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AN EXTREMAL REARRANGEMENT PROPERTY OF STATISTICAL SOLUTIONS OF BURGERS' EQUATION[1]

BY YIMING HU AND W. A. WOYCZYNSKI

Case Western Reserve University

We prove that a certain (centered unimodal) rearrangement of coefficients in the moving average initial input process maximizes the variance (energy density) of the limit distribution of the spatiotemporal random field solution of a nonlinear partial differential equation called Burgers' equation. Our proof is in the spirit of domination principles developed in the book by Kwapien and Woyczynski.

1. Introduction. Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \tag{1.1}$$

where $u = u(t, x)$ and constant $\nu > 0$, which, essentially is a simplified version of the Navier–Stokes equation with the pressure term omitted, is an example of a nonlinear partial differential equation that has been suggested as a simple model of the velocity field for turbulence, for shock waves when $\nu \downarrow 0$, as well as for the distribution of self-gravitating sticky dust in large scale models of the universe [see, e.g., Burgers (1974), Gurbatov, Malahov and Saichev (1990), Shandarin and Zeldovich (1989) and Woyczynski (1993)]. The equation often arises also in the following generic situation. Consider a flow $u(t, x)$ (say, describing the density per unit length of a certain quantity) on the real line with the flux of this quantity through section at x described by another function $\phi(t, x)$. Assume that the flow is subject to a conservation law

$$\frac{\partial}{\partial t}\int_{x_0}^{x_1} u(t, x)\, dx + \phi(t, x_1) - \phi(t, x_0) = 0$$

when $x_0 < x_1$. If we assume that the flux $\phi(t, x) = \Phi(u(t, x))$ depends on the local density only, then as $x_0 \to x_1$, the above conservation law leads to an equation of Riemann type:

$$u_t + \Phi'(u)u_x = 0.$$

If the flux function is permitted to depend additionally on the gradient of the density u according to the question $\phi(t, x) = \Phi(u(t, x)) - \nu u_x(t, x)$, then the

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above conservation law leads to the equation

$$u_t + \Phi'(u)u_x = \nu u_{xx},$$

of which Burgers' equation is a special case.

Let us assume that the initial velocity function $u(0, x) = u_0(x)$. Using the standard Hopf–Cole transformation [see, e.g., Burgers (1974) and Woyczynski (1993)], we obtain that

$$u(t, x) = t^{-1} \frac{Z(t, x)}{I(t, x)}, \tag{1.2}$$

where

$$\begin{aligned} Z(t, x) &= \int_{\mathbb{R}} (x - y) \exp\left(\frac{U_0(y)}{2\nu} - \frac{(x - y)^2}{4\nu t} \right) dy, \\ I(t, x) &= \int_{\mathbb{R}} \exp\left(\frac{U_0(y)}{2\nu} - \frac{(x - y)^2}{4\nu t} \right) dy, \end{aligned} \tag{1.3}$$

and

$$U_0(y) = -\int_{y_0}^{y} u_0(x)\, dx$$

is the initial velocity potential.

In view of its interesting, and sometimes tractable, nonlinear effects, the Burgers equation model, especially with random initial data (in which case its solutions are also referred to as Burgers turbulence, has been very popular in the physical and engineering literature. Over 80 research papers have appeared on this subject just in the last couple of years. Some of that literature was compiled in Woyczynski (1993).

At the mathematical level, several recent papers [see, e.g., Rosenblatt (1987), Bulinski and Molchanov (1991), Hu and Woyczynski (1993), Surgailis and Woyczynski (1993, 1994a, b) and Funaki, Surgailis and Woyczynski (1993)] discussed the asymptotic behavior of the rescaled random field $u(t, x)$ as $t \to \infty$, assuming that the initial data $u_0(x)$ are various stochastic processes (or random fields if the three-dimensional analog of the above Burgers equation is being considered).

In particular, Rosenblatt's paper assumes that u_0 is an asymptotically uncorrelated stationary process and shows that, under some additional technical conditions, a centered and rescaled velocity potential $\int u(t, x)\, dx$ converges weakly to the Brownian motion process. The paper by Bulinski and Molchanov obtains Gaussian scaling limits for initial shot noise and stationary fields with regular spectral density data. A series of papers by Surgailis, Funaki and Woyczynski obtains a classification of (not necessarily Gaussian) scaling limits for a variety of initial processes and random fields, which include Gaussian stationary fields with singular spectral densities and shot noise processes driven by Cox–Gibbs processes, that is, doubly stochastic Poisson processes with random intensity which take into account a potential of interaction between different "bumps." The limits there are taken in the sense of weak convergence of finite-dimensional distributions.

A recent analysis, from the viewpoint of fluid mechanics, of the statistics of decaying Burgers turbulence can be found in Gotoh and Kraichnan (1993). One of the first rigorous results for the Hausdorff dimension of shocks in the zero viscosity limit for Burgers' stochastic flow was obtained by Sinai (1992) in the case of Brownian initial data.

2. Main results. In the present paper, we focus on the following problem. Suppose that the initial velocity potential is a stationary finite moving average process

$$U_0(x) = \sum_{i=1}^{N} c_i \xi_{\lfloor x \rfloor - i}, \tag{2.1}$$

where $\xi, \dots, \xi_{-1}, \xi_0, \xi_1, \dots$ are independent, identically distributed random variables, and $\lfloor x \rfloor$ is the integer part of x. The general question is: How does the coefficient vector $\mathbf{c} = (c_1, c_2, \dots, c_N)$ in the above initial condition affects the distribution of the rescaled solution random field $u(t, x\sqrt{t})$ for large values of t?

Throughout the paper the standing assumption is that $\mathbb{E}e^{a\xi} < \infty$ for $0 \le a < \infty$. Also, without loss of generality, we assume that $\operatorname{Var} \xi = 1$ in (2.1) and $\nu = 1/2$ in (1.1).

To formulate our results, we need to introduce the quantities

$$\begin{aligned}\sigma(\mathbf{c}) = \exp\left(\sum_{i=1}^{N} h(c_i, c_i)\right) + 2\sum_{k=1}^{N-1} \exp\left(\sum_{i=1}^{N-k} h(c_i, c_{i+k})\right) \\ -(2N-1),\end{aligned} \tag{2.2}$$

where

$$h(a, b) = \log \frac{\mathbb{E}e^{(a+b)\xi}}{\mathbb{E}e^{a\xi}\mathbb{E}e^{b\xi}} \tag{2.3}$$

and

$$V = \lim_{t \to \infty} t^{1/2} \sum_{k=-\infty}^{\infty} \left(\int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}} (y - x) \exp\left(-\frac{(y-x)^2}{2} \right) dy \right)^2. \tag{2.4}$$

The quantity V is independent of x. Indeed, if $F(u) = \int_{-\infty}^{u} (y - x) \exp[-(y - x)^2/2]\, dy$, the expression under the limit in (2.4) is equal to $\sum_k \sqrt{t}[F((k+1)/\sqrt{t}) - F(k/\sqrt{t})]^2$. The function F has bounded variation and $F((k+1)/\sqrt{t}) - F(k/\sqrt{t}) = F'(k/\sqrt{t})/\sqrt{t} + O(1/t)$. So the limit behavior in (2.4) is the same as that of $\sum_k F'(k/\sqrt{t})[F((k+1)/\sqrt{t}) - F(k/\sqrt{t})]$, which tends to $\int_{-\infty}^{\infty} (y - x)^2 \exp[-(y - x)^2]\, dy = \sqrt{\pi}/2$, which does not depend on x. This argument is due to an anonymous referee.

Our first result, which will be proved in Section 3, describes the behavior of the one-dimensional distributions of the rescaled solution random field u at large times.

