites on } \wp)\},$$

$$G_3 = \{\text{in } (Q \cap \{\|x\|_1 \le k\}) \setminus \wp \text{ there is no path of } n \text{ sites with colors in } \{\kappa + 1, \kappa, \kappa - 1, \dots, \kappa - (\tfrac{1}{2} + \delta)\kappa\}\},$$

$$G_4 = \{\text{site } z_k + e_1 \text{ has color } \kappa - k, \text{ and all sites in } \{x \ge 0, y \ge 0, x + y = k + 1\} \cup \{x = -1, -1 \le y \le k + 1\} \cup \{-1 \le x \le k + 1, y = -1\} \text{ have color outside } \{\kappa + 1, \kappa, \kappa - 1, \dots, \kappa - 2n\}\},$$

$$
\begin{aligned}
G_5 = \{&\text{there is a path } \wp' \text{ from } z_k + e_1 \text{ to a site outside } B_{2\kappa},\\
&\text{which stays inside } z_k + e_1 + Q,\\
&\text{and such that } \gamma(\wp') \subset \{\kappa - k, \kappa - k + 1, \dots, \kappa - 1\}\},\\
G_6 = \{&\text{no 00, 01 or 11 neighbor pairs outside}\\
&\{x \geq -1, y \geq -1, x + y \leq k + 1\}\},\\
G = G_1 &\cap \cdots \cap G_6.
\end{aligned}
$$

Now suppose that G happens. At time k there is a 1 at z_k that is connected via a path of 0's to the outside of $B_{2\kappa}$. (G_1–G_4 and G_6 ensure that the 1 moves on $\wp$ without colliding with another 1 in the process, whereas G_5 guarantees the path of 0's.) By Lemma 2 (with $t = k$, $u = \kappa - 1$) the last 1 on $\wp'$ at time $\kappa - 1$, w say, is connected to a site outside $B_{2\kappa}$. Moreover, $\mathbf{0}$ is connected to w by a path with colors 0 (at $\mathbf{0}$), $\kappa - 1$, $\kappa - 2, \dots, 1$ (at w). All sites outside $B_{2\kappa}$ have color 0 or $\kappa - 1$ (by G_6). We claim that the same is true for sites in $-Q$. This is so because at time $(\frac{1}{2} + \delta)\kappa$ all the 1's in $B_{5\kappa}$ lie in $\{x + y \geq (\frac{1}{2} + \delta)\kappa - 2n\}$ (by G_3, G_4 and the fact that the 1 on $\wp$ is moving further away from $\mathbf{0}$ at each time step); hence they are too far away to reach $-Q$ by time $\kappa - 1$. We conclude that w is connected to $\mathbf{0}$ by a path that consists entirely of 0's and $\kappa -$ 1's at time $\kappa - 1$. By concatenation we obtain an open loop through $\mathbf{0}$ with index 1, that is, the desired defect.

We need to bound $P_\pi(G)$ from below. Clearly

$$P_\pi(G) = P_\pi(G_1 \cap G_2 \cap G_3) P_\pi(G_4) P_\pi(G_5 \cap G_6).$$

The easiest event to control is G_4:

$$P_\pi(G_4) \geq \frac{1}{\kappa}\left(1 - \frac{2n + 2}{\kappa}\right)^{3\kappa},$$

which, because n is of order $\log \kappa$, goes to 0 as a power of κ. Next, write $G_2 = G_2' \cap G_2''$, where

$$G_2' = \{\text{no } \kappa + 1 \text{ on } \wp\}, \quad G_2'' = \{\text{no } \kappa + 1 \text{ off } \wp \text{ within } \|\cdot\|_1\text{-distance } 2n \text{ of } \wp\}.$$

Then $P_\pi(G_1 \cap G_2 \cap G_3) = P_\pi(G_1) P_\pi(G_2' | G_1) P_\pi(G_2'' \cap G_3 | G_1)$. It is clear that $P_\pi(G_2' | G_1) \geq 1/\kappa$. Now it is in estimating $P_\pi(G_2'' \cap G_3 | G_1)$ that we find it helpful to think of 0 as κ and 1 as $\kappa + 1$. G_1 is then *decreasing* on the configuration off $\wp$: Decreasing any color off $\wp$ cannot destroy this event. The same is true for G_2'' and G_3 as well. So by the Fortuin–Kastelyn–Ginibre (FKG) inequality [16], $P_\pi(G_2'' \cap G_3 | G_1) \geq P_\pi(G_2'') P_\pi(G_3)$, but

$$P_\pi(G_2'') \geq \left(1 - \frac{1}{\kappa}\right)^{4\kappa n^2} \quad \text{and} \quad P_\pi(G_3^c) \leq 25\kappa^2 e^{-an} \leq \frac{25}{\kappa^2}.$$

Finally, returning to the usual color code, observe that G_5 and G_6 are *increasing* events: Increasing any color cannot destroy either. So by FKG again, $P_\pi(G_5 \cap G_6) \geq P_\pi(G_5) P_\pi(G_6)$. Because G_6 can be identified with a supercritical problem in Q, $P_\pi(G_5) \geq \alpha(\varepsilon) > 0$, where $\alpha(\varepsilon)$ is independent of

κ. Of course $P_\pi(G_6) \geq P_\pi$(no 00, 01 or 11 neighbor pairs in $B_{5\kappa}$). Let B_1 (resp. B_2) denote all sites (i, j) in $B_{5\kappa}$ with $i + j$ even (resp. odd). Note that no two sites in B_1 (or B_2) are neighbors of each other. Let $b_1 = \#B_1$. Then, by conditioning on the number n of 0's or 1's in B_1 and using the fact that any set of n sites has at most $4n$ boundary sites, we get

$$P_\pi(G_6) \geq \sum_{n=0}^{b_1} \binom{b_1}{n} \left(\frac{2}{\kappa}\right)^n \left(1 - \frac{2}{\kappa}\right)^{b_1 - n} \left(1 - \frac{2}{\kappa}\right)^{4n}$$

$$= \left(\left(1 - \frac{2}{\kappa}\right)\left(1 + 2\frac{(\kappa - 2)^3}{\kappa^4}\right)\right)^{b_1} = \left(1 - \frac{16}{\kappa^2} + O\left(\frac{1}{\kappa^3}\right)\right)^{b_1}.$$

The expression on the right clearly converges to a positive limit.

