# OPTIMAL INVESTMENT AND CONSUMPTION
# WITH TRANSACTION COSTS

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A complete solution is provided to the infinite-horizon, discounted problem of optimal consumption and investment in a market with one stock, one money market (sometimes called a "bond") and proportional transaction costs. The utility function may be of the form $c^p/p$, where $p < 0$ or $0 < p < 1$, or may be $\log c$. It is assumed that the interest rate for the money market is positive, the mean rate of return for the stock is larger than this interest rate, the stock volatility is positive and all these parameters are constant. The only other assumption is that the value function is finite; necessary conditions for this are given.

In the Appendix (by S. Shreve), the sensitivity of the value function under the assumption $0 < p < 1$ is shown to be of the order of the transaction cost to the 2/3 power. This implies that the *liquidity premium* associated with small transaction costs is also of the order of the transaction cost to the 2/3 power. Because this power is less than 1, the *marginal liquidity premium* turns out to be infinite.

The analysis of this paper and its Appendix relies on the concept of viscosity solutions to Hamilton–Jacobi–Bellman equations. A self-contained treatment of this subject, adequate for the present application, is provided.

## CONTENTS

**1. Introduction.** In [54], Merton initiated the study of financial markets via continuous-time, stochastic models. Then as now, two principal issues were the role of financial intermediaries, such as mutual funds, and the interaction among many agents which leads to price formation. Merton chose to study these issues by first understanding the behavior of a single agent acting as a market price-taker and seeking to maximize expected utility of consumption. The utility function of the agent was assumed to be a power function, and the market was assumed to comprise a risk-free asset with constant rate of return and one or more stocks, each with constant mean rate of return and volatility. Current prices, but no other information, were available to the agent, there were no transaction costs and the assets were infinitely divisible. In this idealized setting, Merton was able to derive a closed-form solution to the stochastic control problem faced by the agent.

As a consequence of his work, Merton found that the stocks can be replaced by a mutual fund such that the agent is indifferent between investing in the risk-free asset and stocks individually or only in the risk-free asset and the mutual fund. Moreover, this mutual fund is independent of the agent's utility function. Thus, the financial intermediary of a mutual fund simplifies the model, but the mutual fund in this particular case is redundant.

In later papers [55], [56], Merton allowed the market coefficients to be nonconstant, depending on a "state" variable. In this context, Merton addressed the issue of price formation, writing down necessary conditions for equilibrium prices. He did not, however, resolve the question of existence of a solution to these conditions. Recent progress on existence and uniqueness of equilibrium can be found in [2], [13], [16], [18], [21], [35], [41], [42], [52] and [53].

Merton's model has been generalized in several directions. The restriction to utility functions of power form was removed in [39]. Market coefficients depending in an adapted way on an underlying Brownian motion were treated in [9], [40] and [58]. The present paper treats yet another generalization of Merton's model: the generalization in which transactions incur costs.

The introduction of proportional transaction costs to Merton's model was first accomplished by Magill and Constantinides [51]. These authors were apparently motivated by a desire to understand why mutual funds exist, and indeed, in a companion paper, Magill [50] argues that these transaction costs can make investment in a mutual fund preferable to investment in stocks individually. Magill and Constantinides also expressed hope that their work would "prove useful in determining the impact of trading costs on capital

market equilibrium." The analysis of transaction cost models has unfortunately not yet progressed to the point where this hope can be realized.

The Merton model with proportional transaction costs has also been used to price contingent claims. Hodges and Neuberger [34] initiated the idea of using such a model to determine what price for a contingent claim would make it attractive to investors. This idea has been further developed in [24], [15] and [8]. Other papers on pricing contingent claims in the presence of transaction costs include [28], [44], [5], [3] and [33]. Duffie and Sun [20] studied an altogether different structure for transaction costs and information accrual.

Grossman and Laroque [29] constructed a related model with transaction costs. Their model is designed so that the so-called *consumption-based capital asset pricing model* relation between consumption and risk premia is not valid. They were motivated by the fact that this relation had been rejected by empirical studies (e.g., [30] and [31]).

The present paper treats the case of a risk-free asset, which we call a *money market*, and one risky asset, a *stock*. In the constant-coefficient model studied by Merton, if there is only one stock, then the optimal portfolio holds a constant proportion of wealth in the stock. This constant, which we call the *Merton proportion*, depends on all the model parameters, but not on the wealth itself. To achieve this optimal portfolio, the agent must engage in continual trading. In the problem with proportional transaction costs, Magill and Constantinides [51] found that "the investor trades in securities when the variation in the underlying security prices forces his portfolio proportions outside a certain region about the optimal proportions in the absence of transaction costs." Under certain model parameters, this assertion is confirmed by our work. We find that, regardless of model parameters, the proportion of wealth held in the stock by the optimal portfolio remains in an interval whose endpoints depend on all the model parameters, but not on the wealth. It can happen, however, when one endpoint of this interval is larger than unity (the optimal portfolio borrows at the risk-free rate in order to invest in stock) that the Merton proportion is so large that it lies outside this interval. In other words, when leverage is optimal, the presence of transaction costs reduces the agent's desire for leverage, and the agent should trade to move from the Merton proportion to a less leveraged position.

The Appendix to this paper concerns the effect on the Merton model of the presence of transaction costs. One way of studying this is to estimate the "liquidity premium," defined to be the amount of increase in the rate of return for the stock which would be required to compensate the investor for the presence of the transaction costs. Constantinides [7] computed upper bounds for the liquidity premium in the model studied here. For transaction costs between 0.5 and 20%, this upper bound turns out to be approximately 0.14 times the transaction cost. Fleming, Grossman, Vila and Zariphopoulou [25] determine the asymptotic behavior of a model with transaction costs and with consumption occurring at the final time only. In their model, they discover that the liquidity premium is of the order of the transaction cost to

the 2/3 power. (See [63] and [22] for the solution of models closely related to that solved by [25]). We show in the Appendix that the 2/3 power behavior observed by [25] is present also in the Merton model with transaction costs. This behavior was apparently not observed in the numerical work of [7] because it appears only for levels of transaction cost smaller than 0.5%.

The model with transaction costs and intermediate consumption was posed in discrete time by Constantinides [6] and in continuous time by Magill and Constantinides [51]. The continuous-time model is now understood to be one of *singular* stochastic control, that is, the optimal solution can be described only in terms of singularly continuous processes. Although [51] shows clear insight into the nature of the optimal policy, the tools of singular stochastic control were unavailable to these authors. The introduction of these tools to the transaction cost problem was first accomplished by Taksar, Klass and Assaf [63], in the context of maximization of the rate of growth of wealth, and later by Davis and Norman [14], in the more difficult context of the Merton model with proportional transaction costs. Shreve, Soner and Xu [61] solve the problem for two risk-free assets paying different rates of interest. Zariphopoulou [66], [67] applied viscosity solution analysis to continuous-time transaction cost problems in which the randomness arises from a Markov chain governing the interest rate of one of the assets. For a recent account of singular stochastic control which includes a study of the transaction cost problem for the Merton model, using the viscosity solution techniques of this paper, we refer the reader to Fleming and Soner [26], Chapter 8.

Davis and Norman [14] provide a precise formulation and analysis, including an algorithm and numerical computations of the optimal policy, for the problem of this paper. Their work is a landmark in the study of transaction cost problems, and our paper is strongly influenced by theirs. Our purpose in revisiting this problem is threefold. First, as explained in the next two paragraphs, the results of [14] are obtained under restrictive and not fully verifiable assumptions. We succeed in removing these assumptions, replacing them by the sole assumption of a finite value function. In the process, we confirm some conjectures of [14] and refute others. Second, the approach of this paper provides a framework in which the liquidity premium estimation can be accomplished. Finally, this problem provides an opportunity to demonstrate once again the power of viscosity solution analysis in mathematical finance and singular stochastic control. In particular, because the viscosity solution approach is designed for *partial* rather than *ordinary* differential equations, it is well suited for an examination of the problem when more than one stock is present; see [1]. This paper is written for the reader who is not familiar with viscosity solutions, but wishes to be. The presentation of viscosity solutions in this paper is sufficient for the present application. For the reader whose appetite is whetted by this paper, we recommend the work of Zariphopoulou et al. [15], [17], [27], [66] and [67].

In order to compare our work to [14], we introduce some notation. A more detailed discussion appears in the next section. There is a *risk-free rate* or *interest rate r* and a single stock with *mean rate of return α* and *volatility σ*.

The utility for consumption is $c^p/p$, where $p < 1$, $p \neq 0$, or $\log c$ if $p = 0$. To make the problem nontrivial, it is assumed that $\alpha > r > 0$. The utility of consumption is discounted at rate $\beta > 0$. Letting $x$ denote the wealth initially invested in the money market, and letting $y$ denote the wealth initially invested in the stock, we denote by $v(x, y)$ the expected, discounted, infinite-horizon utility which can be obtained from optimal consumption when trading in the two assets incurs transaction costs.

A key insight noted by [51] and exploited in [14] is that because of homotheticity of $v$ (Proposition 3.3), the dimension of the problem can be reduced from two to one. In the analysis of the Hamilton–Jacobi–Bellman (HJB) equation for this problem, Davis and Norman consider the function $v(1, x)$ of the single variable $x$. This function is constructed as the solution to a two-point boundary value problem for a second-order ordinary differential equation in $x$. The endpoints $x_0$ and $x_T$ ($x_0 < x_T$) of this problem correspond to the endpoints of the optimal portfolio interval mentioned above, and their determination by the so-called "principle of smooth fit" is part of the two-point boundary problem. Because the ordinary differential equation is degenerate at $x = 0$, it is not apparent that $v(1, x)$ is smooth there. Davis and Norman assume that the Merton proportion $(\alpha - r)/(1 - p)\sigma^2$ is strictly less than 1, which guarantees that $x_0 > 0$ and the problem of degeneracy does not arise. With additional work, it may be possible to extend the analysis of [14] to cover the case $x_0 < 0$. In the present paper, we do not rely on the principle of smooth fit and can thus include the case $x_0 \leq 0$ with no extra work. It is in the case $x_0 < 0$ that we discover the phenomenon mentioned earlier that the presence of transaction costs reduces the agent's desire for leverage. A second issue which must be addressed is whether it is ever optimal to invest only in the money market. If it were, then there would be no right-hand endpoint $x_T$ required by the analysis of [14]. In order to guarantee the existence of $x_T$, Davis and Norman impose *Condition B*, which is the assumption that there is an initial point on a certain arc from which the trajectory of a two-dimensional, nonlinear differential system will reach a desired point. The relationship between Condition B and the model parameters is unclear. Davis and Norman succeed in proving the validity of this condition only when $\alpha - r < \sigma^2(1 - p)/2$ and, supported by numerical evidence, conjecture its validity in full generality. Although we do not study Condition B directly, we do establish that it is never optimal to invest only in the money market and hence the $x_T$ of [14] always exists (Theorem 11.6). Finally, the analysis of [14] is performed under the assumption

$$(1.1) \qquad \beta - rp > \frac{p(\alpha - r)^2}{2\sigma^2(1 - p)},$$

which is necessary and sufficient for the value function in the problem without transaction costs to be finite. Even when this condition is violated, as it is for $0 < p < 1$ and small volatility $\sigma$, the value function in the problem with transaction costs can be finite. In the present paper we make only the

assumption of a finite value function for the problem with transaction costs, and in Section 12 we provide two sufficient conditions for this in addition to (1.1).

In [26], the transaction cost problem is studied and the optimal strategy is constructed for the case $0 < p < 1$. Although parts of the present analysis closely follow [26], in this paper we allow $p$ to be zero or negative, we obtain further regularity on the value function (Section 10), we use this regularity to provide bounds on the location of the free boundaries which provide the basis for the optimal strategy (Section 11) and we examine the sensitivity of the value function to the transaction costs (Appendix).

The method of analysis of this paper depends on the newly developed concept of *viscosity solutions* to Hamilton–Jacobi–Bellman (HJB) equations. The foundational work on this subject is due to Crandall and Lions [10], Crandall, Evans and Lions [11], and Lions [45]. All these papers deal with first-order equations. However, the Hamilton–Jacobi–Bellman equation for a controlled diffusion process gives rise to a second-order equation. The extension of the viscosity theory to second-order equations was obtained in a series of papers by Lions [46]–[48], Jensen [38] and Ishii [37]. The recent survey article by Crandall, Ishii and Lions [12] provides a good account of the viscosity theory, and the application to stochastic control is reported in the book by Fleming and Soner [26]. The use of viscosity solutions in mathematical finance was initiated in the Ph.D. dissertation of Zariphopoulou [66].

The classical approach to stochastic control is to construct a function by ad hoc methods which solves the HJB equation, and then use this equation to verify that the constructed function is the value function. As seen in [14], the construction of this function often requires considerable ingenuity and sometimes the introduction of extraneous conditions. By contrast, the viscosity solution approach is to begin with the value function, assuming only its finiteness, and use the *principle of dynamic programming* (see Section 4) to show that it solves the HJB equation in the viscosity sense. With this toehold on the problem, the regularity of the value function can often be upgraded so that the HJB equation can be interpreted in the classical sense. The upgrading of regularity is not routine, but because the function under study is known to be the value function rather than merely a solution to the HJB equation, both control theory and differential equation arguments can be brought to bear. We also make heavy use of the concavity of the value function; see Section 6. In the present problem, this upgrading is important for two reasons. First, the optimal consumption process is obtained in feedback form on a partial derivative of the value function. In order to ensure that the resulting stochastic differential equation has a solution, it is necessary to obtain local Lipschitz continuity of this partial derivative. Second, using the continuity of a second partial derivative of the value function, in Section 11 we obtain bounds on the location of the endpoints of the interval in which the optimal proportion of wealth lies. These bounds allow us to draw the conclusion, mentioned above, that the Merton proportion is in this interval whenever the Merton proportion is less than 1, but the Merton

proportion can fall outside this interval when it is larger than 1 (the case of leverage).

**2. Formulation of the model.** Except for notational changes, the model formulation is that of [14]. The market under consideration consists of two investment opportunities, which we call a *money market* and a *stock*. (The money market is sometimes called a *bond*, but unlike a real bond, the asset in question has no maturity date and no risk of default.) An investor avails himself of these opportunities by purchasing shares. The price of a share of the money market at time $t \geq 0$ is

$$P_0(t) \triangleq e^{rt},$$

where $r > 0$ is a constant called the *interest rate* or the *risk-free rate*. The price of a share of stock at time $t \geq 0$ is

(2.1) $$P_1(t) \triangleq \exp\left[\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right],$$

where $\alpha > r$ and $\sigma > 0$ are constants called the *mean rate of return* and the *volatility*, respectively, of the stock. The process $\{W(t); \, t \geq 0\}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}_{t \geq 0}, P)$ with $W(0) = 0$ almost surely. We assume that $\mathscr{F} = \mathscr{F}(\infty)$, the filtration $\{\mathscr{F}(t)\}_{t \geq 0}$ is right-continuous and each $\mathscr{F}(t)$ contains all the $P$-null sets of $\mathscr{F}(\infty)$. Equation (2.1) can be written in the more suggestive differential notation

(2.2) $$dP_1(t) = P_1(t)[\alpha \, dt + \sigma \, dW(t)].$$

An agent is given in initial position of $x_0$ dollars invested in the money market and $y_0$ dollars invested in the stock. The agent must choose a consumption/investment policy consisting of three $\{\mathscr{F}(t)\}_{t \geq 0}$-adapted processes $C$, $L$ and $M$. The *consumption process* $C$ is required to be nonnegative and integrable on each finite time interval, that is,

$$C(t) \geq 0, \qquad \int_0^t C(s) \, ds < \infty \qquad \forall t \geq 0, \text{ a.s.}$$

The process $L$ and $M$ are right-continuous with left-hand limits (RCLL), nonnegative and nondecreasing. The cumulative dollar value of all money market shares sold for the purpose of buying stock is recorded by $L$, whereas $M$ records the sale of stock for the purpose of investment in the money market.