THEOREM 2.1. *Let $u(t, x)$ be a solution random field* (1.2) *of Burgers' equation* (1.1). *Then, for each fixed x, as $t \to \infty$, the distribution of $t^{5/4}u(t, x\sqrt{t})$ converges weakly to the normal distribution $N(0, V\sigma(\mathbf{c})/2\pi)$.*

It should be noted that all the limit finite-dimensional limiting distributions of the random field u can also be identified, even for more general stationary initial velocity potentials [see, e.g., Corollary 3.1 of Surgailis and Woyczynski (1993)]. However, in the present paper, the point is to provide a self-contained proof of an explicit formula for the variance of the limiting field in terms of the coefficient vector $\mathbf{c}$. The formula is used subsequently in the proof of the extremal rearrangement property described in Theorem 2.3.

Also observe that in fluid flow with mass density ρ and velocity field $\mathbf{u}(t, \mathbf{x})$, the kinetic energy

$$E_{\text{kin}} = \tfrac{1}{2}\int \rho \| \mathbf{u}(t, \mathbf{x}) \|^2 \, d\mathbf{x},$$

so that the scaled expected energy density for large times is proportional to the variance (for zero mean flows) of the scaling limit field appearing in the above theorem. The equivalence of the limit variance and the asymptotic energy density for Burgers' turbulence provided an initial physical motivation for the work presented in this paper.

The next result, a corollary to Theorem 1.1, gives a monotonicity property of the limiting variance σ in terms of componentwise ordering of the initial coefficient vectors $\mathbf{c}$.

THEOREM 2.2. *If $\mathbf{c} = (c_1, \dots, c_N) \geq 0$, then the variance $\sigma(\mathbf{c}) = \sigma(c_1, \dots, c_N)$ is increasing as a function of each variable $c_1, \dots, c_N$.*

The result is quite natural and its proof can be found also in Section 3.

The final result formulates what we call an *extremal rearrangement property* of the variance $\sigma(\mathbf{c})$ of the scaled limit of the random field u with respect to the permutation transformation on $\mathbf{c}$. It is a rather surprising phenomenon and we have not found anything like it in the literature, so perhaps a few words of explanation are in order.

The result asserts that a certain permutation of coefficients in $\mathbf{c}$ guarantees maximal variance (energy density) of the limiting field. This is clearly a nonlinear phenomenon. If one considers a linear flow such as described by the heat equation

$$\phi_t = \nu \phi_{xx},$$

with the initial condition ϕ_0, then the solution u is expressed by the usual convolution with the Gaussian kernel, and

$$\phi(t, x\sqrt{t}) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \phi_0(y) \exp\left(-\frac{(x\sqrt{t} - y)^2}{4\nu t} \right) dy$$
$$= \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} \phi_0(y\sqrt{t}) \exp\left(-\frac{(x - y)^2}{4\nu} \right) dy.$$

So, first, by the ergodic theorem, as $t \to \infty$, the right-hand side converges to a constant deterministic limit for a large class of stationary initial data [see, e.g., Surgailis and Woyczynski (1993)] and the scaling limit problem is trivial. Second, if we assume the initial datum to be of the form $\phi_0 = \exp(U_0(y)/2\nu)$, where U_0 is as in (2.1), then by (1.3) and Lemma 3.1,

$$\lim_{t\to\infty} \phi(t, x\sqrt{t}) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} t^{-1/2} I(t, x\sqrt{t}) = \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi),$$

which is obviously independent of how the components of the coefficient vector $\mathbf{c}$ are permuted. Thus, in the case of a linear diffusion equation (and other linear flows) the problem of an extremal rearrangement is not meaningful.

The situation turns out to be more interesting for stochastic Burgers' flow, where, as we shall see later on [e.g., (2.2)], the scaling limit of the solution random field does depend on how components of $\mathbf{c}$ are ordered and where, at least in the Gaussian and Poisson cases, we can identify a permutation of $\mathbf{c}$ which maximizes the limiting energy density. A majorization method and Schur convexity ideas are used in the proof, which is supplied in Section 4. The result is as follows:

THEOREM 2.3. *If ξ is either a Gaussian or a Poisson random variable and the coefficient vector $\mathbf{c} = (c_1, \dots, c_N)$ satisfies the centered unimodality condition*

$$0 \le c_1 \le c_N \le c_2 \le c_{N-1} \le c_3 \le c_{N-2} \le c_4 \le \cdots, \tag{2.5}$$

then

$$\sigma(\mathbf{c}) = \max_{\Pi} \sigma(\Pi\mathbf{c}), \tag{2.6}$$

where $\Pi\mathbf{c}$ is a vector obtained from $\mathbf{c}$ by permutation Π of its components.

The case of continuous moving average initial data has been considered recently by Hu and Woyczynski (1994). It is an open question if Theorem 2.3 remains true for moving average initial data generated by independent identically distributed random variables ξ_i with more general distributions.

3. Limiting behavior. In order to prove our basic result on the limiting behavior of statistical solutions of (1.1), we will need several lemmas.

LEMMA 3.1. *In probability,*

$$\lim_{t\to\infty} t^{-1/2} I(t, x\sqrt{t}) = \sqrt{2\pi} \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi). \tag{3.1}$$

PROOF. Observe that

$$\begin{aligned}
&\frac{1}{\sqrt{t}} I(t, x\sqrt{t}) - \sqrt{2\pi} \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi)\\
&\quad = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} \exp\left(\sum_{i=1}^{N} c_i \xi_{\lfloor y \rfloor - i} - \frac{(x\sqrt{t} - y)^2}{2t} \right) dy - \sqrt{2\pi} \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi)\\
&\quad = \int_{-\infty}^{\infty} \exp\left(\sum_{i=1}^{N} c_i \xi_{\lfloor \sqrt{t} u \rfloor - i} - \frac{(x-u)^2}{2} \right) du - \sqrt{2\pi} \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi)\\
&\quad = \sum_{k=-\infty}^{\infty} \prod_{i=1}^{N} \exp(c_i \xi_{k-i}) \int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}} \exp\left(-\frac{(x-u)^2}{2} \right) du - \sqrt{2\pi} \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi)\\
&\quad = \sum_{k=-\infty}^{\infty} \left(\prod_{i=1}^{N} \exp(c_i \xi_{k-i}) - \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi) \right) \int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}} \exp\left(-\frac{(x-u)^2}{2} \right) du\\
&\quad = \sum_{l=0}^{N-1} \eta_l(t),
\end{aligned}$$

where

$$\begin{aligned}
\eta_l(t) = &\sum_{m=-\infty}^{\infty} \left(\prod_{i=1}^{N} \exp(c_i \xi_{mN+l-i}) - \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi) \right)\\
&\times \int_{(mN+l)/\sqrt{t}}^{(mN+l+1)/\sqrt{t}} \exp\left(-\frac{(x-y)^2}{2} \right) du.
\end{aligned}$$

Now, it suffices to show that for each $l = 0, \dots, N-1$, the random variables $\eta_l(t) \to 0$ in probability as $t \to \infty$. Since

$$E \prod_{i=1}^{N} \exp(c_i \xi_{mN+l-i}) = \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi)$$

and since the products on the left-hand side are independent random variables for different m's (for a fixed l) this conclusion is obtained by a straightforward application of Chebyshev's inequality, because for each $l = 0, \dots, N-1$,

$$\begin{aligned}
\operatorname{Var} \eta_l(t) = &\sum_{m=-\infty}^{\infty} \left(\int_{(mN+l)/\sqrt{t}}^{(mN+l+1)/\sqrt{t}} \exp\left(-\frac{(x-u)^2}{2} \right) du \right)^2\\
&\times \operatorname{Var} \prod_{i=1}^{N} \exp(c_i \xi_{mN+l-i}),
\end{aligned}$$

where the variance on the right-hand side above are independent of m and because

$$\sum_{m=-\infty}^{\infty}\left(\int_{(mN+l)/\sqrt{t}}^{(mN+l+1)/\sqrt{t}}\exp\left(-\frac{(x-u)^2}{2}\right)du\right)^2$$
$$\leq \sqrt{2\pi}\max_{-\infty<m<\infty}\int_{(mN+l)/\sqrt{t}}^{(mN+l+1)/\sqrt{t}}\exp\left(-\frac{(x-u)^2}{2}\right)du \leq \sqrt{\frac{2\pi}{t}}. \qquad \square$$

The next lemma computes the mean and the limit variance of the numerator Z in the solution (1.2). Recall that $\nu = 1/2$.