Putting all the estimates together, we conclude that $P_\pi(G) \geq \exp\{-c \log^2 \kappa\} P_\pi(G_1)$. Hence

$$P_\pi(G \text{ happens for some fast } \wp) \geq \sum_{\text{fast } \wp} \exp\{-c \log^2 \kappa\} P_\pi(G_1 \text{ happens for } \wp)$$

$$\geq \exp\{-c \log^2 \kappa\} P(\kappa, \lfloor (p_c + \varepsilon)\kappa \rfloor).$$

Therefore, applying Proposition 1,

$$\liminf_{\kappa \to \infty} \frac{1}{\kappa} \log P_\pi(\gamma_0(\mathbf{0}) = 1, \text{the 1 at } \mathbf{0} \text{ creates a defect in } B_{5\kappa} \text{ by time } \kappa)$$

$$\geq \liminf_{\kappa \to \infty} \frac{1}{\kappa} \log P(\kappa, \lfloor (p_c + \varepsilon)\kappa \rfloor) \geq -\int_0^{p_c + \varepsilon} \Phi(p)\, dp.$$

Sending $\varepsilon \to 0$ and noting that disjoint translates of $B(5\kappa)$ have independent chances of producing a translate of G, (3.5) follows. □

We have performed extensive numerical experiments on the CAM-6 cellular automaton machine [23] in order to estimate the putative nucleation scaling rate $c^* = C^*$ for the basic κ-color GHM γ_t on an $L \times L$ box. Given κ, suppose that $|P_\pi(\gamma_t \equiv 0 \text{ eventually}) - \frac{1}{2}|$ is minimized at $L = L_\kappa$ and that $L_\kappa \approx Ae^{C^*\kappa}$ for κ large. Let us conclude this section by describing the data we have accumulated and the analysis made in order to estimate C^* and A.

A trial consisted of running the basic κ-color GHM on an $L \times L$ box, with free boundaries, starting from uniform product measure π. Each trial lasted until time $t = 2\kappa$, and was considered to have nucleated if at least one site in $\gamma_{2\kappa}$ had color 1. For each pair of values κ and L for which data were collected, we kept track of the number of trials run and the number of those trials achieving nucleation. The values of κ range from 4 to 30. For each κ, corresponding values of L were chosen to capture the range of system behavior from rare nucleation to prevalent nucleation. At least 20 trials were run for each (κ, L) pair. In the regime where nucleation occurred approximately half the time, either more trials were run or the range of L values was covered more thoroughly. Between 480 and 1690 trials were conducted for each κ, more than 1000 in most cases.

The data were used to estimate L_κ for each κ. First, data for extreme values of L were eliminated in order to focus on values of L for which the likelihood of nucleation was neither too small nor too large. Data points for the remaining allowed values of L were then fitted to a best least-squares line, and that line was used to compute the estimate $\hat{L}_\kappa$. Finally, a plot of $\ln \hat{L}_\kappa$ vs. κ was made. The least-squares line for these data was computed, and from this line we estimated C^* and A. To perform this analysis, one must decide rather arbitrarily how to eliminate extreme values of L for each κ. We did so by choosing "minimum and maximum nucleation ratios." The eliminated values were all those less than the smallest value of L whose ratio of nucleations to trials exceeded the minimum nucleation ratio, and also all those greater than the largest value of L whose ratio of nucleations to trials was greater than the maximum nucleation ratio.

Here are the results of the analysis for three choices of the minimum and maximum nucleation ratios. When the minimum and maximum nucleation ratios are 0.4 and 0.6, the estimates are $C^* \approx 0.225$ and $A \approx 1.325$; when the ratios are 0.25 and 0.75, the estimates are $C^* \approx 0.223$ and $A \approx 1.375$; and when the ratios are 0.1 and 0.9 (the data set shown in Figure 2), the

κ	$\hat{L}_\kappa$
4	3
5	4
6	5
7	7
8	8
9	10
10	13
11	16
12	19
13	26
14	31
15	39
16	48
18	72
20	113
22	184
24	272
26	454
28	706
30	1200

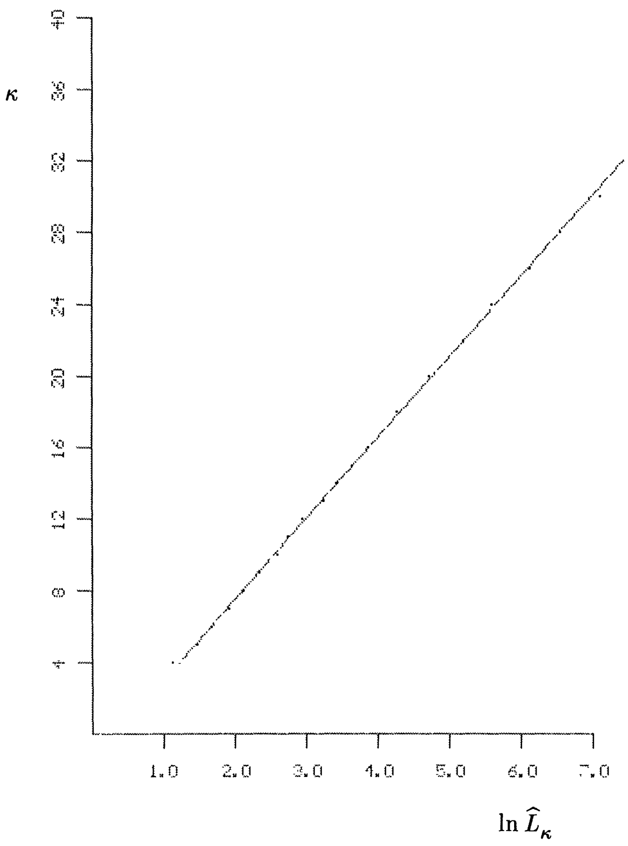


FIG. 2. *Table and least-squares plot for estimation of c^*.*

estimates are $C^* \approx 0.222$ and $A \approx 1.394$. On the basis of our experiments we offer the "ballpark" formula:

$$L_\kappa \approx 1.4 \times e^{0.22\kappa}. \tag{3.6}$$

4. GHM metastability starting from product measures that contain only three colors. Let us now turn to GHM behavior starting from distributions other than primordial soup. As explained in the Introduction, we are particularly interested in other product measures μ that self-organize more efficiently. For simplicity we focus our analysis on μ supported by only three colors. Such measures were considered by Mazelko and Showalter [21]. Our results shed some light on the density of 1's and 2's needed to produce the maximum density of nucleating centers, an issue addressed briefly in [21].

We begin this section by proving a necessary and sufficient condition for uniform local periodicity starting from translation invariant initial μ. Note that if γ_0 does not contain either 0's or 1's, then $\gamma_{\kappa-1} \equiv 0$. The case when γ_0 contains only 0's and 1's, which gives rise to *annihilating rings*, is analyzed in [13]. Our next result shows that on $\mathbb{Z}^2$ every site eventually updates every time whenever μ gives positive density to 0's, 1's and at least one additional color.