We denote by $X(t)$ [respectively, $Y(t)$] the number of dollars invested in the money market (respectively, stock) at time $t$, and we refer to $(X(t), Y(t))$ as the *position* of the agent at time $t$. We set

(2.3) $$X(0^-) = x_0, \qquad Y(0^-) = y_0$$

and let $(X, Y)$ evolve according to the equations

(2.4) $$dX(t) = (rX(t) - C(t)) \, dt - dL(t) + (1 - \mu) \, dM(t),$$

(2.5) $$dY(t) = \alpha Y(t) \, dt + \sigma Y(t) \, dW(t) + (1 - \lambda) \, dL(t) - dM(t).$$

The constants $0 \leq \mu < 1$ and $0 \leq \lambda < 1$ appearing in these equations account for proportional *transaction costs* incurred whenever wealth is moved from one asset to the other. Note that

(2.6)
$$X(0) = x_0 - L(0) + (1 - \mu)M(0),$$
$$Y(0) = y_0 + (1 - \lambda)L(0) - M(0)$$

may differ from $X(0^-), Y(0^-)$ because of a transaction at time zero.

Equations (2.4) and (2.5) capture the idea that investments in the money market grow at interest rate $r$ and investments in the stock fluctuate according to (2.2). A heuristic argument from discrete to continuous time to substantiate this claim appears on page 372 of [43]. More rigorous arguments related to this point can be found in [19], [32] and [65].

Following [14], we define the (open) *solvency region* (see Figure 1)

$$\mathscr{S} \triangleq \left\{ (x, y); \; x + \frac{y}{1 - \lambda} > 0, \; x + (1 - \mu)y > 0 \right\}$$

and we partition the boundary into

$$\partial_1 \mathscr{S} \triangleq \left\{ (x, y); \; y \leq 0, \; x + \frac{y}{1 - \lambda} = 0 \right\},$$
$$\partial_2 \mathscr{S} \triangleq \left\{ (x, y); \; y > 0, \; x + (1 - \mu)y = 0 \right\}.$$

The solvency region is the set of positions from which the agent can move to a position of positive wealth in both assets. If the agent's wealth is on $\partial_2 \mathscr{S}$ and he sells stock in order to pay his debt to the money market, the agent comes to position $(0, 0)$. If the agent's wealth is on $\partial_1 \mathscr{S}$ and he uses funds from the money market to cover his short position in the stock, the agent again arrives at position $(0, 0)$.
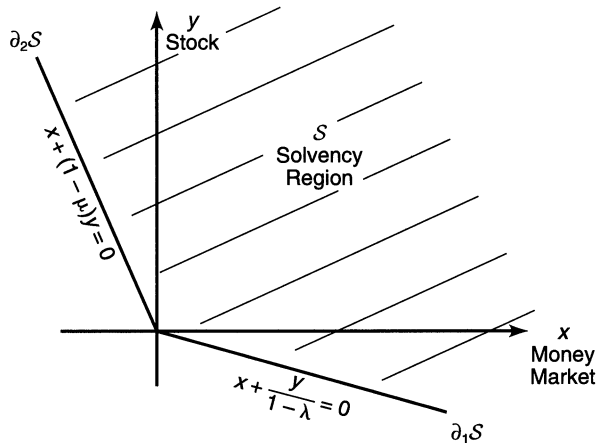


FIG. 1.  *The solvency region.*

We assume that the agent is given an initial position $(x_0, y_0) \in \bar{\mathscr{S}}$. A consumption/investment policy $(C, L, M)$ is *admissible for* $(x_0, y_0)$ if $(X(t), Y(t))$ given by (2.3)–(2.5) is in $\bar{\mathscr{S}}$ for all $t \geq 0$. We denote by $\mathscr{A}(x_0, y_0)$ the set of all such policies.

REMARK 2.1. If $(x_0, y_0) \in \partial \mathscr{S}$, the only admissible policy is to jump immediately to the origin and remain there. In particular, $C$ must be identically zero.

To verify the assertion for $(x_0, y_0) \in \partial_1 \mathscr{S}$, let $(C, L, M) \in \mathscr{A}(x_0, y_0)$ be given and note from (2.6) that $X(0) + Y(0)/(1 - \lambda) = [1 - \mu - 1/(1 - \lambda)]M(0)$, which is not allowed to be negative. Thus, $M(0) = 0$ and $(X(0), Y(0)) \in \partial_1 \mathscr{S}$. Furthermore, (2.4) and (2.5) show that

$$d\left[ e^{-rt}\left( X(t) + \frac{1}{1 - \lambda}Y(t) \right) \right]$$

(2.7)
$$= e^{-rt}\left[ \frac{\alpha - r}{1 - \lambda}Y(t)\, dt + \frac{\sigma}{1 - \lambda}Y(t)\, dW(t) - C(t)\, dt \right.$$

$$\left. + \left( 1 - \mu - \frac{1}{1 - \lambda} \right) dM(t) \right].$$

Let $\tau \triangleq 1 \wedge \inf\{t \geq 0;\, Y(t) \notin (y_0 - 1, 0)\}$ and integrate (2.7) to obtain

$$0 \leq e^{-r\tau}\left( X(\tau) + \frac{1}{1 - \lambda}Y(\tau) \right)$$

$$= \int_0^\tau e^{-rt}\left[ \frac{\alpha - r}{1 - \lambda}Y(t)\, dt - C(t)\, dt + \left( 1 - \mu - \frac{1}{1 - \lambda} \right) dM(t) \right]$$

$$+ \frac{\sigma}{1 - \lambda}\int_0^\tau e^{-rt}Y(t)\, dW(t)$$

$$\leq \frac{\sigma}{1 - \lambda}\int_0^\tau e^{-rt}Y(t)\, dW(t).$$

However, $E\int_0^\tau e^{-rt}Y(t)\, dW(t) = 0$, from which we conclude that

$$\frac{\sigma}{1 - \lambda}\int_0^\tau e^{-rt}Y(t)\, dW(t) = 0 \quad \text{a.s.}$$

Because $\sigma > 0$, this implies $\tau = 0$ almost surely, and because $Y$ is right-continuous and $Y(0) \geq y_0$ [because $M(0) = 0$], we must have $Y(0) \geq 0$. It is easily verified that because $(x_0, y_0) \in \partial_1 \mathscr{S}$, $Y(0) > 0$ is incompatible with $(X(0), Y(0)) \in \bar{\mathscr{S}}$, and we conclude $Y(0) = 0$ almost surely.

For the assertion concerning $(x_0, y_0) \in \partial_2 \mathscr{S}$, one can use a similar argument and the assumption $\sigma > 0$ to obtain $L(0) = 0$ and

$$0 \leq e^{-r\tau}(X(\tau) + (1 - \mu)Y(\tau))$$

$$\leq \sigma(1 - \mu)\int_0^\tau e^{-rt}Y(t)\left( \frac{\alpha - r}{\sigma}\, dt + dW(t) \right),$$

where now $\tau \triangleq 1 \wedge \inf \{t \geq 0; \ Y(t) \in (0, y_0 + 1)\}$. According to Girsanov's theorem, there is a probability measure, mutually absolutely continuous with respect to $P$, under which $\{(\alpha - r)t/\sigma + W(t); \ t \geq 0\}$ is a standard Brownian motion. Under this measure, the nonnegative random variable

$$\sigma(1 - \mu) \int_0^\tau e^{-rt} Y(t) \left( \frac{\alpha - r}{\sigma} \, dt + dW(t) \right)$$

has expectation zero and so is almost surely zero. This implies that $\tau = 0$, almost surely, from which we conclude $Y(0) = 0$.

REMARK 2.2. If $(x_0, y_0) \in \mathcal{S}$, it is possible to jump immediately to the $x$-axis. If $y_0 > 0$, this is accomplished by setting $M(0) = y_0, L(0) = 0$ and then $X(0) = x_0 + (1 - \mu)y_0$. If $y_0 \leq 0$, one can set $M(0) = 0, L(0) = -y_0/(1 - \lambda)$ and obtain $X(0) = x_0 + y_0/(1 - \lambda)$. After jumping to the $x$-axis, one can choose consumption proportional to wealth and make no further transfers between the assets. This results in an admissible policy. Indeed, if $C(t) = \gamma X(t)$, where $\gamma$ is a positive constant, then for all $t \geq 0$,

$$X(t) = \begin{cases} (x_0 + (1 - \mu)y_0)e^{(r - \gamma)t}, & \text{if } y_0 > 0, \\ \left( x_0 + \dfrac{y_0}{1 - \lambda} \right)e^{(r - \gamma)t}, & \text{if } y_0 \leq 0. \end{cases}$$

We now fix a parameter $p < 1$ and introduce the agent's *utility function* $U_p$ defined for all $c \geq 0$ by

$$U_p(c) \triangleq \begin{cases} \dfrac{c^p}{p}, & \text{if } p < 1, p \neq 0, \\ \log c, & \text{if } p = 0, \end{cases}$$

where we set $U_p(0) \triangleq -\infty$ if $p \leq 0$. We also introduce a positive *discount factor* $\beta > 0$.

The following condition will be in force throughout:

STANDING ASSUMPTION 2.3. For all $(x, y) \in \bar{\mathcal{S}}$,

$$\sup_{(C, L, M) \in \mathcal{A}(x, y)} E \int_0^\infty e^{-\beta t} \max\{U_p(C(t)), 0\} \, dt < \infty.$$

If $p < 0$, then this Standing Assumption is clearly satisfied. We shall show in Propostion 5.4 that it is also satisfied when $p = 0$. For $0 < p < 1$, the Standing Assumption can fail. Some sufficient conditions for the Standing Assumption in this case are provided in Remark 5.3 and in Section 12; a necessary condition appears in Proposition 3.4.

We define the *value function* by

$$(2.8) \quad v(x, y) = \sup_{(C, L, M) \in \mathcal{A}(x, y)} E \int_0^\infty e^{-\beta t} U_p(C(t)) \, dt \qquad \forall (x, y) \in \bar{\mathcal{S}}.$$

If $0 < p < 1$, Standing Assumption 2.3 is equivalent to the condition $v(x, y) < \infty$ for all $(x, y) \in \mathscr{S}$.

For future reference, we introduce the function $\tilde{U}_p \colon (0, \infty) \to R$ defined by

$$(2.9) \quad \tilde{U}_p(\tilde{c}) \triangleq \sup_{c>0} \{U_p(c) - c\tilde{c}\} = \begin{cases} \dfrac{1-p}{p} \tilde{c}^{p/(p-1)}, & \text{if } p < 1, \, p \neq 0, \\[2mm] -1 - \log \tilde{c}, & \text{if } p = 0, \end{cases}$$

for all $\tilde{c} > 0$. The supremum in (2.9) is attained by $c = I_p(\tilde{c})$, where

$$(2.10) \qquad\qquad I_p(\tilde{c}) \triangleq \tilde{c}^{1/(p-1)} \qquad \forall \tilde{c} > 0$$

is the inverse of the strictly decreasing function $U_p'$. We have from (2.9) that

$$(2.11) \qquad \tilde{U}_p(\tilde{c}) + c\tilde{c} - U_p(c) \geq 0 \qquad \forall c \geq 0, \, \tilde{c} > 0,$$

and because the function $c \mapsto \tilde{U}_p(\tilde{c}) + c\tilde{c} - U_p(c)$ is convex, we also have for any constant $k > 0$,

$$(2.12) \qquad \tilde{U}_p(\tilde{c}) + c\tilde{c} - U_p(c) \geq (c - k)\big(\tilde{c} - U_p'(k)\big) \qquad \forall c > 0.$$

**3. Elementary properties of the value function.**  In contrast to [14], where the principal activity is to create a solution to the Hamilton–Jacobi–Bellman equation, our principal activity is to establish properties of the value function $v$. In this section, using the definition of $v$ as the value function for a control problem, we establish concavity, continuity and homotheticity of $v$ and provide some bounds.

PROPOSITION 3.1.  *The value function $v$ defined by (2.8) is concave.*

PROOF.  Let $(x_1, y_1)$ and $(x_2, y_2)$ be in $\bar{\mathscr{S}}$, and let $\gamma \in (0, 1)$, $(C_1, L_1, M_1) \in \mathscr{A}(x_1, y_1)$ and $(C_2, L_2, M_2) \in \mathscr{A}(x_2, y_2)$ be given. The linearity of (2.4) and (2.5) implies that

$$\big(\gamma C_1 + (1 - \gamma)C_2, \gamma L_1 + (1 - \gamma)L_2, \gamma M_1 + (1 - \gamma)M_2\big)$$
$$\in \mathscr{A}\big(\gamma x_1 + (1 - \gamma)x_2, \gamma y_1 + (1 - \gamma)y_2\big).$$

Because $U_p$ is concave, we have

$$v\big(\gamma x_1 + (1 - \gamma)x_2, \gamma y_1 + (1 - \gamma)y_2\big)$$
$$\geq E\int_0^\infty e^{-\beta t} U_p\big(\gamma C_1(t) + (1 - \gamma)C_2(t)\big)\, dt$$
$$\geq \gamma E\int_0^\infty e^{-\beta t} U_p\big(C_1(t)\big)\, dt + (1 - \gamma) E\int_0^\infty e^{-\beta t} U_p\big(C_2(t)\big)\, dt.$$

Maximizing the right-hand side over $(C_1, L_1, M_1) \in \mathscr{A}(x_1, y_1)$ and $(C_2, L_2, M_2) \in \mathscr{A}(x_2, y_2)$, we obtain

$$v(\gamma x_1 + (1 - \gamma) x_2, \gamma y_1 + (1 - \gamma) y_2) \geq \gamma v(x_1, y_1) + (1 - \gamma) v(x_2, y_2).$$
$\square$

COROLLARY 3.2.  *The value function $v$ is continuous on $\mathscr{S}$.*

PROOF.  A concave function is continuous on the interior of its domain ([60], Theorem 10.1). $\square$

We show in Corollaries 5.5 and 5.8 that $v$ is continuous on $\partial \mathscr{S}$ as well.