LEMMA 3.2.

$$\mathbb{E}\left(t^{-1/4}Z(t, x\sqrt{t})\right) = 0 \tag{3.2}$$

and

$$\mathrm{Var}\left(t^{-1/4}Z(t, x\sqrt{t})\right) = V\tilde{\sigma}(c), \tag{3.3}$$

where V *is as in* (2.4) *and*

$$\tilde{\sigma}(\mathbf{c}) = \sum_{s=-N}^{N}\left(\prod_{i=-N}^{2N}\mathbb{E}\exp((c_i + c_{i+s})\xi) - \left(\prod_{i=1}^{N}\mathbb{E}\exp(c_i\xi)\right)^2\right). \tag{3.4}$$

PROOF. Property (3.2) is obvious. For convenience let,

$$a_k = a_k(t) = \int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}}(y-x)\exp\left(-\frac{(y-x)^2}{2}\right)dy.$$

Since

$$\mathbb{E}\left(t^{-1/4}Z(t, x\sqrt{t})\right)^2$$
$$= t^{-1/2}\mathbb{E}\left(\int_{-\infty}^{\infty}(y - x\sqrt{t})\exp\left(-\frac{(y-x\sqrt{t})^2}{2t} + \sum_{i=1}^{N}c_i\xi_{\lfloor y\rfloor - i}\right)dy\right)^2$$
$$= t^{1/2}\mathbb{E}\left(\sum_{k=-\infty}^{\infty}a_k(t)\exp\left(\sum_{i=1}^{N}c_i\xi_{k-i}\right)\right)^2,$$

using the fact that

$$\int_{-\infty}^{\infty} (y-x)\exp\left(-\frac{(y-x)^2}{2}\right) dy \prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi) = 0,$$

we have

$$\begin{aligned}\mathbb{E}\big(t^{-1/4}Z(t,x\sqrt{t})\big)^2 &= t^{1/2}\mathbb{E}\left(\sum_{k=-\infty}^{\infty} a_k\left(\exp\left(\sum_{i=1}^{N} c_i \xi_{k-i}\right) - \mathbb{E}\exp\left(\sum_{i=1}^{N} c_i\xi_{k-i}\right)\right)\right)^2\\ &= t^{1/2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_k a_l\left(\mathbb{E}\exp\sum_{i=1}^{N} c_i\xi_{k-i}\exp\sum_{i=1}^{N} c_i\xi_{l-i}\right.\\ &\qquad \left.-\mathbb{E}\exp\sum_{i=1}^{N} c_i\xi_{k-i}\mathbb{E}\exp\sum_{i=1}^{N} c_i\xi_{l-i}\right).\end{aligned}$$

Since, for $|k-l| > N$, $\Sigma_{i=1}^{N} c_i \xi_{k-i}$ and $\Sigma_{i=1}^{N} c_i \xi_{l-i}$ are independent, we have that

$$\begin{aligned}&\mathbb{E}\big(t^{-1/4}Z(t,x\sqrt{t})\big)^2\\ &\quad = t^{1/2}\sum_{k=-\infty}^{\infty}\sum_{|k-l|\le N} a_k a_l\left(\mathbb{E}\exp\sum_{i=1}^{N} c_i\xi_{k-i}\exp\sum_{i=1}^{N} c_i\xi_{l-i}\right.\\ &\qquad\left.-\left(\prod_{i=1}^{N}\mathbb{E}\exp(c_i\xi)\right)^2\right).\end{aligned}$$

Because $c_i = 0$, for $i \notin \{1,\dots,N\}$ and $|k-l| \le N$, we notice that

$$\begin{aligned}\sum_{i=1}^{N} c_i\xi_{k-i} + \sum_{i=1}^{N} c_i\xi_{l-i} &= \sum_{i=-\infty}^{\infty} c_i\xi_{k-i} + \sum_{i=-\infty}^{\infty} c_i\xi_{l-i}\\ &= \sum_{i=-\infty}^{\infty}\big(c_i + c_{i-(k-l)}\big)\xi_{k-i} = \sum_{i=-N}^{2N}\big(c_i + c_{i-(k-l)}\big)\xi_{k-i}.\end{aligned}$$

If we set $s = k - l$, we get that

$$\begin{aligned}&\mathbb{E}\big\{t^{-1/4}Z(t,x\sqrt{t})\big\}^2\\ &\quad = t^{1/2}\sum_{k=-\infty}^{\infty}\sum_{s=-N}^{N} a_k a_{k-s}\mathbb{E}\left(\exp\sum_{i=-N}^{2N}(c_i + c_{i-s})\xi_{k-i}\right.\\ &\qquad\left.-\left(\prod_{i=1}^{N}\mathbb{E}\exp(c_i\xi)\right)^2\right)\\ &\quad = t^{1/2}\sum_{i=-\infty}^{\infty}\sum_{s=-N}^{N} a_k a_{k-s}\left(\prod_{i=-N}^{2N}\mathbb{E}\exp\big((c_i+c_{i-s})\xi_{k-i}\big)\right.\\ &\qquad\left.-\left(\prod_{i=1}^{N}\mathbb{E}\exp(c_i\xi)\right)^2\right).\end{aligned}$$

Notice that

$$\begin{aligned}
&\Bigg|\mathbb{E}\big(t^{-1/4}Z(t, x\sqrt{t}\,)\big)^2 \\
&\quad - t^{1/2}\sum_{k=-\infty}^{\infty} a_k^2 \sum_{s=-N}^{N}\left(\prod_{i=-N}^{2N}\mathbb{E}\exp((c_i + c_{i-s})\xi_{k-i}) - \left(\prod_{i=1}^{N}\mathbb{E}\exp(c_i\xi)\right)^2\right)\Bigg| \\
&\le \max_{s\in\{-N,\dots,N\}}\left|\prod_{i=-N}^{2N}\mathbb{E}\exp((c_i + c_{i-s})\xi) - \left(\prod_{i=1}^{N}\mathbb{E}\exp(c_i\xi)\right)^2\right| t^{1/2} \\
&\quad \times \left|\sum_{k=-\infty}^{\infty} a_k \sum_{s=-N}^{N} a_{k-s} - \sum_{k=-\infty}^{\infty}\sum_{s=-N}^{N} a_k^2\right| \\
&\le C t^{1/2}\Bigg|\sum_{s=-N}^{N}\sum_{k=-\infty}^{\infty} a_k \int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}}\left(\left(u - x - \frac{s}{\sqrt{t}}\right)\exp\left(-\frac{(u - x - s/\sqrt{t}\,)^2}{2}\right)\right. \\
&\qquad\qquad\qquad\qquad \left. -(u-x)\exp\left(-\frac{(u-x)^2}{2}\right)\right) du\Bigg| \\
&\le C\tilde{C}\sum_{s=-N}^{N}\int_{-\infty}^{\infty}\Bigg|\left(u - x - \frac{s}{\sqrt{t}}\right)\exp\left(-\frac{(u - s/\sqrt{t} - x)^2}{2}\right) \\
&\qquad\qquad\qquad\qquad -(u-x)\exp\left(-\frac{(u-x)^2}{2}\right)\Bigg| du
\end{aligned}$$

and that, by the Lebesgue dominated convergence theorem, the last term converges to 0 when $t \to \infty$. In the last inequality we used the fact that

$$|a_k| \le \frac{1}{\sqrt{t}}\max_u\left(|u - x|\exp\left(-\frac{(u-x)^2}{2}\right)\right) = \frac{\tilde{C}}{\sqrt{t}}.$$

Finally, it is easy to check that $\lim_{t\to\infty} t^{1/2}\sum_{-\infty}^{\infty} a_k^2 = V \in (0,\infty)$, which gives Lemma 3.2. □

The proof of the next lemma will require the following estimate in the central limit theorem for dependent random variables, which is due to Bulinski (1987).