THEOREM 5. *Let γ_t be the basic κ-color Greenberg–Hastings model on $\mathbb{Z}^2$ starting from a translation invariant μ, with $\mu_k = P_\mu(\gamma_0(x) = k)$, $0 \le k \le \kappa - 1$. If $\mu_0 > 0$, $\mu_1 > 0$ and $\mu_k > 0$ for some $k \in \{2, \dots, \kappa - 1\}$, then almost surely, for any x, $\gamma_{t+1}(x) = (\gamma_t(x) + 1)$ mod κ eventually in t.*

PROOF. If $\kappa = 3$, then there is a clock in γ_0. If $\kappa \ge 4$, then a finite configuration of the following form produces a clock at time $t = \kappa - 2$:

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	k	k	k	1	k	k	k	k	k	k	k	k	k	1
1	k	0	k	1	0	0	0	0	0	0	0	0	k	1
1	k	0	k	1	k	k	k	k	k	k	k	k	k	1
1	k	0	k	1	1	1	1	1	1	1	1	1	1	1
1	k	0	k	1						1	k	0	k	1
1	k	0	k	1						1	k	0	k	1
1	k	0	k	1						1	k	0	k	1
1	k	0	k	1						1	k	0	k	1
1	k	0	k	1						1	k	0	k	1
1	1	1	1	1	1	1	1	1	1	1	k	0	k	1
1	k	k	k	k	k	k	k	k	k	1	k	0	k	1
1	k	0	0	0	0	0	0	0	0	1	k	0	k	1
1	k	k	k	k	k	k	k	k	k	1	k	k	k	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The diagram shows the case $\kappa = 10$. Sites that are not labeled can have any color. In general, the central block of arbitrary colors consists of $(\kappa - 5) \times (\kappa - 5)$ sites. (It is empty if $\kappa = 4$ or 5.) The idea behind our construction is as follows. Let y be the position of the leftmost uppermost 0. No matter what k is, at time $t = \kappa - 2$ there is a 1 at y and a 0 at $y + e_1$: 1's in the initial configuration form a protective shield at $t = 1, 2, \dots, \kappa - 2$ that no other 1 can penetrate. It does not matter that k may turn to 0 at time 1 because the paths along which 1's that see 0's travel are the fastest possible to y. Once a clock is formed, the proof of uniform local periodicity proceeds just as for the basic CCA in [8]. □

Next on the agenda are two lemmas that give sufficient conditions for death of γ_t on a finite box B_L. Lemma 3 simply states that survival of a finite system is equivalent to eventual defect formation. Lemma 4 then asserts that excitation cannot persist if γ_t is completely dominated by two successive colors except for very small isolated clusters of colors from the rest of the palette. The statements and proofs of these results involve "topological" features of *chains of infection*: the nearest-neighbor paths along which excitation (a 1) travels. Specifically, Lemma 4 relies on the fact that the index of an open loop l is unchanged by a detour involving an open loop of index 0. More precisely, suppose $l = (z_0, z_1, \dots, z_l = z_0)$ and for some $0 \le i \le j \le l$, z_i and z_j are connected by an open path $\wp$. We can form l' by replacing the part of l between z_i and z_j with $\wp$. It is easy to see that $\text{index}(l') = \text{index}(l)$ if the open loop formed by l between z_i and z_j, together with $\wp$, has index 0.

LEMMA 3. *Let γ_t be the basic κ-color Greenberg–Hastings model on a finite box B_L. Suppose that τ is the first time that some $x \in B_L$ is infected for the second time by the same* 1. *Then γ_τ contains a defect. Consequently,* $\{\gamma_t$ *survives*$\} = \{\gamma_t$ *contains a defect for some* $t\}$.

PROOF. Assume that $\gamma_0(x) = 1$. Let this 1 propagate along a self-avoiding loop $l = (x = x_0, x_1, x_2, \dots, x_\tau = x)$. Assume that $t \ge \kappa - 1$ and set $n = t - \kappa + 1$. At time t the color of x_n is 0, and the colors of $x_{n+1}, \dots, x_{n+\kappa-1} = x_t$ are $\kappa - 1, \kappa - 2, \dots, 2, 1$, respectively. Map $x_0, \dots, x_n$ into $\{0, \dots, n\}$ and call every 1 in $\{1, \dots, n\}$ that has either a 1 or a 0 on its left an L, and any 1 that has a 1 or 0 on its right an R. We claim there are always at least as many R's as L's.

Accepting this for a moment, suppose the 1 originating at x first reaches x again at time τ. Then at time $\tau - 1$ the index of l equals $1 + \#R\text{'s} - \#L\text{'s} \ge 1$, so $\gamma_\tau(l)$ is a defect. In a finite GHM that survives, some path of infection must eventually visit the same site twice, so survival is equivalent to defect formation.

The claim is proved by induction. A case by case analysis reveals that under basic GHM dynamics R's and L's may either annihilate in pairs, be created in pairs, an L may leave at 0 or an R may appear from 0. (The same analysis shows that the eventual index of l is equal to the number of times x is infected after first becoming a 1, up to and including the second time it is infected by that same 1.) We omit further details, which appear in [12]. □

LEMMA 4. *Suppose $\kappa \geq 5$ and there is a color k such that every l^∞-cluster (i.e., l^∞-connected component) of $\mathscr{S} = \{x\colon \gamma_0(x) \notin \{k, k+1\}\}$ has at most $(\kappa - 5)/3$ sites. Then every site x has $\gamma_t(x) = 1$ for at most one time t.*

PROOF. The conclusion follows from Lemma 3 once we prove that it is impossible for a defect to be formed. To this end, assume the converse, and let time τ be such that there is a self-avoiding defect $l\colon z_0, z_1, \dots, z_{N+1} = z_0$ in γ_τ. Then l cannot be entirely included in either a single l^∞-cluster of $\mathscr{S}$ (these are too small) or a single l^∞-cluster of $\mathscr{S}^c$ (l would have index 0). Without loss of generality, assume that $z_0 \in \mathscr{S}$, $z_N \in \mathscr{S}^c$. Then there exist i_m, $m = 1, 2, \dots, M+1$, and j_m, $m = 1, 2, \dots, M$, such that $i_1 = 0$, $i_{M+1} = N+1$ and, for $m = 1, 2, \dots, M$, $i_m < j_m < i_{m+1}$, with $z_n \in \mathscr{S}$ for $i_m \leq n \leq j_m - 1$ and $z_n \in \mathscr{S}^c$ for $j_m \leq n \leq i_{m+1} - 1$. Let $\wp_m$ be a self-avoiding open path that connects $z_{i_m - 1}$ (z_N if $m = 1$) with z_{j_m} through $\partial^0 \mathscr{S} = (x + B_1) \cap \mathscr{S}$.