PROPOSITION 3.3.  *The value function $v$ has the homotheticity property*

$$(3.1) \qquad v(\gamma x, \gamma y) = \begin{cases} \gamma^p v(x, y), & if\, p < 1,\, p \neq 0, \\ (1/\beta) \log \gamma + v(x, y), & if\, p = 0, \end{cases}$$

*for all $(x, y) \in \bar{\mathscr{S}}$ and $\gamma > 0$.*

PROOF.  This follows from the fact that $(C, L, M) \in \mathscr{A}(x, y)$ if and only if $(\gamma C, \gamma L, \gamma M) \in \mathscr{A}(\gamma x, \gamma y)$. $\square$

PROPOSITION 3.4.  *Define*

$$C_* = \frac{\beta - rp}{1 - p}.$$

*Then $C_* > 0$ (which is a necessary condition for Standing Assumption 2.3 when $0 < p < 1$). If $p \neq 0$, the value function has the lower bound*

$$(3.2) \quad v(x, y) \geq \begin{cases} \dfrac{1}{p} C_*^{p-1}(x + (1 - \mu)y)^p, & \forall (x, y) \in \bar{\mathscr{S}},\, y \geq 0, \\[3mm] \dfrac{1}{p} C_*^{p-1}\left(x + \dfrac{y}{1 - \lambda}\right)^p, & \forall (x, y) \in \bar{\mathscr{S}},\, y < 0. \end{cases}$$

*If $p = 0$, then*

$$(3.3) \quad v(x, y) \geq \begin{cases} \dfrac{1}{\beta} \log(x + (1 - \mu)y) + \dfrac{1}{\beta} \log \beta + \dfrac{r - \beta}{\beta^2}, \\[2mm] \qquad\qquad\qquad\qquad\qquad \forall (x, y) \in \bar{\mathscr{S}},\, y \geq 0, \\[3mm] \dfrac{1}{\beta} \log\left(x + \dfrac{y}{1 - \lambda}\right) + \dfrac{1}{\beta} \log \beta + \dfrac{r - \beta}{\beta^2}, \\[2mm] \qquad\qquad\qquad\qquad\qquad \forall (x, y) \in \bar{\mathscr{S}},\, y < 0. \end{cases}$$

*On $\partial_1 \mathscr{S} \cup \partial_2 \mathscr{S}$, $v$ coincides with these lower bounds, which are 0 if $0 < p < 1$ and $-\infty$ if $p \leq 0$.*

PROOF.  Remark 2.1 gives us the claimed values of $v$ on $\partial_1 \mathcal{S} \cup \partial_2 \mathcal{S}$. For the lower bounds on $v$, let us first consider the case $0 < p < 1$. Let $\gamma$ be a constant satisfying $\gamma > \max\{0, -(\beta - rp)/p\}$ and evaluate the consumption process in Remark 2.2 to obtain $v(x, y) \geq X^p(0)\gamma^p/[p(\beta - rp + \gamma p)]$, where $X(0)$ is as in Remark 2.2. If $(\beta - rp)/p \leq 0$, we can let $\gamma \downarrow -(\beta - rp)/p$ and this lower bound on $v$ converges to $\infty$ for all $(x, y) \in \mathcal{S}$. Thus, $C_* > 0$ is a consequence of Standing Assumption 2.3. Under this assumption, we note that $C_*$ maximizes $X^p(0)\gamma^p/[p(\beta - rp + \gamma p)]$ over $\gamma > 0$, and this provides (3.2).

For $p < 0$, the derivation of (3.2) is the same, except that now $C_* > 0$ because $\beta > 0$ and $r > 0$. For $p = 0$, the consumption process in Remark 2.2 leads to the lower bound

$$v(x, y) \geq \frac{1}{\beta} \log X(0) + \frac{1}{\beta} \log \gamma + \frac{r - \gamma}{\beta^2},$$

which is maximized by $\gamma = C_* = \beta$.  $\square$

If $(x_0, y_0) \in \overline{\mathcal{S}}$ and $(x, y) \in \overline{\mathcal{S}}$ satisfy [cf. (2.6)]

$$(3.4) \qquad x = x_0 - l + (1 - \mu)m, \qquad y = y_0 + (1 - \lambda)l - m$$

for some $l \geq 0$ and $m \geq 0$, then $(x, y)$ *can be reached from* $(x_0, y_0)$ *by a transaction*. We have the following easy result.

PROPOSITION 3.5.  *If* $(x, y) \in \overline{\mathcal{S}}$ *can be reached from* $(x_0, y_0) \in \overline{\mathcal{S}}$ *by a transaction, then* $v(x, y) \leq v(x_0, y_0)$.

PROPOSITION 3.6.  *Let* $(x, y) \in \mathcal{S}$ *be given. If* $p < 1$, $p \neq 0$, *then for all* $\delta \geq 0$,

$$(3.5) \qquad v(x + \delta, y) \geq \begin{cases} \left( \dfrac{x + \delta + y/(1 - \lambda)}{x + y/(1 - \lambda)} \right)^p v(x, y), & \text{if } y \geq 0, \\[4mm] \left( \dfrac{x + \delta + (1 - \mu)y}{x + (1 - \mu)y} \right)^p v(x, y), & \text{if } y < 0. \end{cases}$$

*If* $p = 0$, *then for all* $\delta \geq 0$,

$$(3.6) \qquad v(x + \delta, y) \geq \begin{cases} \dfrac{1}{\beta} \log\left( \dfrac{x + \delta + y/(1 - \lambda)}{x + y/(1 - \lambda)} \right) + v(x, y), & \text{if } y \geq 0, \\[4mm] \dfrac{1}{\beta} \log\left( \dfrac{x + \delta + (1 - \mu)y}{x + (1 - \mu)y} \right) + v(x, y), & \text{if } y < 0. \end{cases}$$

PROOF.   We consider only the case $y \geq 0$; the case $y < 0$ is similar. Define $\gamma = \delta y/[(1 - \lambda)x + y]$ so that

$$( x + \delta - \gamma, y + (1 - \lambda)\gamma) = \left( \frac{x + \delta + y/(1 - \lambda)}{x + y/(1 - \lambda)} \right)(x, y)$$

is on the same ray from the origin as $(x, y)$. Because $(x + \delta - \gamma, y + (1 - \lambda)\gamma)$ can be reached from $(x + \delta, y)$ by a transaction, we have $v(x + \delta, y) \geq v(x + \delta - \gamma, y + (1 - \lambda)\gamma)$. The first parts of (3.5) and (3.6) now follow from (3.1).  □

COROLLARY 3.7.   *Let $(x_0, y_0) \in \mathscr{S}$ and a differentiable function $\varphi: \mathscr{S} \to R$ satisfying $\varphi(x_0, y_0) = v(x_0, y_0)$ be given. If $\varphi \geq v$ on $\mathscr{S}$ or $\varphi \leq v$ on $\mathscr{S}$, then*

$$\varphi_x(x_0, y_0) \geq \begin{cases} pv(x_0, y_0)(x_0 + y_0/(1 - \lambda))^{-1}, & if \, p < 1, \, p \neq 0, \\ \dfrac{1}{\beta}(x_0 + y_0/(1 - \lambda))^{-1}, & if \, p = 0, \end{cases}$$

*if $y_0 \geq 0$, and*

$$\varphi_x(x_0, y_0) \geq \begin{cases} pv(x_0, y_0)(x_0 + (1 - \mu)y_0)^{-1}, & if \, p < 1, \, p \neq 0, \\ \dfrac{1}{\beta}(x_0 + (1 - \mu)y_0)^{-1}, & if \, p = 0, \end{cases}$$

*if $y_0 < 0$.*

PROOF.   We consider only the case $y_0 \geq 0$ and $p \neq 0$. The other cases are similar. If $\varphi \geq v$, then (3.5) implies that

$$\varphi_x(x_0, y_0) \geq \limsup_{h \downarrow 0} \frac{1}{h}[v(x_0 + h, y_0) - v(x_0, y_0)]$$

$$\geq \lim_{h \downarrow 0} \frac{1}{h}\left[ \left( \frac{x_0 + h + y_0/(1 - \lambda)}{x_0 + y_0/(1 - \lambda)} \right)^p - 1 \right]v(x_0, y_0)$$

$$= pv(x_0, y_0)\left( x_0 + \frac{y_0}{1 - \lambda} \right)^{-1}.$$

If $\varphi \leq v$, then (3.5) implies that

$$\varphi_x(x_0, y_0) \geq \limsup_{h \downarrow 0} \frac{1}{h}[v(x_0, y_0) - v(x_0 - h, y_0)]$$

$$\geq \lim_{h \downarrow 0} \frac{1}{h}\left[ \left( \frac{x_0 + y_0/(1 - \lambda)}{x_0 - h + y_0/(1 - \lambda)} \right)^p - 1 \right]v(x_0 - h, y_0)$$

$$= pv(x_0, y_0)\left( x_0 + \frac{y_0}{1 - \lambda} \right)^{-1}. \qquad □$$

**4. Dynamic programming.** The method of viscosity solutions is designed to exploit the principle of dynamic programming, which we now state. Let $(x_0, y_0) \in \mathscr{S}$ be fixed. For every $(C, L, M) \in \mathscr{A}(x_0, y_0)$, we have

$$E \int_0^\infty e^{-\beta s} U_p(C(s)) \, ds \le v(x_0, y_0).$$

Because of our Standing Assumption, $v(x_0, y_0) < \infty$, it is possible, for every $\varepsilon > 0$, to find a policy $(C^\varepsilon, L^\varepsilon, M^\varepsilon) \in \mathscr{A}(x_0, y_0)$ such that

$$(4.1) \qquad v(x_0, y_0) < \varepsilon + E \int_0^\infty e^{-\beta s} U_p(C^\varepsilon(s)) \, ds.$$

Such a policy will be called $\varepsilon$-optimal. Using the strong Markov property, one can replace the initial time 0 in the above assertions by an arbitrary stopping time $\tau$. The details of this replacement (see, e.g., [4]) involve technical measurability issues. We simply state the result without proof.

PRINCIPLE OF DYNAMIC PROGRAMMING 4.1. Let $(x_0, y_0) \in \mathscr{S}$ be given and let $\tau$ be a stopping time for the underlying filtration $\{\mathscr{F}(t)\}_{t \ge 0}$. For every $(C, L, M) \in \mathscr{A}(x_0, y_0)$, we have

$$(4.2) \quad E\left[ \int_\tau^\infty e^{-\beta s} U_p(C(s)) \, ds | \mathscr{F}(\tau) \right] \le 1_{\{\tau < \infty\}} e^{-\beta \tau} v(X(\tau), Y(\tau)) \quad \text{a.s.,}$$

where $(X(\cdot), Y(\cdot))$ are given by (2.3)–(2.5). Moreover, for each $\varepsilon > 0$, there is a policy $(C^\varepsilon, L^\varepsilon, M^\varepsilon) \in \mathscr{A}(x_0, y_0)$ which agrees with $(C, L, M)$ on $[0, \tau)$, which satisfies

$$(4.3) \qquad L^\varepsilon(\tau) \ge L(\tau), \qquad M^\varepsilon(\tau) \ge M(\tau)$$

and for which

$$(4.4) \quad 1_{\{\tau < \infty\}} e^{-\beta \tau} v(X(\tau), Y(\tau)) < \varepsilon + E\left[ \int_\tau^\infty e^{-\beta s} U_p(C^\varepsilon(s)) \, ds | \mathscr{F}(\tau) \right] \quad \text{a.s.}$$

Note that $(X(\tau), Y(\tau))$ on the left-hand side of (4.4) is determined by $(C, L, M)$. The construction of $(C^\varepsilon, L^\varepsilon, M^\varepsilon)$ takes $(X(\tau), Y(\tau))$ as the initial state, from which an initial jump may occur, as allowed by (4.3). Thus, the position process $(X^\varepsilon(\cdot), Y^\varepsilon(\cdot))$ associated with $(C^\varepsilon, L^\varepsilon, M^\varepsilon)$ agrees with $(X(\cdot), Y(\cdot))$ on $[0, \tau)$ and $(X^\varepsilon(\tau), Y^\varepsilon(\tau))$ can be reached from $(X(\tau), Y(\tau))$ by a transaction.

COROLLARY 4.2. Let $(x_0, y_0) \in \mathscr{S}$ be given and let $\mathscr{O}$ be an open subset of $\mathscr{S}$ containing $(x_0, y_0)$. For $(C, L, M) \in \mathscr{A}(x_0, y_0)$, let $(X, Y)$ be given by (2.3)–(2.5) and define

$$\tau \triangleq \inf\{t \ge 0; (X(t), Y(t)) \notin \overline{\mathscr{O}}\}.$$

*Then, for each* $t \in [0, \infty]$, *we have the optimality equation*

$$v(x, y) = \sup_{(C, L, M) \in \mathscr{A}(x_0, y_0)} E\left[ \int_0^{t \wedge \tau} U_p(C(s)) \, ds \right.$$

(4.5)

$$\left. + 1_{\{t \wedge \tau < \infty\}} e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau)) \right].$$

PROOF. Let $(C, L, M) \in \mathscr{A}(x_0, y_0)$ be given and for $\varepsilon > 0$, let $(C^\varepsilon, L^\varepsilon, M^\varepsilon)$ $\in \mathscr{A}(x, y)$ agree with $(C, L, M)$ on $[0, t \wedge \tau)$ and satisfy (4.4) with $t \wedge \tau$ replacing $\tau$. Then

$$E\left[ \int_0^{t \wedge \tau} U_p(C(s)) \, ds + 1_{\{t \wedge \tau < \infty\}} e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau)) \right]$$

$$\leq \varepsilon + E \int_0^\infty e^{-\beta s} U(C^\varepsilon(s)) \, ds$$

$$\leq \varepsilon + v(x_0, y_0).$$

Letting $\varepsilon \downarrow 0$ and then maximizing over $(C, L, M) \in \mathscr{A}(x_0, y_0)$, we obtain

$$\sup_{(C, L, M)} E\left[ \int_0^{t \wedge \tau} U_p(C(s)) \, ds + 1_{\{t \wedge \tau < \infty\}} e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau)) \right]$$

$$\leq v(x_0, y_0).$$

For the reverse inequality, choose $(C^\varepsilon, L^\varepsilon, M^\varepsilon) \in \mathscr{A}(x_0, y_0)$ satisfying (4.1). Then (4.2) implies

$$v(x_0, y_0) \leq \varepsilon + E\left[ \int_0^{t \wedge \tau} U_p(C^\varepsilon(s)) \, ds \right.$$

$$\left. + 1_{\{t \wedge \tau < \infty\}} e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau)) \right]. \qquad \square$$

REMARK 4.3. Because of Proposition 3.5, (4.5) still holds if the supremum on the right-hand side is restricted to policies $(C, L, M)$ for which $(X(\tau), Y(\tau))$ $\in \partial \mathscr{O}$ on $\{\tau < \infty\}$.

The infinitesimal version of the principle of dynamic programming is the *Hamilton–Jacobi–Bellman (HJB) equation* (4.6). Let $C^2(\mathscr{S})$ denote the set of twice continuously differentiable, real-valued functions on $\mathscr{S}$. For $\varphi \in C^2(\mathscr{S})$, define a second-order differential operator by

$$(\mathscr{L}\varphi)(x, y) = \beta\varphi(x, y) - \tfrac{1}{2}\sigma^2 y^2 \varphi_{yy}(x, y) - \alpha y \varphi_y(x, y) - r x \varphi_x(x, y)$$

$$\forall (x, y) \in \mathscr{S}.$$

The HJB equation for the transaction cost problem of this paper is

$$(4.6) \quad \min\left\{ \mathscr{L}\varphi - \tilde{U}_p(\varphi_x), -(1 - \mu)\varphi_x + \varphi_y, \varphi_x - (1 - \lambda)\varphi_y \right\} = 0,$$

where $\tilde{U}_p$ is defined on $(0, \infty)$ by (2.9) and extended to be $\infty$ on $(-\infty, 0]$. The expectation is that the value function $v$ satisfies (4.6). The chief difficulty is that the necessary derivatives of $v$ are not yet known to exist. Thus, we first show that $v$ satisfies (4.6) in the viscosity sense. This gives us a toehold from which we can obtain the more definitive result that $v$ is twice continuously differentiable on $\mathscr{S}$, except possibly on the positive $y$-axis, and even here all the derivatives of $v$ which appear in (4.6) exist and are continuous. We will thus see that $v$ is a classical solution of (4.6) everywhere on $\mathscr{S}$.