PROPOSITION 3.1. *Let $\{X_j(t),\ j \in U(t)\}$ be an $m(t)$-dependent field on a finite set $U(t) \subset \mathbb{Z}^d$ and let, for some $s \in (2,3]$ and all $t > 0$,*

$$\sup_{j\in U(t)}\left(\mathbb{E}|X_j(t)|^s\right)^{1/s} = C_s(t) < \infty.$$

Then

$$\sup_{x\in\mathbb{R}}\left|P\left(\delta^{-1}(t)\sum_{j\in U(t)}\left(X_j(t)-\mathbb{E}X_j(t)\right)\le x\right)-\Phi(x)\right|$$
$$\le k_0|U(t)|M_s^s(t)m^{d(s-1)}(t)+M_s(t)m^d(t)\log^{(d-1)/2}|U(t)|$$
$$+|U(t)|^{1/2}M_s^2(t)m^{2d/s}(t),$$

where $\delta(t)=(\operatorname{Var}\sum_{j\in U(t)}X_j(t))^{1/2}>0$, $k_0=k_0(d)$, $|U(t)|$ *is the number of points in* $U(t)$, $M_s(t)=\delta^{-1}(t)C_s(t)$ *and* $\Phi(x)$ *is the distribution function of* $N(0,1)$.

The last lemma we need to prove Theorem 2.1 gives an explicit limit distribution for the numerator Z in the solution (1.2).

LEMMA 3.3. *The distribution of* $t^{-1/4}Z(t,x\sqrt{t})$ *converges weakly to* $N(0,V\tilde{\sigma}(\mathbf{c}))$ *as* $t\to\infty$.

PROOF. Throughout this proof, $f\asymp g$ means that

$$0<\liminf_{t\to\infty}\frac{g(t)}{f(t)}\le\limsup_{t\to\infty}\frac{g(t)}{f(t)}<\infty.$$

Let

$$b_k(t)=t^{1/4}\int_{k/\sqrt{t}}^{(k+1)/\sqrt{t}}(y-x)\exp\left(-\frac{(y-x)^2}{2}\right)dy\prod_{i=1}^{N}\exp(c_i\xi_{k-i}).$$

We claim that for each fixed x, as $t\to\infty$,

$$\sum_{k=\lfloor t^{5/8}\rfloor}^{\infty}b_k(t)\to 0\quad\text{and}\quad\sum_{k=-\infty}^{\lfloor -t^{5/8}\rfloor}b_k(t)\to 0,$$

in probability. Indeed,

$$\mathbb{E}\left(\sum_{k=\lfloor t^{5/8}\rfloor}^{\infty}b_k(t)\right)\asymp t^{1/4}\left(\int_{t^{1/8}}^{\infty}(y-x)\exp\left(-\frac{(y-c)^2}{2}\right)dy\right)$$
$$\asymp t^{1/4}\exp(-t^{1/4})\to 0$$

as $t\to\infty$ and

$$\sum_{k=\lfloor t^{5/8}\rfloor}^{\infty}b_k(t)=\sum_{l=0}^{N-1}\sum_{m=0}^{\infty}b_{\lfloor t^{5/8}\rfloor+mN+l}(t).$$

Hence, for each fixed l,

$$\operatorname{Var}\left(\sum_{m=0}^{\infty}b_{\lfloor t^{5/8}\rfloor+mN+l}(t)\right)\asymp t^{1/2}\left(\int_{t^{1/8}}^{\infty}(y-x)\exp\left(-\frac{(y-x)^2}{2}\right)dy\right)^2$$
$$\asymp t^{1/2}\exp(-2t^{1/4})\to 0$$

as $t\to\infty$. Now, an application of Chebyshev's inequality gives our first claim.

The second claim is proved exactly in the same fashion. Finally, since $\{b_k(t)\}$ are N-dependent,

$$\sup_j \left(\mathbb{E}|b_j(t)|^3\right)^{1/3} \asymp t^{-1/4},$$

and since, by Lemma 3.2,

$$\delta(t) = \left(\operatorname{Var} \sum_{k=\lfloor -t^{5/8}\rfloor}^{\lfloor t^{5/8}\rfloor} b_k(t)\right)^{1/2} \asymp V\tilde{\sigma}(\mathbf{c}) > 0,$$

we get that

$$M_3(t) = \frac{1}{\delta(t)} \sup_j \left(\mathbb{E}|b_j(t)|^3\right)^{1/3} \asymp t^{-1/4},$$

so that, using Proposition 3.1, we obtain that

$$\sup_{x\in\mathbb{R}} \left| P\left(\delta^{-1}(t) \sum_{k=\lfloor t^{5/8}\rfloor}^{\lfloor t^{5/8}\rfloor} b_k(t) \le x\right) - \Phi(x)\right|$$
$$\le C\left(t^{5/8}(t^{-1/4})^3 + (t^{5/8})^{1/2}(t^{-1/4})^2\right) \asymp t^{-1/8} \to 0$$

as $t \to \infty$, which gives the conclusion of Lemma 3.3. □

Now, we are in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Since

$$t^{5/4}u(t, x\sqrt{t}) = \frac{t^{-1/4}Z(t, x\sqrt{t})}{t^{-1/2}I(t, x\sqrt{t})},$$

using Lemma 3.1 and Lemma 3.3, we get that

$$t^{5/4}u(t, x\sqrt{t}) \to N\left(0, \frac{V\tilde{\sigma}(\mathbf{c})}{2\pi\left(\prod_{i=1}^N \mathbb{E}e^{c_i\xi}\right)^2}\right).$$

Grouping summands in the expression (3.4), separately for $s = 0, \pm 1, \pm 2, \dots, \pm N$, we get that

$$\begin{aligned}\tilde{\sigma}(\mathbf{c}) = {} & \mathbb{E}e^{2c_1\xi} \cdots \mathbb{E}e^{2c_N\xi} \\ & + 2\mathbb{E}e^{c_1\xi}\mathbb{E}e^{(c_1+c_2)\xi} \cdots \mathbb{E}e^{(c_{N-1}+c_N)\xi}\mathbb{E}e^{c_N\xi} \\ & + 2\mathbb{E}e^{c_1\xi}\mathbb{E}e^{c_2\xi}\mathbb{E}e^{(c_1+c_3)\xi} \cdots \mathbb{E}e^{(c_{N-2}+c_N)\xi}\mathbb{E}e^{c_{N-1}\xi}\mathbb{E}e^{c_N\xi} + \cdots \\ & + 2\mathbb{E}e^{c_1\xi} \cdots \mathbb{E}e^{c_{N-1}\xi}\mathbb{E}e^{(c_1+c_N)\xi}\mathbb{E}e^{c_2\xi} \cdots \mathbb{E}e^{c_N\xi} \\ & + 2(\mathbb{E}e^{c_1\xi} \cdots \mathbb{E}e^{c_N\xi})^2 - (2N+1)\left(\prod_{i=1}^N \mathbb{E}e^{c_i\xi}\right)^2,\end{aligned}$$

and taking into account definitions (2.2) and (2.3), we get, by a simple calculation, that

$$\sigma(\mathbf{c}) = \frac{\tilde{\sigma}(\mathbf{c})}{\left(\prod_{i=1}^{N} \mathbb{E}\exp(c_i \xi)\right)^2}. \qquad \square$$

The function h introduced in (2.3) enjoys certain additional properties which are listed in the following lemma. They will be used later on.