If an l^∞-connected set has size n, then it is easy to prove by induction that its exterior l^∞-boundary is a loop with size at most $4n + 4$. Hence any two points of this boundary can be connected by a path of at most $2n + 4$ sites. It follows that the (open) loop made of $\wp_m$ and the part of l connecting $z_{i_m - 1}$ to z_{j_m} has at most $(\kappa - 5)/3 + 2(\kappa - 5)/3 + 4 < \kappa$ sites, and thus cannot be a defect. We can, therefore, replace all parts of l between $z_{i_m - 1}$ and z_{j_m} by $\wp_m$ without changing the index. However, this leaves a loop through $\mathscr{S}^c$; hence its index at time τ must be 0, a contradiction. □

Using Lemma 4 and Theorem 5 it is not difficult to establish exponential nucleation scaling starting from many three-color product measures as either κ becomes large or the density p of initial excitation tends to 0.

THEOREM 6. *Let γ_t be the basic Greenberg–Hastings model on B_L. Assume that the initial product measure μ contains only three colors. More precisely, suppose there exists a $k \in \{2, \dots, \kappa - 1\}$ such that the densities of μ are given by $\mu_1 = \mu_k = p$ and $\mu_0 = 1 - 2p$ for some $p > 0$. Assume that $p < 0.02$ and $\kappa > 6$. Then there exist constants c and C such that:*

(a) *If $L \leq p^{-c\kappa}$, then $P_\mu(\gamma_t$ dies out$) \to 1$ as $p \to 0$ or $\kappa \to \infty$,*
(b) *If $L \geq p^{-C\kappa}$, then $P_\mu(\gamma_t$ dies out$) \to 0$ as $p \to 0$ or $\kappa \to \infty$.*

PROOF. To prove (a), we let G be the event that there is no l^∞-cluster of $\{x\colon \gamma_0(x) \neq 0\}$ of size greater than $(\kappa - 5)/3$. By Lemma 4, $P_\mu(\gamma_t$ fixates$) \geq P_\mu(G)$. The number of l^∞-clusters with n sites, one of which is the origin, is bounded above by $3\,(13)^n$ (see [16], page 75). Therefore,

$$P_\mu(G^c) \leq (2L+1)^2\, 3(13)^{\kappa/3 - 2}(2p)^{\kappa/3 - 2} \leq 15(26)^{\kappa/3 - 2} p^{\kappa/3 - 2 - 2c\kappa}.$$

Because $\kappa \geq 7$, this last quantity clearly tends to 0 as $p \to 0$ as soon as $c < \frac{1}{6} - \frac{1}{7}$. On the other hand, if $p < 0.02$, then we can choose $c = 0.027$ to conclude that $P_\mu(G) \to 1$ as $\kappa \to \infty$.

For (b) we use the $(\kappa + 5) \times (\kappa + 5)$ construction in the proof of Theorem 5 to see that

$$P_\mu(\gamma_t \text{ dies}) \le \left(1 - p^{20(\kappa+5)}\right)^{4/(p^{C\kappa}(\kappa+5)^2)} \le \exp\left\{-4(\kappa+2)^{-2} p^{20(\kappa+5)-C\kappa}\right\}.$$

Choosing $C > 40$ and assuming $\kappa > 5$, the preceding expression tends to 0 as either $p \to 0$ or $\kappa \to \infty$. □

Suppose now that μ is a product measure consisting of only 0's, 1's and 2's, with densities that depend on κ, and that there is a $\delta > 0$ such that either (i) $\mu_0 < p_c - \delta$ or (ii) $\mu_2 < 1 - p_c - \delta$. Recall that the size of a subcritical percolation cluster has an exponential tail. In case (i), then, because a 1 needs to find a self-avoiding path of $\kappa - 2$ 0's in order to survive, γ_t must die out on small enough exponential boxes. In case (ii) the same conclusion follows from Lemma 4. If we want survival on subexponential boxes, it is therefore necessary that the densities of 0's and 2's be close to p_c and $1 - p_c$, respectively. Our next result, mentioned in the Introduction, deals with this case.

THEOREM 7. *Assume μ is a product measure such that $\mu_0 \ge p_c - 1/\kappa^3$, $\mu_2 \ge 1 - p_c - 1/\kappa^3$, $\mu_1 \ge 1/\kappa^\lambda$ for some $\lambda > 0$ and $\mu_k = 0$ otherwise. Let γ_t be the basic κ-color Greenberg–Hastings model on B_{κ^ν}. For a suitable choice of ν, $P_\mu(\gamma_t$ survives$) \to 1$ as $\kappa \to \infty$.*

As an illustration of this *power-law nucleation*, spiral formation is easy with 170 colors on a 400×400 grid by a judicious choice of initial measure: $\mu_0 = 0.61$, $\mu_1 = 0.0005$ and $\mu_2 = 0.3895$; see Figure 3. We invite the reader to estimate, using (3.6), the size of a square graphics array, with 1000 pixels per inch resolution, that would be required to achieve 170-color basic GHM nucleation from primordial soup. Suffice it to say that L would need to exceed the diameter of our solar system. The following proof gives a poor estimate for the infimum ν^* of power law exponents ν that can be achieved. Rick Durrett (private communication) has a nice proof that $\nu^* \le 2$. The ease with which experiments such as the one in Figure 3 succeed suggests that ν^* may, in fact, be closer to 1.