We shall use the following equation (4.7) several times. Suppose $\varphi \in C^2(\mathscr{S})$, $(x_0, y_0) \in \mathscr{S}$, $(C, L, M) \in \mathscr{A}(x_0, y_0)$ and $(X, Y)$ is given by (2.3)–(2.5). Let $\tau$ be an almost surely finite $\{\mathscr{F}(t)\}_{t \geq 0}$ stopping time. Then Itô's rule for RCLL semimartingales (e.g., [57] and [59]) applied to $e^{-\beta t}\varphi(X(t), Y(t))$ yields

$$
\varphi(x_0, y_0) = e^{-\beta\tau}\varphi(X(\tau), Y(\tau)) + \int_0^\tau e^{-\beta s}(\mathscr{L}\varphi + C(s)\varphi_x)\, ds
$$

$$
\text{(4.7)} \qquad - \sigma \int_0^\tau e^{-\beta s} Y(s)\varphi_y\, dW(s)
$$

$$
+ \int_0^\tau e^{-\beta s}\left[\left(-(1-\mu)\varphi_x + \varphi_y\right) dM^c(s) + \left(\varphi_x - (1-\lambda)\varphi_y\right) dL^c(s)\right]
$$

$$
+ \sum_{0 \leq s \leq \tau} e^{-\beta s}\left[\varphi(X(s-), Y(s-)) - \varphi(X(s), Y(s))\right],
$$

where $\varphi$ and is derivatives are evaluated at $(X(s), Y(s))$ unless otherwise indicated and

$$
L^c(t) \triangleq L(t) - \sum_{0 \leq s \leq t}(L(s) - L(s-)),
$$

$$
M^c(t) \triangleq M(t) - \sum_{0 \leq s \leq t}(M(s) - M(s-))
$$

denote the continuous parts of $L$ and $M$, respectively.

## 5. Upper bounds and continuity of the value function.

Although we have not yet related $v$ to the HJB equation (4.6), we can apply the operators in (4.6) to so-called supersolutions of (4.6) to obtain upper bounds on $v$. For $0 < p < 1$, consider a function $\varphi \colon \mathscr{S} \to [0, \infty)$ of the form

$$
\text{(5.1)} \qquad \varphi(x, y) = \frac{1}{p} A^{p-1}(x + \gamma y)^p \qquad \forall (x, y) \in \bar{\mathscr{S}},
$$

where $A > 0$ and $\gamma$ is a constant satisfying

$$
\text{(5.2)} \qquad 1 - \mu \leq \gamma \leq \frac{1}{1-\lambda}.
$$

Note that (5.2) implies $x + \gamma y \geq 0$ for all $(x, y) \in \bar{\mathscr{S}}$. Direct computation reveals

$$(5.3) \qquad -(1 - \mu)\varphi_x + \varphi_y = A^{p-1}(x + \gamma y)^{p-1}(\gamma - (1 - \mu)) \geq 0 \quad \text{on } \bar{\mathscr{S}},$$

$$(5.4) \qquad \varphi_x - (1 - \lambda)\varphi_y = A^{p-1}(x + \gamma y)^{p-1}(1 - \gamma(1 - \lambda)) \geq 0 \quad \text{on } \bar{\mathscr{S}}$$

and

$$(\mathscr{L}\varphi)(x, y) - \tilde{U}_p(\varphi_x(x, y))$$

$$(5.5) \qquad = A^{p-1}(x + \gamma y)^p \left[ \frac{\beta - rp}{p} - \frac{(\alpha - r)^2}{2\sigma^2(1 - p)} - \frac{1 - p}{p} A \right.$$

$$\left. + \frac{1}{2(1 - p)} \left( \sigma(1 - p)\frac{\gamma y}{x + \gamma y} - \frac{\alpha - r}{\sigma} \right)^2 \right] \quad \forall (x, y) \in \bar{\mathscr{S}}.$$

Regarding $p$ as a variable, consider the equation

$$(5.6) \qquad B(p) \triangleq \frac{\beta - rp}{p} - \frac{(\alpha - r)^2}{2\sigma^2(1 - p)} = 0,$$

which has a unique solution $\bar{p} \in (0, 1)$. Define

$$(5.7) \qquad A(p) = \frac{\beta - rp}{1 - p} - \frac{p(\alpha - r)^2}{2\sigma^2(1 - p)^2} = \frac{p}{1 - p} B(p);$$

if $0 < p < \bar{p}$, then $A(p) > 0$, and replacing $A$ by $A(p)$ in (5.1), we have

$$(5.8) \qquad \mathscr{L}\varphi - \tilde{U}_p(\varphi_x) \geq 0 \quad \text{on } \bar{\mathscr{S}}.$$

PROPOSITION 5.1. *Assume* $0 < p < \bar{p}$. *With* $A(p)$ *defined by* (5.7) *and* $\gamma$ *satisfying* (5.2), *we have*

$$v(x, y) \leq \frac{1}{p} A^{p-1}(p)(x + \gamma y)^p \quad \forall (x, y) \in \bar{\mathscr{S}}.$$

PROOF. Let $\varphi$ be given by (5.1) with $A = A(p)$. Let $(x_0, y_0) \in \mathscr{S}$ and $(C, L, M) \in \mathscr{A}(x_0, y_0)$ be given. Inequalities (5.3) and (5.4) show that $\varphi$ is nonincreasing in the direction of jumps of the corresponding state process, that is,

$$(5.9) \qquad \varphi(X(s), Y(s)) \leq \varphi(X(s-), Y(s-)) \quad \forall s \geq 0.$$

Choose an increasing sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of $\mathscr{S}$ containing $(x_0, y_0)$ and whose union in $\mathscr{S}$, and define $\tau_n \triangleq n \wedge \inf\{t \geq 0; (X(t), Y(t)) \notin K_n\}$. Then

$$E\int_0^{\tau_n} e^{-\beta s} Y(s) \varphi_y(X(s), Y(s)) \, dW(s) = 0$$

for each $n$. From (4.7), (5.3), (5.4), (5.8), (5.9) and (2.11) and the nonnegativity of $\varphi$, we have

$$\varphi(x_0, y_0) \geq E \int_0^{\tau_n} e^{-\beta s} U_p(C(s)) \, ds.$$

Let $n \to \infty$ and then maximize the right-hand side over $(C, L, M)$ to obtain the desired result. $\square$

REMARK 5.2. The function obtained by setting $\gamma = 1$ in Proposition 5.1 is the value function obtained by Merton [54] in the problem with no transaction costs ($\lambda = \mu = 0$).

REMARK 5.3. We have just shown that under the assumption $0 < p < \bar{p}$, the value function is finite and continuous on $\bar{\mathscr{S}}$ (see Proposition 3.4 and Corollary 3.2). In particular, Standing Assumption 2.3 is satisfied.

PROPOSITION 5.4. *Assume $p = 0$. Then Standing Assumption 2.3 is satisfied, and for any $\gamma$ satisfying (5.2), we have*

$$v(x, y) \leq \frac{1}{\beta} \log(x + \gamma y) + \frac{1}{\beta} \log \beta + \frac{r - \beta}{\beta^2} + \frac{(\alpha - r)^2}{2\beta^2 \sigma^2} \qquad \forall (x, y) \in \bar{\mathscr{S}}.$$

PROOF. For $p < 1$, let us temporarily denote by $v_p$ the value function corresponding to $U_p$. Choose $p \in (0, \bar{p})$ and observe that $\log c \leq c^p/p$ for all $c \geq 0$. Therefore, for any $(x, y) \in \bar{\mathscr{S}}$ and $(C, L, M) \in \mathscr{A}(x, y)$,

$$E \int_0^\infty e^{-\beta s} \max\{\log(C(s)), 0\} \, ds \leq E \int_0^\infty e^{-\beta s} \frac{1}{p} C^p(s) \, ds \leq v_p(x, y) < \infty,$$

which establishes Standing Assumption 2.3. In fact, $\log c \leq c^p/p - 1/p$ for all $c \geq 0$, from which follows $v_0 \leq v_p - 1/(\beta p)$. Proposition 5.1 implies

$$v_0(x, y) \leq \lim_{p \downarrow 0} \frac{1}{p} \left[ A^{p-1}(p)(x + \gamma y)^p - \frac{1}{\beta} \right]$$

$$= \frac{1}{\beta} \log(x + \gamma y) + \frac{1}{\beta} \log \beta + \frac{r - \beta}{\beta^2} + \frac{(\alpha - r)^2}{2\beta^2 \sigma^2}. \qquad \square$$

COROLLARY 5.5. *If $p \leq 0$, then $v$ has limit $-\infty$ at $\partial \mathscr{S}$, that is, $v$ is continuous on $\bar{\mathscr{S}}$.*

PROOF. Corollary 3.2 and Proposition 3.4 show that in order to prove the continuity of $v$ on $\bar{\mathscr{S}}$, it suffices to prove that $v$ has limit $-\infty$ at $\partial \mathscr{S}$. If $p = 0$, this follows from Proposition 5.4 by taking $\gamma = 1/(1 - \lambda)$ at $\partial_1 \mathscr{S}$ and $\gamma = 1 - \mu$ at $\partial_2 \mathscr{S}$. If $p < 0$, the inequality $c^p/p \leq \log c - 1/p$ for all $c \geq 0$ can be used in conjunction with Proposition 5.4 to obtain the desired result. $\square$

Finally, we examine the case $0 < p < 1$, which is not fully covered by Propostion 5.1. To do that, we need a brief digression on the manner in which the state process can approach $\partial \mathscr{S}$.

LEMMA 5.6.   *For* $n = 1, 2, \ldots,$ *define*

$$(5.10) \qquad F_n = \left\{ (x, y); x + \frac{y}{1 - \lambda} \geq \frac{1}{n}, \ x + (1 - \mu)y \geq \frac{1}{n} \right\}.$$

*For* $(x_0, y_0) \in \mathscr{S}$ *and* $(C, L, M) \in \mathscr{A}(x_0, y_0),$ *let* $(X, Y)$ *be given by* (2.3)–(2.5) *and define*

$$(5.11) \qquad \nu_n \triangleq \inf\{t \geq 0; (X(t), Y(t)) \notin F_n\},$$

$$(5.12) \qquad \nu \triangleq \inf\{t \geq 0; (X(t), Y(t)) = (0, 0)\}.$$

*Then* $\nu_n \uparrow \nu$ *almost surely as* $n \to \infty.$

PROOF.   Define $\nu_\infty = \lim_{n \to \infty} \nu_n.$ Then clearly $\nu_\infty \leq \nu,$ and we have only to prove the reverse inequality. Suppose $\nu_\infty < \infty.$ Then $(X(\nu_\infty -), Y(\nu_\infty -)) \in \partial \mathscr{S}.$ The argument in Remark 2.1 shows that $(X(\nu_\infty), Y(\nu_\infty)) = (0, 0).$ Therefore, $\nu \leq \nu_\infty.$ $\square$

PROPOSITION 5.7.   *Assume* $0 < p < 1.$ *Choose* $\delta \in (0, 1 - \mu)$ *such that*

$$0 < 1 - \mu - \delta < \frac{(1 - \mu)(1 - p)\sigma^2}{2(\alpha - r)}$$

*and define*

$$A \triangleq \left[ \frac{pv(-\delta, 1)}{(1 - \mu - \delta)^p} \right]^{1/(p-1)}, \qquad B \triangleq (pv(1, 0))^{1/(p-1)}.$$

*Then* $A \leq C_*$ *and* $B \leq C_*$ *[where* $C_* = (\beta - rp)/(1 - p) > 0$ *because of Proposition 3.4], and we have the upper bounds*

$$(5.13) \quad v(x, y) \leq \begin{cases} \dfrac{1}{p} A^{p-1}(x + (1 - \mu)y)^p, \\[2ex] \qquad\qquad\qquad\qquad\quad if\, x \leq 0, \ -\dfrac{x}{1 - \mu} \leq y \leq -\dfrac{x}{\delta}, \\[2ex] \dfrac{1}{p} B^{p-1}\left(x + \dfrac{y}{1 - \lambda}\right)^p, \\[2ex] \qquad\qquad\qquad\qquad\quad if\, x \geq 0, \ -(1 - \lambda)x \leq y \leq 0. \end{cases}$$

PROOF. From Proposition 3.4, we have

$$\frac{1}{p}A^{p-1}(1 - \mu - \delta)^p = v(-\delta, 1) \geq \frac{1}{p}C_*^{p-1}(1 - \mu - \delta)^p,$$

$$\frac{1}{p}B^{p-1} = v(1,0) \geq \frac{1}{p}C_*^{p-1},$$

from which we conclude that $A \leq C_*$, $B \leq C_*$.

We derive only the first bound in (5.13); the proof of the second is similar. Define

(5.14) $$D = \left\{(x, y); x < 0, -\frac{x}{1 - \mu} < y < -\frac{x}{\delta}\right\} \subset \mathscr{S}$$

and define $\varphi$ on $\overline{D}$ by

(5.15) $$\varphi(x, y) \triangleq \frac{1}{p}A^{p-1}(x + (1 - \mu)y)^p \qquad \forall (x, y) \in D.$$

Note that $\varphi = v$ on $\partial D$. Define $\varphi = v$ on $\mathscr{S} \setminus \overline{D}$. Just as in (5.3) and (5.4), we have

(5.16) $$-(1 - \mu)\varphi_x + \varphi_y \geq 0, \qquad \varphi_x - (1 - \lambda)\varphi_y \geq 0 \quad \text{on } D.$$

Furthermore, for $(x, y) \in D$

$$(\mathscr{L}\varphi)(x, y) - \tilde{U}_p(\varphi_x(x, y))$$

$$= A^{p-1}(x + (1 - \mu)y)^p\left[\frac{\beta - rp}{p} - \frac{(\alpha - r)^2}{2\sigma^2(1 - p)} - \frac{1 - p}{p}A\right.$$

(5.17) $$\left. + \frac{1}{2(1 - p)}\left(\sigma(1 - p)\frac{(1 - \mu)y}{x + (1 - \mu)y} - \frac{\alpha - r}{\sigma}\right)^2\right]$$

$$\geq A^{p-1}(x + (1 - \mu)y)^{p-1}(1 - \mu)y$$

$$\times \left[\frac{1}{2}(1 - p)\sigma^2\frac{(1 - \mu)y}{x + (1 - \mu)y} - (\alpha - r)\right]$$

$$\geq 0$$

because $A \leq C_*$ and, by the choice of $\delta$,

$$\frac{y}{x + (1 - \mu)y} > \frac{1}{1 - \mu - \delta} > \frac{2(\alpha - r)}{(1 - \mu)(1 - p)\sigma^2} \quad \text{on } D.$$

Let $(x_0, y_0) \in D$ and $(C, L, M) \in \mathscr{A}(x_0, y_0)$ be given, and define $F_n$, $\nu_n$ and $\nu$ by (5.10)–(5.12). Define also

$$H_n \triangleq \{(x, y) \in \overline{D}; y \leq n\},$$

$$\tau_n \triangleq \inf\{t \geq 0; (X(t), Y(t)) \notin H_n\},$$

$$\tau \triangleq \inf\{t \geq 0; (X(t), Y(t)) \notin \overline{D}\},$$

so that $\lim_{n \to \infty} \tau_n = \tau$ almost surely. We show that

$$(5.18) \qquad \{\tau < \infty\} = \bigcup_{n=1}^{\infty} \{\nu_n \wedge \tau_n = \tau \le n\}.$$

It is clear that $\{\tau < \infty\}$ contains the union. For the reverse containment, assume $\tau(\omega) < \infty$ for some $\omega$. Then $\tau(\omega) < \nu(\omega)$, for otherwise $(X(\cdot, \omega), Y(\cdot, \omega))$ would reach and stick at the origin before exiting $\overline{D}$ and then $\tau(\omega)$ would be $\infty$. According to Lemma 5.6, we must then have $\tau(\omega) < \nu_n(\omega)$ for sufficiently large $n$. Choose $n$ so large that $\tau(\omega) \le n$, $\tau(\omega) < \nu_n(\omega)$ and $\{Y(t, \omega); 0 \le t \le \tau(\omega)\}$ does not exceed $n$. Then $\tau_n(\omega) = \tau(\omega)$, we have $\omega \in \{\nu_n \wedge \tau_n = \tau \le n\}$ and (5.18) is proved.