LEMMA 3.4. *If $a, b \geq 0$, then $h(a, b) \geq 0$ and, additionally, the function $h(a, b)$ is increasing in each variable.*

PROOF. The condition $h(a, b) \geq 0$ is equivalent to the condition $\mathbb{E}e^{(a+b)\xi} \geq \mathbb{E}e^{a\xi}\mathbb{E}e^{b\xi}$. Let η be an independent copy of ξ. Since $(e^{a\xi} - e^{a\eta})(e^{b\xi} - e^{b\eta}) \geq 0$ in view of the monotonicity of the functions e^{ax} and e^{bx}, we get that $\mathbb{E}(e^{a\xi} - e^{a\eta})(e^{b\xi} - e^{b\eta}) \geq 0$, which is equivalent to $h(a, b) \geq 0$.

As far as the monotonicity of h is concerned, notice that

$$\frac{\partial}{\partial a} h(a, b) = \frac{\partial}{\partial a} \log \frac{\mathbb{E}e^{(a+b)\xi}}{\mathbb{E}e^{a\xi}\mathbb{E}e^{b\xi}} = \frac{\mathbb{E}\xi e^{(a+b)\xi}}{\mathbb{E}e^{(a+b)\xi}} - \frac{\mathbb{E}\xi e^{a\xi}}{\mathbb{E}e^{a\xi}},$$

so that it suffices to prove that $f(x) = \mathbb{E}\xi e^{x\xi}/\mathbb{E}e^{x\xi}$ is increasing in $x > 0$. The property follows from the fact that

$$\frac{\partial f(x)}{\partial x} = \frac{\mathbb{E}\xi^2 e^{x\xi} \cdot \mathbb{E}e^{x\xi} - \left(\mathbb{E}\xi e^{x\xi}\right)^2}{\left(\mathbb{E}e^{x\xi}\right)^2} = \mathbb{E}\left(\xi^2\left(\frac{e^{x\xi}}{\mathbb{E}e^{x\xi}}\right)\right) - \left(\mathbb{E}\left(\xi\frac{e^{x\xi}}{\mathbb{E}e^{x\xi}}\right)\right)^2 \geq 0.$$

$\square$

PROOF OF THEOREM 2.2. The proof follows directly from Theorem 2.1 and Lemma 3.4. $\square$

4. Extremal property.

In this section, we prove Theorem 2.3—the main result of this paper. The proof, which is somewhat involved, will be based on a domination technique in the spirit of domination principles of Kwapien and Woyczynski (1992) and it relies on the notion of Schur convexity. We begin with some notation and a couple of definitions.

In what follows, we denote by $(x_{(i)})_{i=1}^N$ and, respectively, $(x_{[i]})_{i=1}^N$, the order statistics and the reverse order statistics of the finite sequence $(x_i)_{i=1}^N$, which thus satisfy, respectively, the conditions $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)}$ and $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[N]}$.

DEFINITION 4.1. Let $\mathbf{x} = (x_1, \dots, x_N) \geq 0$ and $\mathbf{y} = (y_1, \dots, y_N) \geq 0$. We shall say that $\mathbf{x} \succ \mathbf{y}$ (read "$\mathbf{x}$ dominates $\mathbf{y}$") if $\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]}$, $k = 1, \dots, N-1$, and $\sum_{i=1}^N x_{[i]} = \sum_{i=1}^N y_{[i]}$. Equivalently, $\mathbf{x} \succ \mathbf{y}$ if and only if $\sum_{i=1}^k x_{(i)} \leq \sum_{i=1}^k y_{(i)}$, $k = 1, \dots, N-1$, and $\sum_{i=1}^N x_{(i)} = \sum_{i=1}^N y_{(i)}$.

The following property of the order relation $\succ$ follows directly from its definition.

PROPOSITION 4.1. *Let* $\mathbf{a} = (a_i)_{i=1}^N$, $\mathbf{b} = (b_i)_{i=1}^N$, $\mathbf{c} = (c_i)_{i=1}^N$ *and* $\mathbf{d} = (d_i)_{i=1}^N$. *If sequences* (b_i) *and* (d_i) *are either both nonincreasing or both nondecreasing in* i, $\mathbf{a} \prec \mathbf{b}$ *and* $\mathbf{c} \prec \mathbf{d}$, *then* $\mathbf{a} + \mathbf{c} \prec \mathbf{b} + \mathbf{d}$.

Functions of several variables which are monotone with respect to the above order relation are called Schur-convex. More formally, we have the following definition.

DEFINITION 4.2. Let $I \subset \mathbb{R}$ be an interval. A function $f(\mathbf{x})$: $I^n \to \mathbb{R}$ is said to be Schur-convex if $\mathbf{x} \succ \mathbf{y}$ implies that $f(\mathbf{x}) \geq f(\mathbf{y})$.

It is clear that any Schur-convex function is a symmetric function of its variables. The following, easy to establish, result [see also Schur (1923) and Hardy, Littlewood and Pólya (1929)] shows one of the reasons why the notion of Schur convexity is so useful.

PROPOSITION 4.2. *If* $I \subset \mathbb{R}$ *is an interval, and* φ: $I \to \mathbb{R}$ *is a convex function, then* $\phi(\mathbf{x}) = \sum_{i=1}^n \varphi(x_i)$ *is Schur-convex on* I^n.

Thus, in particular, we get the following two examples of Schur-convex functions which are of direct interest in what follows.

LEMMA 4.1. *The functions* $H(\mathbf{x}) = \sum_{i=1}^n e^{x_i}$ *and* $G(\mathbf{x}) = \sum_{i=1}^n \log(\mathbb{E} e^{x_i \xi})$ *are Schur-convex on* $[0, \infty)$.

PROOF. For $H(\mathbf{x})$ it is obvious. As far as $G(\mathbf{x})$ is concerned, $d \log(\mathbb{E} e^{t\xi})/dt = \mathbb{E}\xi e^{t\xi}/\mathbb{E} e^{t\xi}$, and we already know by Lemma 3.4 that $\mathbb{E}\xi e^{t\xi}/\mathbb{E} e^{t\xi}$ is nondecreasing. Now, we get Lemma 4.1 using Proposition 4.2. □

REMARK 4.1. Although the following observation is not directly used in what follows, it gives a good insight into properties of the function h entering in the formula determining $\sigma(\mathbf{c})$, which are related to our domination principle. So, let $h(a, b)$ be the function defined in (2.3). If $\mathbf{u} = (u_1, \dots, u_n) \geq 0$ and $\mathbf{v} = (v_1, \dots, v_n) \geq 0$, then

$$\sum_{i=1}^n h(u_i, v_i) \leq \sum_{i=1}^n h(u_{[i]}, v_{[i]}).$$

Indeed, since

$$\sum_{i=1}^{n} h(u_i, v_i) = \sum_{i=1}^{n} \log \mathbb{E} \exp((u_i + v_i)\xi) - \sum_{i=1}^{n} \log \mathbb{E} e^{u_i \xi} - \sum_{i=1}^{n} \log \mathbb{E} e^{v_i \xi},$$

$$\sum_{i=1}^{n} h(u_{[i]}, v_{[i]}) - \sum_{i=1}^{n} h(u_i, v_i) = \sum_{i=1}^{n} \log \mathbb{E} \exp((u_{[i]} + v_{[i]})\xi) - \sum_{i=1}^{n} \log \mathbb{E} \exp((u_i + v_i)\xi)$$

and $(u_{[1]} + v_{[1]}, \dots, u_{[n]} + v_{[n]}) \succ (u_1 + v_1, \dots, u_n + v_n)$, we get the result using Proposition 4.1 and Lemma 4.1.