PROOF OF THEOREM 7. Our goal is a lower bound on the probability that a defect is formed by time $\kappa - 2$ in $B_{2\kappa}$. We need to show how 0's enable 1's to survive while 2's prevent the 1's from making rings. Write $B_r = B_{2\kappa} \cap \{x_1 > 0\}$, $B_{lu} = B_{2\kappa} \cap \{x_1 < 0, x_2 > 0\}$, $B_{ld} = B_{2\kappa} \cap \{x_1 < 0, x_2 < 0\}$ and $\partial B = B_{2\kappa} \setminus B_{2\kappa-1}$. Consider the following events:

$$G_1 = \{\gamma_0(\mathbf{0}) = 1, \gamma_0(x) \ne 1 \text{ for } x \in B_{4\kappa} \setminus \{\mathbf{0}\}\},$$

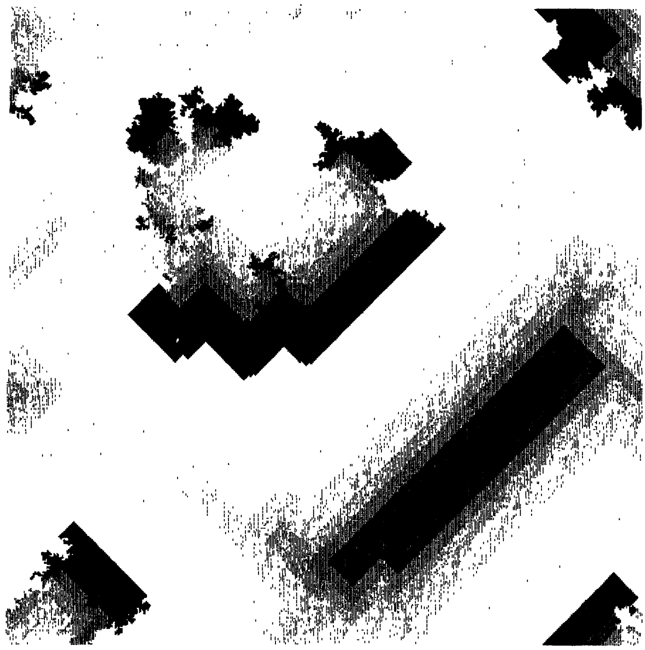
$$G_2 = \{\text{there is a path of 0's in } B_r \text{ that connects } \mathbf{0} \text{ to } \partial B\},$$

$$G_3 = \{\text{there is an } l^\infty\text{-path of 2's in } B_{lu} \text{ that connects } \mathbf{0} \text{ to } \partial B\},$$

$$G_4 = \{\text{there is an } l^\infty\text{-path of 2's in } B_{ld} \text{ that connects } \mathbf{0} \text{ to } \partial B\},$$

$$G_5 = \{\gamma_0(x) = 2 \text{ for } x = -e_1, -e_1 + e_2, -e_1 - e_2\},$$

$$G = G_1 \cap \cdots \cap G_5.$$



(a)

(b)

FIG. 3. *Power-law nucleation scaling*: *The basic GHM with* $\kappa = 170$ *at* (a) $t = 200$ *and* (b) $t = 500$.

We claim that $G \subset \{\exists$ a defect at time $\kappa - 2\}$. To see this, let $\wp_1$ be a self-avoiding path of 0's that guarantees G_2 and let $\wp_2$ and $\wp_3$ be self-avoiding l^∞-paths of 2's that guarantee G_3 and G_4, respectively. Denote by B_0 the region in $B_{2\kappa}$ below (and including) $\wp_2$ and above (and including) $\wp_3$. This region is connected: Each point in B_0 is connected via a path to $\mathbf{0}$. Moreover, at time $\kappa - 2$ all points in B_0 have color 0. This is so because all 1's except the 1 at $\mathbf{0}$ are unable to reach B_0 (due to G_1), while the 1 at $\mathbf{0}$ is also prevented from reaching B_0 (due to G_3, G_4 and G_5).

Let $\wp_1$ consist of sites $\mathbf{0} = z_0, z_1, \dots, z_m \in \partial B$, with $\|z_{i+1} - z_i\|_1 = 1$, $i = 0, \dots, m-1$. Let m_0 be the largest i for which $\gamma_{\kappa-2}(z_i) = 1$. Then $\gamma_{\kappa-2}(z_i) = 0$ for $i > m_0$ by Lemma 2. It follows that there exists a path of sites $\mathbf{0} = z'_0, z'_1, \dots, z'_{\kappa-2} = z_{m_0}$ such that $\gamma_{\kappa-2}(z'_i) = \kappa - i - 1$ for $i = 0, \dots, \kappa - 2$. Define the loop l to consist of $z'_0, \dots, z'_{k-2}, z_{m_0+1}, \dots, z_m$, a path in ∂B that connects z_m to any point $z \in B_0$, and finally a path in B_0 that connects z to $\mathbf{0}$. This path has index 1.

Let us now estimate $P_\mu(G)$. First, we use the following results due to Russo [22]. Fix integers $i, j > 0$, make sites in the box $B_{i,j} = [0, i-1] \times [0, j-1]$ occupied independently with probability p_c and let $p_{i,j}$ (resp. $p^*_{i,j}$) be the probability that there is a path (resp. l^∞-path) of occupied (resp. unoccupied) sites in $B_{i,j}$ that connects $\{x_1 = 0\}$ to $\{x_1 = i - 1\}$. Then there exists a positive α so that for each i,

$$p_{i,2i} \geq \alpha \quad \text{and} \quad p^*_{i,2i} \geq \alpha.$$

This enables us to find a lower bound for the probability q_i that $\mathbf{0}$ is connected via unoccupied sites in $B_{i,i}$ to $\{x_1 = i-1\} \cup \{x_2 = i-1\}$. Let l be the smallest integer such that $2^l > i$. Then (see [22])

$$q_i \geq p_{1,2} p_{2,4} \cdots p_{2^{l-1},2^l} \geq \alpha^l \geq \alpha^{(\log i/\log 2)+1} = \alpha i^{\log \alpha/\log 2}.$$

Denote by γ'_0 the configuration in which there is a 1 at $\mathbf{0}$, but all other sites are 0 with probability p_c and 2 with probability $1 - p_c$. Write μ' for the law of γ'_0. Under a natural coupling $\tilde{P}$,

$$\tilde{P}(\gamma_0(x) = \gamma'_0(x) \text{ for all } x \in B_{4\kappa} \setminus \{\mathbf{0}\}) \geq 1 - \frac{4(4\kappa+1)^2}{\kappa^3},$$

so it suffices to find bounds for $P_{\mu'}(G_2), \dots, P_{\mu'}(G_5)$ starting from γ'_0. However, it is easy to see that

$$P_{\mu'}(G_2) \geq \frac{\alpha}{4\kappa+1}, \qquad P_{\mu'}(G_3 | G_5) = P_{\mu'}(G_4 | G_5) \geq q_{2\kappa} \quad \text{and}$$
$$P_{\mu'}(G_5) = (1-p_c)^3.$$

Hence

$$P_{\mu'}(G) \geq c_0 \kappa^{(2\log\alpha/\log 2) - 1 - \lambda}$$

for some c_0. The probability that a translate of G occurs somewhere in the box B_{κ^ν} is at least

$$1 - \left(1 - c_0 \kappa^{2\log\alpha/\log 2 - 1 - \lambda}\right)^{4\kappa^\nu/(4\kappa+1)},$$

which converges to 1 as soon as $\nu > 2 + \lambda - 2\log\alpha/\log 2$. □

There are other three-color combinations that yield power law metastability. We leave it as a puzzle for the reader to determine what initial densities one should select when the colors involved are $\kappa - 1$, 0 and 1.