Inequalities (5.16) show that (5.9) must hold. Moreover, for $0 \le s \le \nu_n \wedge \tau_n$, $Y(s)\varphi_y(X(s), Y(s))$ is bounded, so

$$E \int_0^{n \wedge \nu_n \wedge \tau_n} e^{-\beta s} Y(s) \varphi_y(X(s), Y(s)) \, dW(s) = 0.$$

From these facts, (4.7), (5.16), (5.17) and (2.11), we obtain

$$\varphi(x_0, y_0) \ge E \exp\left[-\beta(n \wedge \nu_n \wedge \tau_n)\right] \varphi(X(n \wedge \nu_n \wedge \tau_n), Y(n \wedge \nu_n \wedge \tau_n))$$
$$+ E \int_0^{n \wedge \nu_n \wedge \tau_n} e^{-\beta s} U_p(C(s)) \, ds$$
$$\ge E\left[1_{\{\nu_n \wedge \tau_n = \tau \le n\}} e^{-\beta\tau} v(X(\tau), Y(\tau))\right]$$
$$+ E \int_0^{n \wedge \nu_n \wedge \tau_n} e^{-\beta s} U_p(C(s)) \, ds,$$

where the second inequality uses the fact that $\varphi \ge v$ on $\mathscr{S} \setminus \overline{D}$. Letting $n \to \infty$, we can use the monotone convergence theorem to establish

$$\varphi(x_0, y_0) \ge E\left[1_{\{\tau < \infty\}} e^{-\beta\tau} v(X(\tau), Y(\tau)) + \int_0^{\tau} e^{-\beta s} U_p(C(s)) \, ds\right].$$

Maximizing the right side over $(C, L, M) \in \mathscr{A}(x_0, y_0)$ and invoking the optimality equation (4.5), we derive the first part of (5.13). $\square$

COROLLARY 5.8. *If* $0 < p < 1$, *then* $v$ *has limit* $0$ *at* $\partial\mathscr{S}$, *that is,* $v$ *is continuous on* $\overline{\mathscr{S}}$.

PROOF. Corollary 3.2 and Proposition 3.4 show that in order to prove the continuity of $v$ on $\overline{\mathscr{S}}$, it suffices to show for every $(x_0, y_0) \in \partial\mathscr{S}$ that

$$\limsup_{\substack{(x, y) \to (x_0, y_0) \\ (x, y) \in \mathscr{S}}} v(x, y) \le 0.$$

For $(x_0, y_0) \in \partial\mathscr{S} \setminus \{(0, 0)\}$, this follows from Proposition 5.7. For $(x_0, y_0) = (0, 0)$, it follows from homotheticity (Proposition 3.3). $\square$

Intuition dictates that because the stock is risky and has higher rate of return than the money market, it is never advantageous to sell the stock short. Thus, the value function in the wedge,

$$G = \left\{ (x, y); \, y < 0, \, x + \frac{y}{1 - \lambda} > 0 \right\},$$

below the $x$-axis should agree with the value function on the $x$-axis evaluated at the position reached by a transaction, that is,

$$v(x, y) = v\left( x + \frac{y}{1 - \lambda}, 0 \right) \qquad \forall (x, y) \in G.$$

The similar result (5.19) is true in a wedge whose one boundary is $\partial_2 \mathscr{S}$. We prove these things here under the assumption $0 < p < 1$.

THEOREM 5.9. *Assume* $0 < p < 1$. *With the notation of Proposition 5.7, we have*

$$
\begin{aligned}
(5.19) \quad v(x, y) &= v\left( \frac{-\delta(x + (1 - \mu)y)}{1 - \mu - \delta}, \frac{x + (1 - \mu)y}{1 - \mu - \delta} \right) \\
&= \frac{1}{p} A^{p-1}(x + (1 - \mu)y)^p
\end{aligned}
$$

*if* $x \le 0$, $-x/(1 - \mu) \le y \le -x/\delta$, *and*

$$(5.20) \quad v(x, y) = v\left( x + \frac{y}{1 - \lambda}, 0 \right) = \frac{1}{p} B^{p-1}\left( x + \frac{y}{1 - \lambda} \right)^p$$

*if* $(x, y) \in \overline{G}$.

PROOF. We prove (5.19); the proof of (5.20) is the same.

Let $D$ be given by (5.14) and $\varphi$ by (5.15). Let $(x_0, y_0) \in D$ be given and define

$$x \triangleq x_0 + (1 - \mu)\left( \frac{-\delta y_0 - x_0}{1 - \mu - \delta} \right) = \frac{-\delta(x_0 + (1 - \mu)y_0)}{1 - \mu - \delta},$$

$$y \triangleq y_0 - \left( \frac{-\delta y_0 - x_0}{1 - \mu - \delta} \right) = \frac{x_0 + (1 - \mu)y_0}{1 - \mu - \delta}.$$

Then $(x, y) \in \partial D$ and $(x, y)$ can be reached from $(x_0, y_0)$ by a transaction. Propositions 3.5 and 3.3 imply

$$v(x_0, y_0) \ge v(x, y) = v(-\delta y, y) = y^p v(-\delta, 1)$$

$$= \frac{1}{p} A^{p-1}(1 - \mu - \delta)^p y^p = \varphi(x_0, y_0).$$

The reverse inequality comes from Proposition 5.7. □

THEOREM 5.10.   *Assume $p = 0$. Choose $\delta \in (0, 1 - \mu)$ such that*

$$0 < 1 - \mu - \delta < \frac{(1 - \mu)\sigma^2}{2(\alpha - r)}.$$

*Then*

$$v(x, y) = v\left(\frac{-\delta(x + (1 - \mu)y)}{1 - \mu - \delta}, \frac{x + (1 - \mu)y}{1 - \mu - \delta}\right)$$

$$= \frac{1}{\beta} \log\left(\frac{x + (1 - \mu)y}{1 - \mu - \delta}\right) + v(-\delta, 1)$$

*if $x \leq 0$, $-x/(1 - \mu) \leq y \leq -x/\delta$, and*

$$v(x, y) = v\left(x + \frac{y}{1 - \lambda}, 0\right) = \frac{1}{\beta} \log\left(x + \frac{y}{1 - \lambda}\right) + v(1, 0)$$

*if $(x, y) \in \overline{G}$.*

PROOF.   Again, we prove only the first claim. Let $D$ be given by (5.14), and define $\varphi$ on $\overline{D}$ by

$$\varphi(x, y) \triangleq \frac{1}{\beta} \log\left(\frac{x + (1 - \mu)y}{1 - \mu - \delta}\right) + v(-\delta, 1),$$

so that $\varphi = v$ on $\partial D$. Extend $\varphi$ to the rest of $\overline{\mathscr{S}}$ by setting $\varphi = v$ on $\overline{\mathscr{S}} \backslash D$. It is easily verified that (5.16) holds and

$$(\mathscr{L}\varphi)(x, y) - \tilde{U}_0(\varphi_x(x, y))$$

$$= -\log(1 - \mu - \delta) + \beta v(-\delta, 1)$$

$$+ \frac{(1 - \mu)y}{\beta(x + (1 - \mu)y)}\left[\frac{\sigma^2(1 - \mu)y}{2(x + (1 - \mu)y)} - (\alpha - r)\right] - \frac{r}{\beta} + 1 - \log \beta$$

$$\geq -\log(1 - \mu - \delta) + \beta v(-\delta, 1) - \frac{r}{\beta} + 1 - \log \beta \qquad \forall (x, y) \in D$$

because

$$\frac{y}{x + (1 - \mu)y} > \frac{1}{1 - \mu - \delta} > \frac{2(\alpha - r)}{(1 - \mu)\sigma^2} \quad \text{in } D.$$

However, the first part of (3.3) shows that

$$-\log(1 - \mu - \delta) + \beta v(-\delta, 1) - \frac{r}{\beta} + 1 - \log \beta \geq 0,$$

so we have

$$\mathscr{L}\varphi - \tilde{U}_0(\varphi_x) \geq 0 \quad \text{in } D.$$

Let $(x_0, y_0) \in D$ and $\varepsilon > 0$ be given. Choose an $\varepsilon$-optimal $(C, L, M) \in \mathscr{A}(x_0, y_0)$. Then, for any stopping time $\rho$, the Principle of Dynamic Program-

ming 4.1 implies

$$
\begin{aligned}
E\Bigg[ 1_{\{\rho < \infty\}} e^{-\beta \rho} v(X(\rho), Y(\rho)) + \int_0^\rho e^{-\beta s} \log C(s)\, ds \Bigg] \\
\geq E \int_0^\infty e^{-\beta s} \log C(s)\, ds \geq v(x_0, y_0) - \varepsilon.
\end{aligned}
$$
(5.21)

Let $F_n$, $\nu_n$, $\nu$, $H_n$, $\tau_n$ and $\tau$ be as in the proof of Proposition 5.7. Because $v = -\infty$ on $\partial \mathscr{S}$, (5.21) implies $\nu = \infty$ almost surely. From Lemma 5.6 we have $\lim_{n \to \infty} \nu_n = \infty$ almost surely.

Just as in the proof of Proposition 5.7, we derive the inequality

$$
\begin{aligned}
\varphi(x_0, y_0) \geq E \exp[-\beta(n \wedge \nu_n \wedge \tau_n)] \varphi(X(n \wedge \nu_n \wedge \tau_n), Y(n \wedge \nu_n \wedge \tau_n)) \\
+ E \int_0^{n \wedge \nu_n \wedge \tau_n} e^{-\beta s} \log C(s)\, ds.
\end{aligned}
$$
(5.22)

We wish to conclude that $\varphi(x_0, y_0) \geq v(x_0, y_0)$, but because $\varphi$ takes negative values, we cannot argue as in the proof of the Proposition 5.7. Instead, define the constant

$$
k = -v(-\delta, 1) + \frac{1}{\beta} \log(1 - \mu - \delta) + \frac{1}{\beta} \log \beta + \frac{r - \beta}{\beta^2} + \frac{(\alpha - r)^2}{2 \beta^2 \sigma^2}.
$$

From Proposition 5.4 we have $v \leq \varphi + k$ on $D$. Furthermore, $v = \varphi$ on $\overline{\mathscr{S}} \setminus D$. These facts, together with (5.21), imply

$$
\begin{aligned}
E \exp[-\beta(n \wedge \nu_n \wedge \tau_n)] \varphi(X(n \wedge \nu_n \wedge \tau_n), Y(n \wedge \nu_n \wedge \tau_n)) \\
+ E \int_0^{n \wedge \nu_n \wedge \tau_n} e^{-\beta s} \log C(s)\, ds \\
\geq -k E\big[\exp[-\beta(n \wedge \nu_n \wedge \tau_n)] 1_{\{n \wedge \nu_n \wedge \tau_n < \tau\}}\big] \\
+ E \exp[-\beta(n \wedge \nu_n \wedge \tau_n)] v(X(n \wedge \nu_n \wedge \tau_n), Y(n \wedge \nu_n \wedge \tau_n)) \\
+ E \int_0^{n \wedge \nu_n \wedge \tau_n} e^{-\beta s} \log C(s)\, ds \\
\geq -k E\big[\exp[-\beta(n \wedge \nu_n \wedge \tau_n)] 1_{\{n \wedge \nu_n \wedge \tau_n < \tau\}}\big] + v(x_0, y_0) - \varepsilon.
\end{aligned}
$$
(5.23)

As $n \to \infty$, $1_{\{n \wedge \nu_n \wedge \tau_n < \tau\}} \to 1_{\{\tau = \infty\}}$ [see (5.18)], and on this set, $n \wedge \nu_n \wedge \tau_n \to \infty$. Combining (5.22) and (5.23), we have $\varphi(x_0, y_0) \geq v(x_0, y_0) - \varepsilon$. Letting $\varepsilon \downarrow 0$, we see that $\varphi \geq v$ on $D$.

The second part of (3.1) shows that $\varphi$ also can be written as

$$
\varphi(x_0, y_0) = v\left( \frac{-\delta(x_0 + (1 - \mu) y_0)}{1 - \mu - \delta}, \frac{x_0 + (1 - \mu) y_0}{1 - \mu - \delta} \right) \qquad \forall (x_0, y_0) \in \overline{D}.
$$

Just as in the proof of Theorem 5.9, $v(x_0, y_0) \geq \varphi(x_0, y_0)$.  □

For $p < 0$, the results analogous to Theorem 5.9 also hold; see Corollaries 8.7 and 8.8 and the formulas in Theorem 6.9.

## 6. Convex analysis of the value function.

**6. Convex analysis of the value function.** Because the value function $v$ is concave on its convex domain $\overline{\mathscr{S}}$, we can study it by the methods of convex analysis. In this section we show how to partition $\mathscr{S}$ into three convex comes corresponding to the three expressions on the left-hand side of the HJB equation (4.6).

We define the subdifferential

$$\partial v(x, y) \triangleq \left\{ (\delta_x, \delta_y) \in R^2 ; v(\xi, \eta) \leq v(x, y) + \delta_x(\xi - x) + \delta_y(\eta - y) \right.$$

$$\left. \forall (\xi, \eta) \in \mathscr{S} \right\}$$

for each $(x, y) \in \mathscr{S}$. Because $v$ is concave and finite on $\mathscr{S}$, $\partial v(x, y)$ is a nonempty, compact, convex set. The function $v$ is differentiable at a point $(x, y)$ if and only if $\partial v(x, y)$ is a singleton, and in this case $\partial v(x, y) = \{(v_x(x, y), v_y(x, y))\}$.

LEMMA 6.1. *Let* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *be a sequence in* $\mathscr{S}$ *with limit* $(x_0, y_0) \in \mathscr{S}$. *If* $(\delta_x^n, \delta_y^n) \in \partial v(x_n, y_n)$ *for every* $n$, *then* $\{(\delta_x^n, \delta_y^n)\}_{n=1}^{\infty}$ *is bounded and every limit point of the sequence* $\{(\delta_x^n, \delta_y^n)\}_{n=1}^{\infty}$ *is in* $\partial v(x_0, y_0)$.