Now, the strategy of the proof of Theorem 2.3 in the Gaussian case is as follows. Initially, we will take a closer look at the sequence

$$r_k(\mathbf{c}) := \sum_{i=1}^{k} c_i c_{i+N-k} = R_{N-k}(\mathbf{c}), \qquad k = 1, \dots, N-1, \tag{4.1}$$

where $R_k(\mathbf{c})$ is the covariance function of the initial moving average process U_0 [see (2.1)]:

$$R_k(\mathbf{c}) = E(U_0(k)U_0(0)) - EU_0(k)EU_0(0) = \sum_{i=1}^{N-k} c_i c_{i+k}, \tag{4.2}$$

and show that the centered unimodality of $\mathbf{c}$ [the fundamental assumption (2.5) in Theorem 2.3] makes the sequence $r_1(\mathbf{c}), \dots, r_{N-1}(\mathbf{c})$ maximal, in terms of the order relation $\succ$, among sequences $r_1(\Pi\mathbf{c}), \dots, r_{N-1}(\Pi\mathbf{c})$, where $\Pi\mathbf{c}$ is an arbitrary permutation of components of $\mathbf{c}$.

This goal is accomplished in a series of five lemmas (4.2–4.6) which prove the above domination principle by establishing first similar domination principles established for simpler sequences.

Once this $\succ$-domination property of $r_1(\mathbf{c}), \dots, r_{N-1}(\mathbf{c})$ for centered unimodal $\mathbf{c}$ is obtained, the proof of Theorem 2.3 can be concluded by producing a Schur-convex function H such that

$$\sigma(\mathbf{c}) = H(\mathbf{r}(\mathbf{c})).$$

The proof of Theorem 2.3 in the Poisson case follows a variation of the above outline.

The first in the series of five preparatory lemmas demonstrates that for a centered unimodal $\mathbf{c}$, the sequence $r_1(\mathbf{c}), \dots, r_{N-1}(\mathbf{c})$ is nondecreasing.

LEMMA 4.2. *Assume that* $\mathbf{c} = (c_1, \dots, c_N) \geq 0$ *satisfies the centered unimodality condition* (2.5). *Then* $r_1(\mathbf{c}) \leq \cdots \leq r_{N-1}(\mathbf{c})$, *that is*,

$$(r_k(\mathbf{c}))_{k=1}^{N-1} = (r_{(k)}(\mathbf{c}))_{k=1}^{N-1}.$$

PROOF. The lemma is obviously true in the case $N = 2$. We prove the general case by induction on N. Suppose that the lemma is true for all dimensions less than or equal to $N - 1$ and all $\mathbf{v} \geq 0$ which satisfy condition (2.5). The proof that Lemma 4.2 is valid for the dimension equal to N will be carried out as follows:

First, it is easy to check that, for $k = 1, \dots, N - 1$,

$$r_k(c_1, c_2, \dots, c_N) = r_k(c_N, c_{N-1}, \dots, c_1). \tag{4.3}$$

Next, let $M = \min_k c_k$ and define $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_N)$ such that

$$\tilde{c}_i = c_{N-i+1} - M.$$

When $\mathbf{c}$ satisfies condition (2.5), then $\tilde{c}_N = 0$ and $\hat{c} = (\tilde{c}_1, \dots, \tilde{c}_{N-1})$ also satisfies condition (2.5). It can be easily checked that $r_1(\tilde{c}) = 0$ and $r_{k+1}(\tilde{c}) = r_k(\hat{c})$, for $k = 1, \dots, N - 2$ (the above property will be used again in the proof of Lemma 4.6). Using the inductive assumption, we have that

$$r_1(\tilde{c}) \leq r_2(\tilde{c}) \leq \cdots \leq r_{N-1}(\tilde{c}). \tag{4.4}$$

Finally, we notice that

$$r_k(\mathbf{c}) = kc_1^2 + c_1 \sum_{i=1}^{k} (\tilde{c}_i + \tilde{c}_{i+N-k}) + r_k(\tilde{c})$$

and that $\sum_{i=1}^{k}(\tilde{c}_i + \tilde{c}_{i+N-k})$ is increasing in k for $k = 1, \dots, N - 1$, so that $r_k(\mathbf{c})$ is also increasing in k for $k = 1, \dots, N - 1$. □

In Lemma 4.3 we find a $\succ$-dominating sequence for sequences $r_1(\mathbf{c}), \dots, r_{N-1}(\mathbf{c})$ in the case where the components of $\mathbf{c}$ are either 0 or 1. In what follows, $\boldsymbol{\chi}_D = (\chi_D(1), \dots, \chi_D(N))$, where χ_D is the indicator function of a finite set $D \subset \{1, \dots, N\}$ and $|D|$ is the cardinality of D.

LEMMA 4.3. *If $D \subset \{1, \dots, N\}$ is a nonempty set, then*

$$(r_1(\boldsymbol{\chi}_D), \dots, r_{N-1}(\boldsymbol{\chi}_D)) \prec (0, \dots, 0, 1, 2, \dots, |D| - 1), \tag{4.5}$$

where r_k's are defined in (4.1).

PROOF. We proceed by induction on the rows of the array $(N, |D|)$. It is easy to check that Lemma 4.3 is true for $N = 2$, $|D| = 2$ and for any N if $|D| = 1$. Our inductive assumption is that the lemma is true for all $N \leq K - 1$, $|D|$ such that $|D| = 1, 2, \dots, N$, and for $N = K$, $|D| \leq l - 1$ for a certain $l \leq K$. We only need to prove the validity of the lemma for $N = K$, $|D| = l$. We will consider two cases depending on whether $\chi_D(1) = 0$ or 1.

If $\chi_D(1) = 0$, then by the definition of $r_k(\boldsymbol{\chi}_D)$, the case is reducible to the case $N = K - 1$, $|D| = l$, which holds true by the inductive assumption.