5. Metastability of a random Greenberg–Hasting model. The random Greenberg–Hastings model (RGHM) $\tilde{\gamma}_t$ is a Markov process on $\{0, \dots, \kappa - 1\}^{\mathbb{Z}^2}$. Besides $\mathscr{N}$ and κ, there is an additional parameter β that enters into the generator. Let k be the color of a site x at time t [i.e., $\tilde{\gamma}_t(x) = k$]. If $k \geq 1$, then a jump to $(k+1) \bmod \kappa$ occurs at exponential rate 1, whereas if $k = 0$, a jump to 1 occurs at rate $\beta \cdot \#\{y \in \mathscr{N}_x \colon \tilde{\gamma}_t(y) = 1\}$. This process is easily constructed using graphical representation [3]. Let us assume that $\mathscr{N} = \{z \colon \|z\|_1 = 1\}$.

Note that the case $\kappa = 2$ is the well-known contact process (see [3]), whereas $\kappa = \infty$ (i.e., no "regrowth" transition $\kappa - 1 \to 0$) is the Cox–Durrett forest fire model studied in [2]. This last reference proves existence of a critical value β_c for forest fire survival starting from a single 1 (burning tree) at the origin in a forest of 0's (susceptible trees): $P(\tilde{\gamma}_t(x) = 1 \text{ for some } x) \to 0$ as $t \to \infty$ for $\beta < \beta_c$, whereas $P(\tilde{\gamma}_t \text{ survives}) > 0$ for $\beta > \beta_c$. In the latter case the ring of fire spreads linearly in time and acquires an asymptotic shape. On the other hand, Durrett and Neuhauser [5] have proved (for a slightly different model with regrowth, but the same techniques apply to our process) that if $\beta > \beta_c$, then for *any* $\kappa < \infty$ the random GHM $\tilde{\gamma}_t$ has a stationary distribution that assigns no mass to all 0's.

Our first result for $\tilde{\gamma}_t$ is a random analog of Theorem 1. On any box B_L, regardless of the initial distribution, $\tilde{\gamma}_t$ is eventually trapped by all 0's. So we emphasize the growth rate of the time τ by which this happens for large L as $\kappa \to \infty$. On sufficiently small exponential boxes the extinction time grows at most linearly. On large enough exponential boxes, and assuming that β is suitably large, the extinction time grows at least as an exponential of an exponential. Similar metastability results for the basic one-dimensional contact process were proved in [15].

THEOREM 8. *Let $\tilde{\gamma}_t$ be the random Greenberg–Hastings model on B_L with excitation parameter β and κ colors, started from π (uniform κ-color product measure on B_L). Let $\tau = \min\{t \geq 0 \colon \tilde{\gamma}_t \equiv 0\}$.*

(a) *There exist constants a and c such that $P_\pi(\tau \leq a\kappa) \to 1$ as $\kappa \to \infty$ whenever $L \leq e^{c\kappa}$.*

(b) *Assume that $\beta > \beta_c$. Then there exist constants b and C such that $P_\pi(\tau \leq e^{Le^{-b\kappa}}) \to 0$ as $\kappa \to \infty$ whenever $L \geq e^{C\kappa}$.*

PROOF. (a) It is enough to fix a site with color 1, say **0**, and show that there exists a $\delta < 1$ such that the probability this site is still the ancestor of the color of some site at time $\delta\kappa$ is exponentially small. Let $G_1(t, k)$ be the event that there exists a self-avoiding path $0 = z_0, \dots, z_k$, such that a successor of the 1 at **0** moves on this path and reaches z_k at time t, and let G_1 be the event that some $G_1(\delta\kappa, k)$ with $k \geq [\delta/(2\log 3)]\kappa = \hat{\delta}\kappa$ happens. Also, let G_2 be the event that no site in B_{κ^2} changes color κ times. If G_2 happens but G_1 does not, then the 1 at **0** can only survive until time $t = \delta\kappa$ if there is a self-avoiding path $z_0, \dots, z_j$, $j < \hat{\delta}\kappa$, such that $\gamma_t(z_j) = 1$, the 1 moves along this path during the time interval $[0, t]$ and the leading 1 on this path survives (i.e., infects successive sites before changing to 2). Therefore,

$$P_\pi(\text{the 1 at } \mathbf{0} \text{ survives up to time } \delta\kappa, G_1^c)$$
$$\leq 4 \sum_{j \leq \hat{\delta}\kappa} 3^j e^{-\delta\kappa} + P_\pi(G_2^c) \leq 6e^{-\delta\kappa/2} + 4\kappa^4 e^{-c_1(\delta)\kappa},$$

where $c_1(\delta) > 0$ for $\delta < 1$. We need this estimate to eliminate the possibility that a 1 might survive on some path with a length that is sublinear in κ.

Let us now proceed to estimate the likelihood of G_1. For fixed $a \leq \hat{\delta}$, $P_\pi(G_1) \leq P_\pi(G_1(t, a\kappa)$ happens for some $t \leq \delta\kappa)$. Assuming G_2, this last event occurs only if the leading 1 on a path $0 = z_0, \dots, z_{\delta\kappa}$ survives. On our fixed path, denote this event by G_3. For $1 \leq i \leq \delta\kappa$, let X_i be the first time that $\tilde{\gamma}_t(z_i) = 0$. Also, let $Y_1, Y_2, \dots$ be i.i.d. mean 1 exponential random variables and let $Z_1, Z_2, \dots$ be i.i.d. mean $1/\beta$ exponential random variables, all independent of one another. Then standard large deviations estimates imply that for each $r > 1$,

$$P_\pi\left(\sum_{k \leq a\kappa} Y_k \geq ra\kappa\right) \leq \exp\{-c_1(r)a\kappa\},$$

$$P_\pi\left(\sum_{k \leq a\kappa} Z_k \geq ra\frac{1}{\beta}\kappa\right) \leq \exp\{-c_2(r)a\kappa\}$$

for some $c_1(r), c_2(r)$ that both converge to ∞ as $r \to \infty$. It follows that

$$\begin{aligned} P_\pi(G_3) &\leq P_\pi(X_1 \leq Y_1, X_2 \leq Y_1 + Z_1 + Y_2, \dots, \\ &\qquad X_{a\kappa} \leq Y_1 + Z_1 + Y_2 + Z_2 + \cdots + Y_{a\kappa-1} + Z_{a\kappa-1} + Y_{a\kappa}) \\ &\leq P_\pi\left(X_1 \leq r\left(1 + \frac{1}{\beta}\right)a\kappa, \dots, X_{a\kappa} \leq r\left(1 + \frac{1}{\beta}\right)a\kappa\right) \\ &\qquad + \exp\{-c_1(r)a\kappa\} + \exp\{-c_2(r)a\kappa\} \\ &\leq \left(2r\left(1 + \frac{1}{\beta}\right)a\right)^{a\kappa} + \exp\{-c_1(r)a\kappa\} + \exp\{-c_2(r)a\kappa\}, \end{aligned}$$

where the last inequality uses the fact that $P_\pi(X_1 \leq c\kappa) \leq 2c$ for κ sufficiently large. Finally, $P_\pi(G_1) \leq 4 \cdot 3^{a\kappa} P_\pi(G_3) + P_\pi(G_2^c)$, so choosing r so that $c_i(r) > \log 3$ $(i = 1, 2)$, and then a small, shows that G_1 has exponentially small probability, as desired.