PROOF. Choose $\varepsilon > 0$ so that the open ball $B_\varepsilon(x_0, y_0)$ of radius $\varepsilon$ centered at $(x_0, y_0)$ contains the sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ and the closed ball $\overline{B}_\varepsilon(x_0, y_0)$ is a subset of $\mathscr{S}$. The boundedness of $v$ on $\overline{B}_\varepsilon(x_0, y_0)$ can be used to establish the boundedness of $\{(\delta_x^n, \delta_y^n)\}_{n=1}^{\infty}$. By definition, we have for each $n \geq 1$,

$$v(\xi, \eta) \leq v(x_n, y_n) + \delta_x^n(\xi - x_n) + \delta_y^n(\eta - y_n) \quad \forall (\xi, \eta) \in \mathscr{S}.$$

Passing to the limit, we see that every limit point of $\{(\delta_x^n, \delta_y^n)\}_{n=1}^{\infty}$ is in $\partial v(x_0, y_0)$. □

PROPOSITION 6.2. *Let* $\mathscr{O}$ *be an open subset of* $\mathscr{S}$. *The value function* $v$ *is of class* $C^1$ *in* $\mathscr{O}$ *if and only if* $\partial v(x, y)$ *is a singleton for every* $(x, y) \in \mathscr{O}$.

PROOF. The only nontrivial part of this proposition is the claim that $\partial v(x, y)$ being a singleton for every $(x, y) \in \mathscr{O}$ implies that $v_x$ and $v_y$ are continuous in $\mathscr{O}$. Assume the singleton property. Fix $(x_0, y_0) \in \mathscr{S}$ and let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $\mathscr{O}$ converging to $(x_0, y_0)$. According to Lemma 6.1, the sequence $\{(v_x(x_n, y_n), v_y(x_n, y_n))\}_{n=1}^{\infty}$ is bounded and $(v_x(x_0, y_0), v_y(x_0, y_0))$ is its only limit point. This shows continuity of $v_x$ and $v_y$ at $(x_0, y_0)$. □

Let $(x, y) \in \mathscr{S}$ and $(\delta_x, \delta_y) \in \partial v(x, y)$ be given. Define

$$\varphi(\xi, \eta) = v(x, y) + \delta_x(\xi - x) + \delta_y(\eta - y) \quad \forall (\xi, \eta) \in \mathscr{S},$$

so that $\varphi \geq v$ on $\mathscr{S}$. According to Proposition 3.5, we have for each $\gamma > 0$ satisfying $(x + \gamma, y_0 - (1 - \lambda)\gamma) \in \mathscr{S}$,

$$\varphi(x, y) = v(x, y) \leq v(x + \gamma, y - (1 - \lambda)\gamma)$$

$$\leq \varphi(x + \gamma, y - (1 - \lambda)\gamma)$$

$$= \varphi(x, y) + \gamma[\delta_x - (1 - \lambda)\delta_y],$$

and so

(6.1)     $\delta_x - (1 - \lambda)\delta_y \geq 0$     $\forall(\delta_x, \delta_y) \in \partial v(x, y), \forall(x, y) \in \mathcal{S}.$

A similar argument involving $(x - (1 - \mu)\gamma, y + \gamma)$ shows that

(6.2)     $-(1 - \mu)\delta_x + \delta_y \geq 0$     $\forall(\delta_x, \delta_y) \in \partial v(x, y), \forall(x, y) \in \mathcal{S}.$

Finally, Corollary 3.7 shows that $\delta_x > 0$, and coupling this with (6.2), we obtain

(6.3)     $\delta_x > 0, \quad \delta_y > 0$     $\forall(\delta_x, \delta_y) \in \partial v(x, y), \forall(x, y) \in \mathcal{S}.$

For $(x, y) \in \mathcal{S}, (\delta_x, \delta_y) \in \partial v(x, y)$ and $\gamma \in R$ sufficiently close to 1, the homotheticity of Proposition 3.3 implies

$$(\gamma^p - 1)v(x, y) = v(\gamma x, \gamma y) - v(x, y) \leq (\gamma - 1)x\delta_x + (\gamma - 1)y\delta_y$$

if $p < 1$, $p \neq 0$, and

$$\frac{1}{\beta} \log \gamma = v(\gamma x, \gamma y) - v(x, y) \leq (\gamma - 1)x\delta_x + (\gamma - 1)y\delta_y$$

if $p = 0$. In either case, divide by $(\gamma - 1)$ and let $\gamma$ approach 1, both from the left and from the right, to obtain

(6.4)     $x\delta_x + y\delta_y = \begin{cases} pv(x, y), & \text{if } p < 1, p \neq 0, \\ \dfrac{1}{\beta}, & \text{if } p = 0. \end{cases}$

Finally, let $(x, y) \in \mathcal{S}$ and $(\delta_x, \delta_y) \in \partial v(x, y)$ be given. Let $(\xi, \eta) \in \mathcal{S}$ be given and let $\gamma$ be a positive number. If $p \neq 0$, we have

$$v(\xi, \eta) = \gamma^p v\left(\frac{\xi}{\gamma}, \frac{\eta}{\gamma}\right)$$

$$\leq \gamma^p \left[v(x, y) + \delta_x\left(\frac{\xi}{\gamma} - x\right) + \delta_y\left(\frac{\eta}{\gamma} - y\right)\right]$$

$$= v(\gamma x, \gamma y) + (\gamma^{p-1}\delta_x)(\xi - \gamma x) + (\gamma^{p-1}\delta_y)(\eta - \gamma y),$$

which shows that $(\gamma^{p-1}\delta_x, \gamma^{p-1}\delta_y) \in \partial v(\gamma x, \gamma y)$. If $p = 0$, we have

$$v(\xi, \eta) = \frac{1}{\beta} \log \gamma + v\left(\frac{\xi}{\gamma}, \frac{\eta}{\gamma}\right)$$

$$\leq \frac{1}{\beta} \log \gamma + v(x, y) + \delta_x\left(\frac{\xi}{\gamma} - x\right) + \delta y\left(\frac{\eta}{\gamma} - y\right)$$

$$= v(\gamma x, \gamma y) + \frac{\delta_x}{\gamma}(\xi - \gamma x) + \frac{\delta_y}{\gamma}(\eta - \gamma y),$$

and we reach the same conclusion. Thus, $\gamma^{p-1} \partial v(x, y) \subset \partial v(\gamma x, \gamma y)$ for all $(x, y) \in \mathcal{S}$ and $\gamma > 0$. Replacing $\gamma$ by $1/\gamma$, $x$ by $\gamma x$ and $y$ by $\gamma y$, we obtain

the reverse set containment:

(6.5)        $\gamma^{p-1} \partial v(x, y) = \partial v(\gamma x, \gamma y)$      $\forall (x, y) \in \mathscr{S}, \forall \gamma > 0.$

We now use subdifferentials to partition $\mathscr{S}$ into three convex cones. We being by observing that if $(x, y)$ and $(\bar{x}, \bar{y})$ are in $\mathscr{S}$ and $(\delta_x, \delta_y) \in \partial v(x, y)$, $(\bar{\delta}_x, \bar{\delta}_y) \in \partial v(\bar{x}, \bar{y})$, then

$$v(\bar{x}, \bar{y}) \le v(x, y) + \delta_x(\bar{x} - x) + \delta_y(\bar{y} - y)$$

$$\le v(\bar{x}, \bar{y}) + \bar{\delta}_x(x - \bar{x}) + \bar{\delta}_y(y - \bar{y}) + \delta_x(\bar{x} - x) + \delta_y(\bar{y} - y),$$

so

(6.6)                $\left(\delta_x - \bar{\delta}_x\right)(x - \bar{x}) + \left(\delta_y - \bar{\delta}_y\right)(y - \bar{y}) \le 0.$

Let us define for $(x, y) \in \mathscr{S}$,

$$\theta^+(x, y) \triangleq \max\left\{-(1 - \mu)\delta_x + \delta_y; (\delta_x, \delta_y) \in \partial v(x, y)\right\},$$

$$\theta^-(x, y) \triangleq \min\left\{-(1 - \mu)\delta_x + \delta_y; (\delta_x, \delta_y) \in \partial v(x, y)\right\}.$$

The functions $\theta^{\pm}$ are nonnegative because of (6.2), and the above maxima and minima are attained because $\partial v(x, y)$ is compact. We parametrize a half-line originating on $\partial_1 \mathscr{S}$ and parallel to $\partial_2 \mathscr{S}$ by

$$(x(\rho), y(\rho)) = (1 - (1 - \mu)\rho, -(1 - \lambda) + \rho)      \forall \rho \ge 0,$$

and define

$$\rho_0 \triangleq \inf\{p > 0; \theta^-(x(\rho), y(\rho)) = 0\},$$

where $\rho_0 = \infty$ if this set is empty.

LEMMA 6.3.  *For $0 < \rho < \bar{\rho} < \infty$, we have*

(6.7)                        $\theta^+(x(\bar{\rho}), y(\bar{\rho})) \le \theta^-(x(\rho), y(\rho)).$

*If $\rho_0 \in (0, \infty)$, then $\theta^-(x(\rho_0), y(\rho_0)) = 0$ and*

(6.8)                        $\theta^+(x(\rho), y(\rho)) = 0$      $\forall \rho > \rho_0.$

PROOF.  For $0 < \rho < \bar{\rho} < \infty$, let $(\delta_x, \delta_y) \in \partial v(x(\rho), y(\rho))$ and $(\bar{\delta}_x, \bar{\delta}_y) \in \partial v(x(\bar{\rho}), y(\bar{\rho}))$ be given. From (6.6) we have

$$-(1 - \mu)\bar{\delta}_x + \bar{\delta}_y \le -(1 - \mu)\delta_x + \delta_y.$$

Maximizing the left-hand side over $(\bar{\delta}_x, \bar{\delta}_y) \in \partial v(x(\bar{\rho}), y(\bar{\rho}))$ and minimizing the right-hand side over $(\delta_x, \delta_y) \in \partial v(x(\rho), y(\rho))$, we obtain (6.7).

Suppose $\rho_0 \in (0, \infty)$. Because $\theta^- \le \theta^+$, (6.7) implies that $\rho \mapsto \theta^-(x(\rho), y(\rho))$ is nonincreasing, and we must have $\theta^-(x(\rho), y(\rho)) = 0$ for all $\rho > \rho_0$. For $n \ge 1$, choose $(\delta_x^n, \delta_y^n) \in \partial v(x(\rho_0 + 1/n), y(\rho_0 + 1/n))$ such that $-(1 - \mu)\delta_x^n + \delta_y^n = 0$. According to Lemma 6.1, a subsequence of $\{(\delta_x^n, \delta_y^n)\}_{n=1}^{\infty}$ converges to a point $(\delta_x, \delta_y) \in \partial v(x(\rho_0), y(\rho_0))$. We must have $-(1 - \mu)\delta_x + \delta_y = 0$, so $\theta^-(x(\rho_0), y(\rho_0)) = 0$. Equation (6.8) now follows from (6.7).  □

We partition $\mathscr{S}$ into two open, convex (possibly empty) cones:

$$\text{SS} \triangleq \{(x, y); (\gamma x, \gamma y) = (x(\rho), y(\rho))$$
$$\text{for some } \gamma > 0 \text{ and some } \rho \in (\rho_0, \infty)\},$$
$$\mathscr{S} \backslash \overline{\text{SS}} \triangleq \{(x, y); (\gamma x, \gamma y) = (x(\rho), y(\rho))$$
$$\text{for some } \gamma > 0 \text{ and some } \rho \in (0, \rho_0)\}.$$

PROPOSITION 6.4. *We have*

$$-(1 - \mu)\delta_x + \delta_y = 0 \qquad \forall(\delta_x, \delta_y) \in \partial v(x, y), \forall(x, y) \in SS.$$

PROOF. Let $(x, y) \in \text{SS}$ be given and choose $\gamma > 0$, $\rho > \rho_0$ such that $(\gamma x, \gamma y) = (x(\rho), y(\rho))$. From (6.8) we have $\theta^+(\gamma x, \gamma y) = 0$, and (6.5) implies $\theta^+(x, y) = \gamma^{1-p}\theta^+(\gamma x, \gamma y)$. □

In an analogous manner, we can parametrize a half-line originating on $\partial_2\mathscr{S}$ and parallel to $\partial_1\mathscr{S}$ by

$$(\tilde{x}(\rho), \tilde{y}(\rho)) = (-(1 - \mu) + \rho, 1 - (1 - \lambda)\rho) \qquad \forall \rho > 0$$

and define

$$\tilde{\rho}_0 \triangleq \inf\{\rho > 0; \delta_x - (1 - \lambda)\delta_y = 0 \text{ for some } (\delta_x, \delta_y) \in \partial v(\tilde{x}(\rho), \tilde{y}(\rho))\}.$$

We partition $\mathscr{S}$ into a different pair of open, convex (possibly empty) cones:

$$\text{SMM} \triangleq \{(x, y); (\gamma x, \gamma y) = (x(\rho), y(\rho))$$
$$\text{for some } \gamma > 0 \text{ and some } \rho \in (\tilde{\rho}_0, \infty)\},$$
$$\mathscr{S} \backslash \overline{\text{SMM}} \triangleq \{(x, y); (\gamma x, \gamma y) = (x(\rho), y(\rho))$$
$$\text{for some } \gamma > 0 \text{ and some } \rho \in (0, \tilde{\rho}_0)\}.$$

Analogously to Proposition 6.4, one can show the following proposition:

PROPOSITION 6.5. *We have*

$$\delta_x - (1 - \lambda)\delta_y = 0 \qquad \forall(\delta_x, \delta_y) \in \partial v(x, y), \forall(x, y) \in SMM.$$

COROLLARY 6.6. $\text{SS} \cap \text{SMM} = \varnothing$.

PROOF. The equations $(1 - \mu)\delta_x + \delta_y = 0$ and $\delta_x - (1 - \lambda)\delta_y = 0$ imply $\delta_x = \delta_y = 0$, which is inconsistent with (6.3). □

COROLLARY 6.7. *The value function is $C^1$ in* SS ∪ SMM.

PROOF. In SS we have $-(1 - \mu)\delta_x + \delta_y = 0$ and (6.4). This pair of equations has a unique solution for $(\delta_x, \delta_y)$ [in terms of $v(x, y)$]. Now apply Proposition 6.2. The same argument can be used in SMM. □

If the cone SS is nonempty, then $\partial_2 \mathscr{S}$ is a subset of its boundary. Likewise, if SMM is nonempty, then $\partial_1 \mathscr{S}$ is part of its boundary. Thus, $\mathscr{S} \setminus (\text{SS} \cup \text{SMM})$ is also a convex cone. We denote by NT ("no transaction") the interior of this cone, that is, $\text{NT} = \mathscr{S} \setminus (\overline{\text{SS} \cup \text{SMM}})$.