If $\chi_D(1) = 1$, then we introduce a new vector

$$\mathbf{d} = (d_1, \dots, d_N) = (0, \chi_D(2), \dots, \chi_D(N)), \tag{4.6}$$

so that $|\{i\colon d_i = 1\}| = l - 1$. Using the inductive assumption, we have that

$$(r_1(\mathbf{d}), \dots, r_{N-1}(\mathbf{d})) \prec (0, \dots, 0, 1, \dots, l-2). \tag{4.7}$$

Since $\chi_D(1) = 1$,

$$r_k(\chi_D) = \sum_{i=1}^{k} \chi_D(i)\chi_D(i+N-k) = d_{1+N-k} + r_k(\mathbf{d})$$

and

$$(d_N, \dots, d_2) \prec (\mathbf{0}_{N-l}, \mathbf{1}_{l-1}), \tag{4.8}$$

where $\mathbf{0}_i$ is the vector with i components all equal to 0, and $\mathbf{1}_i$ is defined in a similar fashion. Therefore, by Proposition 4.1 and from (4.7) and (4.8), we get that

$$(r_1(\chi_D), \dots, r_{N-1}(\chi_D)) \prec (0, \dots, 0, 1, 2, \dots, l-1), \tag{4.9}$$

which completes the proof of the lemma. □

REMARK 4.2. Notice that when χ_D satisfies condition (2.5), then

$$(r_1(\chi_D), \dots, r_{N-1}(\chi_D)) = (0, \dots, 0, 1, 2, \dots, |D| - 1).$$

In the next two lemmas we will explore the $\succ$-domination by centered unimodal sequences $\mathbf{c}$, starting with 0, 1-sequences, for auxiliary functions w_k, which are defined as follows: For $\mathbf{c} = (c_1, \dots, c_N)$, $\mathbf{d} = (d_1, \dots, d_N)$ and $k = 1, \dots, N-1$,

$$w_k(\mathbf{c}, \mathbf{d}) := \sum_{i=1}^{k} (c_i + c_{i+N-k}) d_i d_{i+N-k}. \tag{4.10}$$

It is easy to see that

$$w_k(\mathbf{c}_1 + \mathbf{c}_2, \mathbf{d}) = w_k(\mathbf{c}_1, \mathbf{d}) + w_k(\mathbf{c}_2, \mathbf{d}) \tag{4.11}$$

and

$$w_k(\alpha \mathbf{c}, \mathbf{d}) = \alpha w_k(\mathbf{c}, \mathbf{d}). \tag{4.12}$$

LEMMA 4.4. *Let $D_1 \subset D \subset \{1, 2, \dots, N\}$ and let Π be a permutation on the set $\{1, \dots, N\}$ such that both*

$$\Pi \chi_{D_1} = (\chi_{D_1}(\Pi(1)), \dots, \chi_{D_1}(\Pi(N)))$$

and

$$\Pi \chi_D = (\chi_D(\Pi(1)), \dots, \chi_D(\Pi(N)))$$

satisfy the centered unimodality condition (2.5). *Then*

$$\left(w_k(\chi_{D_1}, \chi_D)\right)_{k=1}^{N-1} \prec \left(w_k(\Pi\chi_{D_1}, \Pi\chi_D)\right)_{k=1}^{N-1}. \tag{4.13}$$

PROOF. The proof will be carried out in three steps.

STEP 1. Let $\mathbf{e}^j = (e_1^j, \dots, e_N^j) = (0, \dots, 0, 1, 0, \dots, 0)$, $j = 1, \dots, N$. Then $\chi_{D_1} = \sum_{j=1}^{N} \chi_{D_1}(j)\mathbf{e}^j$ and, using property (4.11) and (4.12), we have, for each

$k = 1, \dots, N-1$,

$$w_k(\chi_{D_1}, \chi_D) = \sum_{j=1}^{N} \chi_{D_1}(j) w_k(\mathbf{e}^j, \chi_D). \tag{4.14}$$

By a simple calculation, we obtain that, for each $k = 1, \dots, N-1$ and each $j = 1, \dots, N$,

$$\begin{aligned} w_k(\mathbf{e}^j, \chi_D) &= \sum_{i=1}^{k} (e_i^j + e_{i+N-k}^j) \chi_D(i) \chi_D(i+N-k) \\ &= \chi_D(j)(\chi_D(j+N-k) + \chi_D(j-N+k)). \end{aligned} \tag{4.15}$$

Putting (4.15) into (4.14), we get

$$\begin{aligned} &(w_1(\chi_{D_1}, \chi_D), \dots, w_{N-1}(\chi_{D_1}, \chi_D)) \\ &\quad = \sum_{i \in D} \chi_{D_1}(i) \chi_D(i) ((0, \dots, 0, \chi_D(1), \chi_D(2), \dots, \chi_D(i-1)) \\ &\qquad + (0, \dots, 0, \chi_D(N), \chi_D(N-1), \dots, \chi_D(i+1))). \end{aligned} \tag{4.16}$$

STEP 2. Define a sequence of vectors

$$\mathbf{a}_i = (\mathbf{0}, \mathbf{1}_{\sum_{s=1}^{i-1} \chi_D(s)}) + (\mathbf{0}, \mathbf{1}_{\sum_{s=i+1}^{N} \chi_D(s)}) \in \mathbb{R}^{N-1}.$$

Since

$$(0, \dots, 0, \chi_D(1), \dots, \chi_D(i-1)) \prec (\mathbf{0}, \mathbf{1}_{\sum_{s=1}^{i-1} \chi_D(s)})$$

and

$$(0, \dots, 0, \chi_D(N), \dots, \chi_D(i+1)) \prec (\mathbf{0}, \mathbf{1}_{\sum_{s=i+1}^{N} \chi_D(s)}),$$

using Proposition 4.1, (4.16) and the monotonicity of $\mathbf{a}_i$'s, we get that

$$\{w_k(\chi_{D_1}, \chi_D)\}_{k=1}^{N-1} \prec \sum_{i \in D_1} \mathbf{a}_i = \sum_{i=1}^{N} \chi_{D_1}(i) \chi_D(i) \mathbf{a}_i. \tag{4.17}$$

STEP 3. From the definition of $\mathbf{a}_i$, we can see that, for $i \in D$, $\chi_D(i)\mathbf{a}_i \neq \mathbf{0}$ and $\chi_D(i)\mathbf{1} \cdot \mathbf{a}_i = |D| - 1$. Also, if $i \neq j$, $i, j \in D$, then

$$\chi_D(i)(\mathbf{0}, \mathbf{1}_{\sum_{s=1}^{i-1} \chi_D(s)}) \neq \chi_D(j)(\mathbf{0}, \mathbf{1}_{\sum_{s=1}^{j-1} \chi_D(s)})$$

and

$$\chi_D(i)(\mathbf{0}, \mathbf{1}_{\sum_{s=i+1}^{N} \chi_D(s)}) \neq \chi_D(j)(\mathbf{0}, \mathbf{1}_{\sum_{s=j+1}^{N} \chi_D(s)}).$$

On the other hand, for $i \notin D$, we have that $\chi_D(i)\mathbf{a}_i = \mathbf{0}$. Therefore

$$\sum_{i \in D_1} \mathbf{a}_i \prec \sum_{i=1}^{|D_1|} (\mathbf{0}, \mathbf{1}_{|D| - \lfloor (|D| - |D_1|)/2 \rfloor + i - 1}) + (\mathbf{0}, \mathbf{1}_{\lfloor (|D| - |D_1|)/2 \rfloor - i}) \tag{4.18}$$

and we can easily check that the right-hand side of (4.18) is equal to $(w_k(\Pi \chi_{D_1}, \Pi \chi_D))_{k=1}^{N-1}$. This completes the proof of Lemma 4.4. □

LEMMA 4.5. *Let* $D \subset \{1, \dots, N\}$ *and let* $\mathbf{c} \geq 0$ *be such that* $D_1 := \{i: c_i \neq 0\} \subset D$. *If* Π *is a permutation of* $\{1, \dots, N\}$ *such that both* $\Pi \boldsymbol{\chi}_D = (\chi_D(\Pi(i)))_{i=1}^N$ *and* $\Pi \mathbf{c} = (c_{\Pi(i)})_{i=1}^N$ *satisfy condition* (2.5), *then*

$$(w_k(\mathbf{c}, \boldsymbol{\chi}_D))_{k=1}^{N-1} \prec (w_k(\Pi\mathbf{c}, \Pi\boldsymbol{\chi}_D))_{k=1}^{N-1}. \tag{4.19}$$