(b) We use the comparison of $\tilde{\gamma}_t$ to a one-dependent oriented percolation from [5]. Fix b and write $L_0 = Le^{-b\kappa}$. Given $x \in B_{L_0}$ and a time t, call (x, t) *occupied* if $x + B_{\exp\{b\kappa\}}$ contains at least $e^{b\kappa/2}$ 1's at time t. Provided that b is large enough, an argument in [5] implies that the set $\{(m, n) \in \mathbb{Z}^2: m + n$ is even and $((m, 0), t_n)$ is occupied$\}$ dominates a one-dependent oriented percolation with density of wet sites close to 1. Here t_n is a suitably chosen sequence of times that increase to ∞, and $t_n \geq c_3 e^{b\kappa} n$ for some constant c_3.

Now let $X_1, X_2, \ldots$ be i.i.d. random variables with $P_\pi(X_1 = 1) = 1/\kappa = 1 - P_\pi(X_1 = 0)$. If κ is sufficiently large, then

$$\begin{aligned}\log P_\pi((x,0) \text{ is not occupied})& \\ &\leq \log P_\pi\big(X_1 + X_2 + \cdots + X_{e^{b\kappa}} < e^{b\kappa/2}\big)\\ &\leq e^{b\kappa/2} + e^{b\kappa}\log\left(1 - \frac{1}{\kappa}(1 - e^{-1})\right) \leq -e^{b\kappa/2}.\end{aligned}$$

It now follows that with probability converging to 1 as $\kappa \to \infty$, all sites in B_{L_0} are occupied.

Finally, using a standard contour argument, it is possible to choose a constant C' so that oriented percolation, started with occupied sites $(2i, 0)$, $-L_0 \leq i \leq L_0$, survives up to time $e^{C'L_0}$ with probability converging to 1 as $L_0 \to \infty$. This implies that $\tilde{\gamma}_t$ remains alive on B_L, $L > e^{C\kappa}$, up to time $e^{C'L_0}$ with probability converging to 1 as $\kappa \to \infty$, provided of course that $C > b$. □

What happens if $\beta < \beta_c$? In this case the forest fire of [2] dies out, and the next lemma shows that it does so at an exponential rate. Let $U = \{x: \tilde{\gamma}_t(x) = 1 \text{ for some } t\}$ be the set of points that are ever on fire.

LEMMA 5. *If $\kappa = \infty$, $\beta < \beta_c$ and $\tilde{\gamma}_0(z) = 1_{\{z = \mathbf{0}\}}$, then there exists a constant $a > 0$ so that*

$$P(U \cap \{z: \|z\|_\infty \geq L\} \neq \varnothing) \leq e^{-aL}.$$

PROOF. Call the probability in question p_L. Let $e(x, y)$, $x \in \mathbb{Z}^2$, $y \in \mathscr{N}_x$, be independent exponential random variables with mean $1/\beta$ and, independently, let $T(x)$, $x \in \mathbb{Z}^2$, be independent exponential random variables with expectation 1. We call a bond $x \to y$ *open* if $e(x, y) < T(x)$. Then (see [2]) U is exactly the set of all sites that can be reached from $\mathbf{0}$ by a (directed) path of open bonds. Denote by R_L the probability of a left to right open crossing of B_L. It follows easily from [2] that for any $\beta < \beta_c$ there exists a $c > 0$ such that $R_L \leq e^{-cL}$. Moreover, a standard percolation argument (cf. [16]) implies that $R_L \geq [1 - (1 - p_L)^{1/4}]^2$, hence $p_L \leq 4\sqrt{R_L}$, and the exponential estimate is proved. □

Suppose $\tilde{\gamma}_0(x) = 1$. Call y a *primary successor* at time $t \geq 0$ if y is connected via a space–time infection path to the 1 at x at time 0, but this path does not pass through any space–time point that has changed κ or more times (i.e., 1's that are regrown on "new" 0's are not counted as successors of the original 1). Let A_t be the set of 1's that are primary successors at some time $s \le t$. Denote by B_t the set of all sites in the forest fire started from $\tilde{\gamma}_0(z) = 1_{\{z=x\}}$ that have caught fire by time t. Two more lemmas are required for our final theorem. The first describes a key coupling between A_t and B_t; the second derives a straightforward large deviations estimate.

LEMMA 6. *For any initial measure on* $\mathbb{Z}^2 \setminus \{x\}$, B_t *is distributionally above* A_t, *that is, there is a coupling of* A_t *and* B_t *such that* $A_t \subseteq B_t$.

PROOF. For any bond $x \to y$, and $k \ge 1$, let $e_k(x, y)$ be exponential random variables with mean β^{-1}, independent for distinct bonds and values of k. As before, let $T(x)$, $x \in \mathbb{Z}^2$, be independent exponential random variables with mean 1. In the forest fire model we let $S(y)$ be the time when y first becomes 1, whereas in the GHM we let $S(y)$ be the time that y becomes a primary successor of x. (This can only happen once in either case.) Make the 1 change to 2 at time $S(x) + T(x)$ in either case. During this lifetime, y tries to infect its neighbor z at all times $S(y) + e_1(y, z), S(y) + e_1(y, z) + e_2(y, z), \dots$ that are smaller than $S(y) + T(y)$. If the color of z is 0 at one of these times, then z makes the transition to 1. This defines the dynamics of the forest fire. Use any graphical representation to define additional transitions in the GHM. By induction, we need to prove that if $A_t \subseteq B_t$ and a site z is added to A_t at time t, then it has been added to B_s at some time $s \le t$. However, if z has not been on fire by time t, then its color in the forest fire model is 0, so it must become a 1 at this time. Hence the inclusion is preserved by this transition. □