PROPOSITION 6.8. *We have*

$$(1 - \mu)\delta_x + \delta_y > 0, \qquad \delta_x - (1 - \lambda)\delta_y > 0$$

$$\forall (\delta_x, \delta_y) \in \partial v(x, y), \forall (x, y) \in \text{NT}.$$

PROOF. Let $(x, y) \in \text{NT}$ and $(\delta_x, \delta_y) \in \partial v(x, y)$ be given. Choose $\gamma > 0$ and $\rho \in (0, \rho_0)$ such that $(\gamma x, \gamma y) = (x(\rho), y(\rho))$. From (6.5) we have $(\gamma^{p-1}\delta_x, \gamma^{p-1}\delta_y) \in \partial v(\gamma x, \gamma y)$. From the definition of $\rho_0$, we have

$$-(1 - \mu)\delta_x + \delta_y = \gamma^{1-p}\left[ -(1 - \mu)\gamma^{p-1}\delta_x + \gamma^{p-1}\delta_y \right]$$

$$\geq \gamma^{1-p}\theta^-(x(\rho), y(\rho)) > 0.$$

We show similarly that $\delta_x - (1 - \lambda)\delta_y > 0$. □

In conclusion, we have defined three nonintersecting, open, convex cones SS, SMM and NT with the property that $\overline{\mathscr{S}} = \overline{\text{SS} \cup \text{SMM} \cup \text{NT}}$. We have established that $v$ is $C^1$ in $\text{SS} \cup \text{SMM}$, and we have obtained linear equations involving $v_x$ and $v_y$ in SS and SMM. These equations, coupled with the homotheticity of Proposition 3.3, mandate that the form of $v$ in SS and SMM must be as in the next theorem.

THEOREM 6.9. *If $p < 1$, $p \neq 0$, then there are constants $A > 0$, $B > 0$ such that*

$$v(x, y) = \frac{1}{p}A^{p-1}(x + (1 - \mu)y)^p \qquad \forall (x, y) \in \text{SS},$$

$$v(x, y) = \frac{1}{p}B^{p-1}\left(x + \frac{y}{1 - \lambda}\right)^p \qquad \forall (x, y) \in \text{SMM}.$$

*If $p = 0$, then there are constants $A$, $B$ such that*

$$v(x, y) = \frac{1}{\beta}\log(x + (1 - \mu)y) + A \qquad \forall (x, y) \in \text{SS},$$

$$v(x, y) = \frac{1}{\beta}\log\left(x + \frac{y}{1 - \lambda}\right) + B \qquad \forall (x, y) \in \text{SMM}.$$

REMARK 6.10. The results of Theorem 6.9 are consistent with Theorems 5.9 and 5.10. Indeed, for $0 \leq p < 1$, Theorems 5.9 and 5.10 show that SS and SMM are nonempty, and SMM contains $\{(x, y) \in \mathscr{S}; y < 0\}$. The sets SS and SMM also have these properties when $p < 0$, as will be shown in Corollaries 8.7 and 8.8.

**7. Viscosity solutions.** In order to proceed further with the analysis of $v$, we must introduce the notion of viscosity solution of a second-order differential equation. We define this concept in $n$-dimensional space and use it in one- and two-dimensional space. For more information on viscosity solutions, we refer the reader to the survey article by Crandall, Ishii and Lions [12] and the book by Fleming and Soner [26]. For the application of the theory of viscosity solutions to problems in mathematical finance, see Zariphopoulou [66], [67], Fleming and Zariphopoulou [27], Duffie, Fleming and Zariphopoulou [17] and Davis, Panas and Zariphopoulou [15]. This section, which is included only for the sake of completeness, contains only known results and slight variations of known results. In particular, the proof of Theorem 7.7 is very similar to the proof Theorem 2 in Section 5 of [15].

Let $L(R^n, R)$ denote the set of $n$-dimensional vectors and $S(R^n, R^n)$ the set of $n \times n$ symmetric matrices. For $A, B \in S(R^n, R^n)$, we write $A \geq B$ to mean that $A - B$ is nonnegative definite. The gradient of a function $w$ of $n$ variables will be denoted by $Dw$ and the matrix of second-order partial derivatives will be denoted by $D^2 w$.

Let $\mathscr{O}$ be a connected, open subset of $R^n$ and let $F\colon \mathscr{O} \times R \times L(R^n, R) \times S(R^n, R^n) \to R$ be continuous and have the property

$$(7.1) \qquad\qquad F(x, z, \delta, A) \leq F(x, z, \delta, B)$$

whenever $x \in \mathscr{O}$, $z \in R$, $\delta \in L(R^n, R)$, $A, B \in S(R^n, R^n)$ and $A \geq B$. We consider the second-order differential equation

$$(7.2) \qquad\qquad F(x, w, Dw, D^2 w) = 0 \quad \text{on } \mathscr{O},$$

where $w$ is a function from $\mathscr{O}$ to $R$.

EXAMPLE 7.1. Let $F\colon \mathscr{S} \times R \times L(R^2, R) \times S(R^2, R^2) \to R$ be defined by

$$F\left( (x, y), z, (\delta_x, \delta_y), \begin{bmatrix} q_{xx} & q_{xy} \\ q_{xy} & q_{yy} \end{bmatrix} \right)$$

$$= \max\Big\{ \beta z - \tfrac{1}{2}\sigma^2 q_{yy} - \alpha y \delta_y - r x \delta_x - \tilde{U}_p(\delta_x),$$

$$-(1 - \mu)\delta_x + \delta_y, \, \delta_x - (1 - \lambda)\delta_y \Big\}.$$

Then (7.1) is satisfied and (7.2) is the HJB equation (4.6).

DEFINITION 7.2 [10]–[12]. Let $w\colon \mathscr{O} \to R$ be continuous. We say $w$ is a *viscosity subsolution* of (7.2) if, for every $x_0 \in \mathscr{O}$ and for every $\varphi \in C^2(\mathscr{O})$ satisfying $\varphi \geq w$ on $O$ and $\varphi(x_0) = w(x_0)$, we have

$$(7.3) \qquad\qquad F\big(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \leq 0.$$

We say $w$ is a *viscosity supersolution* of (7.2) if, for every $x_0 \in \mathscr{O}$ and for every $\varphi \in C^2(\mathscr{O})$ satisfying $\varphi \leq w$ on $O$ and $\varphi(x_0) = w(x_0)$, we have

$$(7.4) \qquad\qquad F\big(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \geq 0.$$

We say $w$ is a *viscosity solution* of (7.2) if it is both a viscosity subsolution and a viscosity supersolution.

REMARK 7.3.   If a viscosity solution $w$ of (7.2) is in $C^2(\mathscr{O})$, then $w$ is a solution in the classical sense. To see this, take the test function $\varphi$ in Definition 7.2 to be $w$ itself and use (7.3) and (7.4) to obtain (7.2).

Moreover, if $w \in C^2(\mathscr{S})$ is a classical solution of (7.2), then it is also a viscosity solution. To see the subsolution property, let $\varphi \in C^2(\mathscr{S})$ satisfy $\varphi \geq w$ and $\varphi(x_0) = w(x_0)$. Because $\varphi - w$ attains a global minimum at $x_0$, we must have $D\varphi(x_0) = Dw(x_0)$ and $D^2\varphi(x_0) \geq D^2 w(x_0)$. Using (7.1), we have

$$F\big(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \leq F\big(x_0, w(x_0), Dw(x_0), D^2 w(x_0)\big) = 0.$$

A similar argument establishes the supersolution property.   □

We see then that the notion of viscosity solution is a generalization of the notion of classical solution to equations of the form (7.2), when (7.1) is satisfied. In this section, without yet knowing that the value function $v$ for the problem of Section 2 is $C^2$, or even $C^1$, we succeed in showing that $v$ is a viscosity solution of the HJB equation (4.6). Before doing this, for later use we give a "local" form of the definition of viscosity solution.

LEMMA 7.4.   *Let $\mathscr{O}$ be an open subset of $R^n$ and let $w: \mathscr{O} \to R$ be continuous. Then there is a $C^\infty$ function $W: \mathscr{O} \to R$ such that $|w| \leq W$ on $\mathscr{O}$.*

PROOF.   We recall that if $A$ and $B$ are disjoint, closed subsets of $R^n$, then there is a continuous function $f: R^n \to [0, 1]$ which takes the value 1 everywhere on $A$ and takes the value 0 everywhere on $B$ (Urysohn's lemma). Indeed, for any closed set $F$, we can define the continuous function $d(\cdot; F)$ by $d(x; F) \triangleq \min\{|y - x|; y \in F\}$, and then a function $f$ with the above properties is given by

$$f(x) = \begin{cases} 1, & \forall x \in A, \\ \dfrac{d(x; A) \wedge d(y; B)}{d(x; A)}, & \forall x \in A^c. \end{cases}$$

Let $\mathscr{O}$ and $w$ be as in the lemma. Define

$$A_0 \triangleq \big\{ x \in R^n; d(x; \mathscr{O}^c) \geq 1 \big\}, \qquad B_0 \triangleq \big\{ x \in R^n; d(x; \mathscr{O}^c) \leq 1/2 \big\}$$

and for $k = 1, 2, \ldots,$ define

$$A_k = \big\{ x \in R^n; 2^{-k} \leq d(x; \mathscr{O}^c) \leq 2^{-k+1} \big\},$$
$$B_k = \big\{ x \in R^n; d(x; \mathscr{O}^c) \leq 2^{-k-1} \text{ or } d(x; \mathscr{O}^c) \geq 2^{-k+2} \big\}.$$

Let $f_k: R^n \to [0, 1]$ be a continuous function taking the value 1 on $A_k$ and 0 on $B_k$. Define the continuous function $w_k: R^n \to R$ by

$$w_k(x) \triangleq \begin{cases} f_k(x)|w(x)|, & \forall x \in \mathcal{O}, \\ 0, & \forall x \in \mathcal{O}^c. \end{cases}$$

Using the notation $B_\varepsilon(x) \triangleq \{x \in R^n; |x| < \varepsilon\}$, we can define

$$\tilde{w}_k(x) \triangleq \max\{w_k(y); y \in \overline{B}_{2^{-k-3}}(x)\}.$$

Let $\zeta$ be a nonnegative, $C^\infty$ function with support in $B_{2^{-k-3}}(0)$ and satisfying $\int_{R^n} \zeta(x)\, ds = 1$. We mollify $\tilde{w}_k$ by defining

$$W_k(x) \triangleq \int_{R^n} \tilde{w}_k(y)\zeta(x - y)\, dy \qquad \forall x \in R^n.$$

Then $W_k \geq w_k$ and $W_k$ is $C^\infty$ on $R^n$. Furthermore, for $k \geq 1$, $W_k$ has support in the set $\{x; 2^{-k-2} \leq d(x; \mathcal{O}^c) \leq 17 \cdot 2^{-k-2}\}$.

Finally, we set $W \triangleq \sum_{k=0}^\infty W_k$. In each compact subset of $\mathcal{O}$, this sum has only finitely many nonzero terms. Thus, $W$ is $C^\infty$ on $\mathcal{O}$. Because $\mathcal{O} = \bigcup_{k=0}^\infty A_k$, we have

$$W \geq |w| \cdot \sum_{k=0}^\infty f_k \geq |w| \quad \text{on } \mathcal{O}. \qquad \square$$

THEOREM 7.5.    *Let $w: \mathcal{O} \to R$ be continuous. Then $w$ is a viscosity subsolution of (7.2) if and only if, for every $x_0 \in \mathcal{O}$ and for every $\varphi \in C^2(\mathcal{O})$ such that $w - \varphi$ has a local maximum at $x_0$, we have*

(7.5)                    $$F(x_0, w(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

*Also, $w$ is a viscosity supersolution of (7.2) if and only if, for every $x_0 \in \mathcal{O}$ and for every $\varphi \in C^2(\mathcal{O})$ such that $w - \varphi$ has a local minimum at $x_0$, we have*

$$F(x_0, w(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

PROOF.    It is clear that the local conditions on $\varphi$ set forth in this theorem are satisfied by the test function $\varphi$ in Definition 7.2. Thus, only the "only if" part of each "if and only if" statement requires proof. We prove the assertion concerning subsolutions; the proof of the supersolution assertion is completely analogous.

Let us suppose that $w$ is a viscosity of (7.2), let $x_0 \in \mathcal{O}$ be given and let $\varphi \in C^2(\mathcal{S})$ be such that $w - \varphi$ has a local maximum at $x_0$. We assume for the moment that this local maximum is strict, so for some $\varepsilon > 0$, we have $B_\varepsilon(x_0) \subset \mathcal{O}$, where $B_\varepsilon(x_0) = \{x; |x - x_0| < \varepsilon\}$, and

$$\varphi(x) + w(x_0) - \varphi(x_0) > w(x) \qquad \forall x \in B_\varepsilon(x_0) \setminus \{x_0\}.$$

There is a $C^2$ function $g: [0, \infty) \to [0, 1]$ such that $g = 1$ on $[0, \varepsilon/2]$ and $g = 0$ on $[\varepsilon, \infty)$. Let $W$ be a $C^2$ function dominating $w$, and define

$$\tilde{\varphi}(x) = g(|x - x_0|)[\varphi(x) + w(x_0) - \varphi(x_0)] + (1 - g(|x - x_0|))W(x).$$

Then $\tilde{\varphi} \in C^2(\mathcal{O})$ satisfies $\tilde{\varphi} \geq w$ on $\mathcal{O}$ and $\tilde{\varphi}(x_0) = w(x_0)$. Therefore, $F(x_0, \tilde{\varphi}(x_0), D\tilde{\varphi}(x_0), D^2\tilde{\varphi}(x_0)) \leq 0$. Because $D\varphi(x_0) = D\tilde{\varphi}(x_0)$ and $D^2\varphi(x_0) = D^2\tilde{\varphi}(x_0)$, (7.5) holds.

If $w - \varphi$ has a nonstrict local maximum at $x_0$, we introduce the function

$$\varphi^\delta = \varphi(x) - \delta e^{-|x - x_0|^2},$$

where $\delta > 0$. Then $w - \varphi^\delta$ has a strict local maximum at $x_0$, so

(7.6)             $F(x_0, w(x_0), D\varphi^\delta(x_0), D^2\varphi^\delta(x_0)) \leq 0.$

However, $D\varphi^\delta(x_0) = D\varphi(x_0)$ and $D^2\varphi^\delta(x_0) = D\varphi(x_0) + 2\delta I$, where $I$ is the $n \times n$ identity. Letting $\delta \downarrow 0$ in (7.6) and using the continuity of $F$ in its last argument, we obtain (7.5). □

COROLLARY 7.6.   *Let $w: \mathcal{O} \to R$ be a continuous viscosity solution to (7.2). If $w$ is twice differentiable at a point $x_0 \in \mathcal{O}$, then $w$ satisfies (7.2) in the classical sense at that point.*

PROOF.   For $\varepsilon > 0$, define $\varphi \in C^2(\mathcal{O})$ by

$$\varphi(x) = w(x_0) + Dw(x_0)(x - x_0)$$
$$+ \tfrac{1}{2}(D^2 w(x_0)(x - x_0)) \cdot (x - x_0)$$
$$+ \varepsilon \|x - x_0\|^2 \qquad \forall x \in C^2(\mathcal{O}).$$

Then $w - \varphi$ has a local maximum at $x_0$, so (7.5) holds. Let $\varepsilon \downarrow 0$ to obtain

$$F(x_0, w(x_0), Dw(x_0), D^2 w(x_0)) \leq 0.$$

The reverse inequality follows from an analogous argument. □

THEOREM 7.7.   *The value function $v$ defined by (2.8) is a viscosity solution of the HJB equation (4.6) on $S$.*

We divide the proof of this theorem into two lemmas.