PROOF. Define recursively pairs $(\delta(k), D_k)$, so that

$$D_1 = \{i: c_i \neq 0\} \quad \text{and} \quad \delta(1) = \min\{c(i): i \in D_1\},$$

$$D_{l+1} = \left\{i: c(i) - \sum_{m=1}^{l} \delta(m) \neq 0, i \in D_l\right\}$$

and

$$\delta(l+1) = \min\left\{c(i) - \sum_{m=1}^{l} \delta(m): i \in D_{l+1}\right\}.$$

If $D_k = \varnothing$, then $\delta(k) = 0$. Now, we can rewrite $\mathbf{c}$ as

$$\mathbf{c} = \sum_{i=1}^{N} \delta(i) \chi_{D_i}. \tag{4.20}$$

In view of (4.11) and (4.12), we have that

$$w_k(\mathbf{c}, \boldsymbol{\chi}_D) = \sum_{i=1}^{N} \delta(i) w_k(\boldsymbol{\chi}_{D_i}, \boldsymbol{\chi}_D). \tag{4.21}$$

Notice that, by Lemma 4.4, for each fixed i, we have that

$$\left(w_k(\boldsymbol{\chi}_{D_i}, \boldsymbol{\chi}_D)\right)_{k=1}^{N-1} \prec \left(w_k(\Pi\boldsymbol{\chi}_{D_i}, \Pi\boldsymbol{\chi}_D)\right)_{k=1}^{N-1}.$$

Also, from Lemma 4.2, we know that for any fixed i, $w_k(\Pi\boldsymbol{\chi}_{D_i}, \Pi\boldsymbol{\chi}_D)$ is increasing in k. By Proposition 4.1,

$$(w_k(\mathbf{c}, \boldsymbol{\chi}_D))_{k=1}^{N-1} \prec \left(\sum_{i=1}^{N} \delta(i) w_k(\Pi\boldsymbol{\chi}_{D_i}, \Pi\boldsymbol{\chi}_D\right)_{k=1}^{N-1}$$

and we can easily check that

$$(w_k(\Pi\mathbf{c}, \Pi\boldsymbol{\chi}_D))_{k=1}^{N-1} = \left(\sum_{i=1}^{N} \delta(i) w_k(\Pi\boldsymbol{\chi}_{D_i}, \Pi\boldsymbol{\chi}_D\right)_{k=1}^{N-1}, \tag{4.22}$$

which completes the proof of Lemma 4.5. □

The last lemma contains the central domination idea.

LEMMA 4.6. *If* $\mathbf{c}$ *satisfies the centered unimodality condition* (2.5), *and the sequence* $r_k(\mathbf{c})$, $k = 1, \dots, N-1$, *is as defined in* (4.1), *then, for any permutation on the components of* $\mathbf{c}$,

$$(r_i(\Pi\mathbf{c}))_{i=1}^{N-1} \prec (r_i(\mathbf{c}))_{i=1}^{N-1}. \tag{4.23}$$

PROOF. Let $D = \{i: c_i \neq 0\}$, $D^c = \{1, \dots, N\} - D$. For a fixed N and $|D| = 1$, Lemma 4.6 is trivially satisfied. Thus, we proceed by induction on $|D|$.

Suppose that the lemma is true for $|D| \le h - 1$. Introduce $M = \min\{c_i: c_i \neq 0\}$ and $\tilde{c} = (\tilde{c}_i)$ such that

$$\tilde{c}_i = \begin{cases} c_i - M, & \text{if } i \in D, \\ 0, & \text{if } i \in D^c. \end{cases}$$

Let $D_1 = \{i: \tilde{c}_i \neq 0\}$. Then, clearly, $D_1 \subset D$ and

$$\begin{aligned} r_k(\mathbf{c}) &= \sum_{i=1}^{k} c_i c_{i+N-k} = \sum_{i=1}^{k} c_i c_{i+N-k}\, \chi_D(i)\, \chi_D(i+N-k) \\ &= \sum_{i=1}^{k} \tilde{c}_i \tilde{c}_{i+N-k} + y_k^1(\tilde{c}, \chi_D) + y_k^2(\chi_D), \end{aligned}$$

where, for $k = 1, \dots, N-1$,

$$y_k^1(\tilde{c}, \chi_D) = M \sum_{i=1}^{k} (\tilde{c}_i + \tilde{c}_{i+N-k})\, \chi_D(i)\, \chi_D(i+N-k)$$

and

$$y_k^2(\chi_D) = M^2 \sum_{i=1}^{k} \chi_D(i)\, \chi_D(i+N-k).$$

Since $|\{i: \tilde{c}_i \neq 0\}| \le h - 1$, by the inductive assumption and the property mentioned in the proof of Lemma 4.2, we have that

$$(r_i(\Pi\tilde{c}))_{i=1}^{N-1} \prec (r_i(\tilde{c}))_{i=1}^{N-1}.$$

On the other hand, by Lemma 4.5 and again, by the property used in the proof of Lemma 4.2, we get that

$$\left(y_i^1\left(\Pi\tilde{c}, \Pi\chi_y^1\right)\right)_{i=1}^{N-1} \prec \left(y_i^1(\tilde{c}, \chi_D)\right)_{i=1}^{N-1},$$

which, in view of Lemma 4.3 and, once more by the property used in the proof of Lemma 4.2, gives that

$$\left(y_i^2(\Pi\chi_D)\right)_{i=1}^{N-1} \prec \left(y_i^2(\chi_D)\right)_{i=1}^{N-1}.$$

Therefore, Lemma 4.2 and Proposition 4.1 imply (4.23), which completes the proof of Lemma 4.6. □

Now we are in a position to prove our main theorem.

PROOF OF THEOREM 2.3. *The Gaussian case*. Recall that, by (2.2),

$$\sigma(\mathbf{c}) = 2 \sum_{k=1}^{N-1} \left(\exp \sum_{i=1}^{k} h(c_i, c_{i+N-k}) \right) + \exp\left(\sum_{i=1}^{N} h(c_i, c_i) \right) - (2N - 1),$$

and observe that $\exp(\sum_{i=1}^{N} h(c_i, c_i)) - (2N - 1)$ is invariant under a permutation Π of components $\mathbf{c}$. Also, since $\xi \sim N(0,1)$, we have that $h(a,b) = ab$ and

$$\sum_{k=1}^{N-1} \exp\left(\sum_{i=1}^{k} h(c_i, c_{i+N-k}) \right) = \sum_{k=1}^{N-1} \exp(r_k(\mathbf{c})),$$

where $r_k(\mathbf{c})$ is given by (4.1). Thus, if $\mathbf{c}$ satisfies the centered unimodality condition (2.5), then by Lemma 4.6,

$$(r_1(\mathbf{c}), \dots, r_{N-1}(\mathbf{c})) \succ (r_1(\Pi\mathbf{c}), \dots, r_{N-1}(\Pi\mathbf{c})).$$

Since $H(\mathbf{v}) = \sum_{i=1}^{N-1} e^{v_i}$ is Schur-convex, Proposition 4.2 yields that $\sigma(\mathbf{c}) \geq \sigma(\Pi\mathbf{c})$.

The Poisson case. Observe that the only property of h required by the above Gaussian case argument is that it is of the form $h(a,b) = v(a)v(b)$, where $v(x)$, $x \in [0,\infty)$, is increasing and $v(0) = 0$. Indeed, this follows from the fact that condition (2.5) depends only on the order of $\{c_i\}_{i=1}^{N}$ and the $v(c_i)$'s preserve the order of the c_i's.

For a Poisson random variable ξ, we have $h(a,b) = v(a)v(b)$ with $v(x) = e^x - 1$, which is increasing and satisfies $v(0) = 0$. The previous remark applies, and the proof of Theorem 2.3 is complete. □

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CENTER FOR STOCHASTIC AND CHAOTIC PROCESSES
IN SCIENCE AND TECHNOLOGY
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106