LEMMA 7. *Fix* m *large. Suppose that* $\tilde{\gamma}_0(x) = 1$. *The probability that this* 1 *has a successor* y *at time* t *with* $\|y - x\|_1 > mt$ *is bounded above by*

$$\frac{4(\beta + m)}{m - 2\beta}\left(\frac{9\beta}{\beta + m}\right)^{mt}.$$

PROOF. Let $X_1, X_2, \dots$ be independent exponential random variables with mean $1/\beta$. The probability of the event in question is bounded above by

$$\sum_{l \ge mt} 4 \cdot 3^l P(X_1 + \cdots + X_l \le t) \le \sum_{t \ge mt} 4 \cdot 3^l e^{mt} E\left[(e^{-mX_1})^l\right]$$

$$\le 4e^{mt} \sum_{l \ge mt} \left(\frac{3\beta}{\beta + m}\right)^l,$$

which easily gives the result. □

We conclude by showing that the RGHM dies out quickly whenever $\beta < \beta_c$ and κ is large. Thus there are two kinds of behavior on exponential boxes $B_{e^{c\kappa}}$ for large κ: If either β or c is too small, then $\tilde{\gamma}_t$ typically dies out in linear time (in κ), but if β and c are large enough, then $\tilde{\gamma}_t$ is likely to live for a very long time. Note that we have not addressed the critical behavior when $\beta = \beta_c$; it seems hard to even guess what happens in that case.

THEOREM 9. *Let $\tilde{\gamma}_t$ be the random Greenberg–Hastings model with excitation parameter β and κ colors. Assume that $\beta < \beta_c$.*

(a) *On $\mathbb{Z}^2$, there exists a κ_0 so that for $\kappa \ge \kappa_0$ and any $\tilde{\gamma}_0$, $\tilde{\gamma}_t$ dies out strongly. That is,*

$$P_\pi\big(\tilde{\gamma}_t(x) = 0 \text{ for all sufficiently large } t\big) = 1 \quad \text{for each } x.$$

(b) *For the process on B_L, given any $c > 0$ there exists an $a > 0$ such that if $L < e^{c\kappa}$, then*

$$P_\pi(\tau \le a\kappa) \to 1 \quad \text{as } \kappa \to \infty.$$

PROOF. Assume there are 1's at l sites $x_1, \dots, x_l$ initially. Fix time $t = \kappa/4$. We claim κ can be chosen large enough that for all $\kappa' \ge \kappa$ the expected total number of successors of these 1's at time t is less than $l/2$ (regardless of the colors at sites other than $x_1, \dots, x_l$.) To see this, denote by N_i the total number of successors at time t of the 1 at x_i. We need to show that $E_\pi[N_i] \le \frac{1}{2}$ for κ sufficiently large. Let S_i be the event that the 1 at x_i has a primary successor at time t. If $l \ge \kappa^4$, then

$$\begin{aligned} P_\pi(N_i \ge l) &\le P_\pi\left(\text{the 1 at } x_i \text{ has a successor } y \text{ at time } t\colon \|y - x_i\|_1 > \frac{\sqrt{l}}{3}\right) \\ &\le C\left(\frac{9\beta}{\beta + 4\kappa/3}\right)^{\sqrt{l}/3} \le C\left(\frac{9\beta}{\kappa}\right)^{\sqrt{l}/3} \end{aligned}$$

for some constant C, by Lemma 7. If, on the other hand, $1 \le l \le \kappa^4$, then we write

$$P_\pi(N_i \ge l) \le P_\pi(S_i) + P_\pi(N_i \ge 1, S_i^c)$$

and estimate

$$\begin{aligned} P_\pi(S_i) &\le P_\pi\big(U \cap (x_i + B_{\kappa^{1/4}})^c \ne \varnothing\big) + P_\pi\big(S_i, U \cap (x_i + B_{\kappa^{1/4}})^c = \varnothing\big) \\ &\le e^{-a\kappa^{1/4}} + P_\pi\big(\text{a site in } (x_i + B_{\kappa^{1/4}}), \text{ after it first becomes 1,} \\ &\qquad\qquad \text{stays in that state for at least } \sqrt{\kappa}/20 \text{ time units}\big) \\ &\le e^{-a\kappa^{1/4}} + 5\kappa^{1/2} e^{-\sqrt{\kappa}/20}, \end{aligned}$$

where we have used Lemmas 5 and 6. Also,

$$\begin{aligned} P_\pi(N_i \ge 1, S_i^c) &\le P_\pi\big(\text{the 1 at } x_i \text{ has a successor } y \text{ at time } t\colon \|y - x_i\|_1 \ge \kappa^2\big) \\ &\quad + P_\pi\big(\text{some site } y\colon \|y - x_i\|_1 < \kappa^2 \text{ changes } \kappa \\ &\qquad\qquad \text{times during } [0, t]\big) \\ &\le C\left(\frac{9\beta}{\beta + \kappa/4}\right)^{\kappa^2} + 5\kappa^4 e^{-\lambda\kappa} \end{aligned}$$

for some positive C and λ, by Lemma 7 and an easy large deviations estimate. Taken together, these bounds yield a $C > 0$ such that

$$E_\pi[N_i] = \sum_{l=1}^{\infty} P_\pi(N_i \geq l) \leq C\kappa^4 e^{-a\kappa^{1/4}} + C\sum_{l=1}^{\infty}\left(\frac{9\beta}{\kappa}\right)^{\sqrt{l}/3},$$

which can clearly be made smaller than $\frac{1}{2}$ if κ is large enough.

Now let $\kappa' \geq \kappa$ and fix a 1 at x in $\tilde{\gamma}_0$. Let M_n be the number of successors at time $n\kappa/4$, $n = 1, 2, \dots$. By induction, $E[M_n] \leq (\frac{1}{2})^n$, and hence

$$\begin{aligned} P_\pi \text{ (the 1 at } x \text{ has a successor at time } t) &\leq P_\pi(M_{\lfloor 4t/\kappa \rfloor} \geq 1) \\ &\leq E_\pi\left[M_{\lfloor 4t/\kappa \rfloor}\right] \leq 2c \cdot 2^{-4t/\kappa}. \end{aligned}$$

Because $\gamma_t(x) = 1$ implies that either an initial 1 at y with $\|x - y\|_1 < t^2$ has a successor at time t, or else an initial 1 at y with $\|x - y\|_1 \geq t^2$ has a successor at x at time t, (a) is an easy consequence of Borel–Cantelli and Lemma 7. Finally, (b) follows by choosing $a > (\kappa/2\log 2)c$. □

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