LEMMA 7.8.   *The value function $v$ is a viscosity supersolution of (4.6) on $S$.*

PROOF.   Let $(x_0, y_0) \in \mathcal{S}$ and $\varphi \in C^2(\mathcal{S})$ be given with

$$\varphi \leq v \quad \text{on } \mathcal{S}, \qquad \varphi(x_0, y_0) = v(x_0, y_0).$$

For $\gamma > 0$ sufficiently small, $(x_0 - \gamma, y_0 + (1 - \lambda)\gamma) \in \mathcal{S}$ and Proposition 3.5 implies

$$\varphi(x_0 - \gamma, y_0 + (1 - \lambda)\gamma) - \varphi(x_0, y_0)$$
$$\leq v(x_0 - \gamma, y_0 + (1 - \lambda)\gamma) - v(x_0, y_0) \leq 0.$$

Divide by $\gamma$ and let $\gamma \downarrow 0$ to obtain

$$\varphi_x(x_0, y_0) - (1 - \lambda)\varphi_y(x_0, y_0) \geq 0.$$

A similar argument shows that

$$-(1 - \mu)\varphi_x(x_0, y_0) + \varphi_y(x_0, y_0) \geq 0.$$

It remains only to show that

(7.7)
$$(\mathscr{L}\varphi)(x_0, y_0) - \tilde{U}_p(\varphi_x(x_0, y_0)) \geq 0.$$

Choose $\varepsilon > 0$ so that $\bar{B}_\varepsilon(x_0, y_0) \subset \mathscr{S}$. For $c > 0$, let $(C, L, M)$ be a policy in $\mathscr{A}(x_0, y_0)$ with $L \equiv 0$, $M \equiv 0$ and $C(s) = c$ for $0 \leq s \leq \tau_\varepsilon$, where

$$\tau_\varepsilon \triangleq \varepsilon \wedge \inf\{t \geq 0; (X(t), Y(t)) \notin \bar{B}_\varepsilon(x_0, y_0)\}.$$

Itô's rule [see (4.7)] implies

$$v(x_0, y_0) = \varphi(x_0, y_0) = Ee^{-\beta\tau_\varepsilon}\varphi(X(\tau_\varepsilon), Y(\tau_\varepsilon)) + E\int_0^{\tau_\varepsilon} e^{-\beta s}(\mathscr{L}\varphi + c\varphi_x)\,ds.$$

Using the relationship between $v$ and $\varphi$ and the optimality equation (4.5), we obtain

$$v(x_0, y_0) \geq E\left[\int_0^{\tau_\varepsilon} e^{-\beta s}U_p(c)\,ds + e^{-\beta\tau_\varepsilon}v(X(\tau_\varepsilon), Y(\tau_\varepsilon))\right]$$

$$\geq E\left[\int_0^{\tau_\varepsilon} e^{-\beta s}U_p(c)\,ds + e^{-\beta\tau_\varepsilon}\varphi(X(\tau_\varepsilon), Y(\tau_\varepsilon))\right]$$

$$= v(x_0, y_0) - E\int_0^{\tau_\varepsilon} e^{-\beta s}[\mathscr{L}\varphi - U_p(c) + c\varphi_x]\,ds.$$

It follows that

$$E\int_0^{\tau_\varepsilon} e^{-\beta s}[(\mathscr{L}\varphi)(X_s, Y_s) - U_p(c) + c\varphi_x(X_s, Y_s)]\,ds \geq 0$$

for all $\varepsilon$ sufficiently small. This can happen only if

$$\max_{(x, y)\in\bar{B}_\varepsilon(x_0, y_0)}[(\mathscr{L}\varphi)(x, y) - U_p(c) + c\varphi_x(x, y)] \geq 0,$$

and as $\varepsilon \downarrow 0$, we see that

$$(\mathscr{L}\varphi)(x_0, y_0) - U_p(c) + c\varphi_x(x_0, y_0) \geq 0.$$

Minimization of this expression over $c > 0$ leads to (7.7). $\square$

LEMMA 7.9. *The value function $v$ is a viscosity subsolution of (4.6) on S.*

PROOF. Let $(x_0, y_0) \in \mathscr{S}$ and $\varphi \in C^2(\mathscr{S})$ be given with $\varphi \geq v$ on $\mathscr{S}$ and $\varphi(x_0, y_0) = v(x_0, y_0)$. We argue by contradiction.

Assume the subsolution inequality

$$\min\{\mathscr{L}\varphi - \tilde{U}_p(\varphi_x), -(1 - \mu)\varphi_x + \varphi_y, \varphi_x - (1 - \lambda)\varphi_y\} \leq 0 \quad \text{at } (x_0, y_0)$$

fails. Then there are constants $\gamma > 0$, $\varepsilon > 0$ such that the set

$$H \triangleq \{(x, y); |x - x_0| \leq \varepsilon, |y - y_0| \leq \varepsilon\}$$

is a subset of $\mathscr{S}$, and on $H$,

(7.8)     $\mathscr{L}\varphi - \tilde{U}_p(\varphi_x) \geq \gamma$,     $-(1 - \mu)\varphi_x + \varphi_y \geq \gamma$,     $\varphi_x - (1 - \lambda)\varphi_y \geq \gamma$.

For $(C, L, M) \in \mathscr{A}(x_0, y_0)$, define $\tau \triangleq \inf\{t \geq 0; (X(t), Y(t)) \notin H\}$. According to the optimality equation (4.5), for each $t \in [0, \infty)$,

(7.9)
$$v(x_0, y_0) = \sup_{(C, L, M) \in \mathscr{A}(x_0, y_0)} E\left[\int_0^{t \wedge \tau} e^{-\beta s} U_p(C(s))\, ds + e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau))\right].$$

In light of Proposition 3.5, it does not reduce the above supremum to restrict it to policies satisfying

(7.10)                    $(X(\tau_\varepsilon), Y(\tau_\varepsilon)) \in \partial H$   on $\{\tau < \infty\}$.

Let $(C, L, M)$ be such a policy. From Itô's rule [see (4.7)] and (7.8) we have

$$\varphi(x_0, y_0) \geq E e^{-\beta(t \wedge \tau)} \varphi(X(t \wedge \tau), Y(t \wedge \tau)) + E\int_0^{t \wedge \tau} e^{-\beta s}\left(\mathscr{L}\varphi - \tilde{U}_p(\varphi_x)\right) ds$$

$$+ E\int_0^{t \wedge \tau} e^{-\beta s}\left(\tilde{U}_p(\varphi_x) + C(s)\varphi_x\right) ds$$

$$+ \gamma E\int_0^{t \wedge \tau} e^{-\beta s}(dL(s) + dM(s))$$

(7.11)
$$\geq E e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau)) + E\int_0^{t \wedge \tau} e^{-\beta s} U_p(C(s))\, ds$$

$$+ \gamma e^{-\beta t} E\left[(t \wedge \tau) + L(t \wedge \tau) + M(t \wedge \tau)\right]$$

$$+ e^{-\beta t} E\int_0^{t \wedge \tau}\left[\tilde{U}_p(\varphi_x) + C(s)\varphi_x - U_p(C(s))\right] ds$$

$$\geq \nu(t) + E\left[e^{-\beta(t \wedge \tau)} v(X(t \wedge \tau), Y(t \wedge \tau)) + \int_0^{t \wedge \tau} e^{-\beta s} U_p(C(s))\, ds\right],$$

where

$$\nu(t) = \inf_{\substack{(C, L, M) \in \mathscr{A}(x_0, y_0) \\ (7.10)\ \text{holds}}} \left\{\gamma e^{-\beta t} E\left[(t \wedge \tau) + L(t \wedge \tau) + M(t \wedge \tau)\right]\right.$$

$$\left. + e^{-\beta t} E\int_0^{t \wedge \tau}\left[\tilde{U}_p(\varphi_x) + C(s)\varphi_x - U_p(C(s))\right] ds\right\}.$$

Maximize the right-hand side of (7.11) over $(C, L, M) \in \mathscr{A}(x_0, y_0)$ such that (7.10) holds and use (7.9) to obtain

$$\varphi(x_0, y_0) \geq \nu(t) + v(x_0, y_0) = \nu(t) + \varphi(x_0, y_0).$$

We obtain our contradiction by showing that for $t > 0$ sufficiently small, $\nu(t) > 0$.

Let us begin with the definitions

$$A(t) = \max_{0 \le s \le t} W(s), \qquad a(t) = \min_{0 \le s \le t} W(s).$$

According to Doob's maximal martingale inequality (e.g., [43], Theorem 1.3.8), for $\delta > 0$,

$$P\{A(t) \ge \delta\} \le \frac{1}{\delta^2} EA^2(t) \le \frac{4}{\delta^2} EW^2(t) = \frac{4t}{\delta^2},$$

and similarly, $P\{a(t) \le -\delta\} \le 4t/\delta^2$. Therefore, for all $\delta > 0$,

$$(7.12) \quad P\{A(t) - a(t) \ge \delta\} \le P\left\{A(t) \ge \frac{\delta}{2}\right\} + P\left\{a(t) \le -\frac{\delta}{2}\right\} \le \frac{32t}{\delta^2}.$$

Define the decreasing family of sets

$$F(t) \triangleq \Big\{ \max\big\{ \exp\big[|\alpha - \sigma^2/2|t + \sigma(A(t) - a(t))\big] - 1,$$

$$1 - \exp\big[-|\alpha - \sigma^2/2|t - \sigma(A(t) - a(t))\big]\big\}$$

$$\le \min\Big\{ \frac{\varepsilon}{2(|y_0| \vee 1)}, 1 \Big\} \Big\}, \qquad t \ge 0.$$

Inequality (7.12) shows we can choose $T > 0$ so

$$(7.13) \qquad P\{F(T)\} \ge \frac{1}{2}, \qquad e^{rT} - 1 \le \frac{\varepsilon}{2(|x_0| \vee 1)}, \qquad e^{rT} \le 2.$$

From (2.3)–(2.5) we have the formulas

$$X(t) = x_0 e^{rt} + \int_0^t e^{r(t-s)}[-C(s)\, ds + (1 - \mu)\, dM(s) - dL(s)],$$

$$Y(t) = y_0 \exp\big((\alpha - \tfrac{1}{2}\sigma^2)t + \sigma W(t)\big)$$

$$+ \int_0^t \exp\big((\alpha - \tfrac{1}{2}\sigma^2)(t - s)$$

$$+ \sigma(W(t) - W(s))\big)[-dM(s) + (1 - \lambda)\, dL(s)],$$

from which follow

$$|X(t) - x_0| \le (e^{rt} - 1)|x_0| + e^{rt}\int_0^t C(s)\, ds + e^{rt}[(1 - \mu)M(t) + L(t)],$$

$$|Y(t) - y_0| \le |y_0|\max\big\{ \exp\big(|\alpha - \tfrac{1}{2}\sigma^2|t + \sigma(A(t) - a(t))\big) - 1,$$

$$1 - \exp\big(-|\alpha - \tfrac{1}{2}\sigma^2|t - \sigma(A(t) - a(t))\big)\big\}$$

$$+ \exp\big(|\alpha - \tfrac{1}{2}\sigma^2|t + \sigma(A(t) - a(t))\big)[M(t) + (1 - \lambda)L(t)].$$

With $T > 0$ chosen to satisfy (7.13), on the set $F(t)$ we have for $0 \le t \le T$,

$$(7.14) \quad |X(t) - x_0| \le \frac{\varepsilon}{2} + 2\int_0^t C(s)\, ds + 2[(1 - \mu)M(t) + L(t)],$$

$$(7.15) \quad |Y(t) - y_0| \le \frac{\varepsilon}{2} + 2[M(t) + (1 - \lambda)L(t)].$$

From Corollary 3.7 we see that $I_p(\varphi_x)$ is bounded on $H$, where $I_p$ is defined by (2.10). Choose $k > \max_H I_p(\varphi_x)$ and set $\eta = \min_H \varphi_x - U_p'(k)$, which is strictly positive. Then (2.12) implies

$$(7.16) \qquad \tilde{U}_p(\varphi_x) + C(s)\varphi_x - U_p(C(s)) \geq \eta(C(s) - k)^+.$$

Finally, choose $t \in (0, T \wedge \varepsilon/(16k))$ and consider $\omega \in F(t)$. Define

$$Z(\omega) = \gamma e^{-\beta t}[t \wedge \tau(\omega) + M(t \wedge \tau(\omega), \omega) + L(t \wedge \tau(\omega), \omega)]$$

$$+ e^{-\beta t}\int_0^{t \wedge \tau(\omega)}[\tilde{U}_p(\varphi_x(X(s, \omega), Y(s, \omega)))$$

$$+ C(s, \omega)\varphi_x(X(s, \omega), Y(s, \omega)) - U_p(C(s, \omega))]\, ds.$$

We consider several cases.

*Case 1:* $\tau(\omega) \geq t$. We have $Z(\omega) \geq \gamma e^{-\beta t}t$.

*Case 2:* $\tau(\omega) < t$. In this case, either $|X(\tau(\omega), \omega) - x_0| \geq \varepsilon$ or $|Y(\tau(\omega), \omega) - y_0| \geq \varepsilon$. We consider these possibilities separately.

*Subcase 2A:* $\tau(\omega) < t$ and $|X(\tau(\omega), \omega) - x_0| \geq \varepsilon$. In light of (7.14), we must have either

$$2\int_0^{\tau(\omega)}C(s, \omega)\, ds \geq \frac{\varepsilon}{4} \quad \text{or} \quad 2[(1 - \mu)M(\tau(\omega), \omega) + L(\tau(\omega), \omega)] \geq \frac{\varepsilon}{4}.$$

In the latter case,

$$Z(\omega) \geq \gamma e^{-\beta t}[M(\tau(\omega), \omega) + L(\tau(\omega), \omega)] \geq \gamma e^{-\beta t}\frac{\varepsilon}{8}.$$

In the former case,

$$\frac{\varepsilon}{8} \leq \int_0^{\tau(\omega)}C(s, \omega)\, ds \leq kt + \int_0^{\tau(\omega)}(C(s, \omega) - k)^+\, ds$$

$$\leq \frac{\varepsilon}{16} + \int_0^{\tau(\omega)}(C(s, \omega) - k)^+\, ds.$$

Then (7.16) implies

$$Z(\omega) \geq e^{-\beta t}\frac{\eta\varepsilon}{16}.$$

*Subcase 2B:* $\tau(\omega) < t$ and $|Y(\tau(\omega), \omega) - y_0| \geq \varepsilon$. Inequality (7.15) implies

$$2[M(\tau(\omega), \omega) + (1 - \lambda)L(\tau(\omega), \omega)] \geq \frac{\varepsilon}{2}$$

and so

$$Z(\omega) \geq \gamma e^{-\beta t}\frac{\varepsilon}{4}.$$

We have shown in every case that

$$Z(\omega) \geq e^{-\beta t}\min\left\{\gamma t, \frac{\gamma\varepsilon}{8}, \frac{\eta\varepsilon}{16}\right\} > 0 \qquad \forall \omega \in F(t),$$

and since $P(F(t)) \geq 1/2$, we have

$$\nu(t) \geq \frac{1}{2} e^{-\beta t} \min\left\{ \gamma t, \frac{\gamma \varepsilon}{8}, \frac{\eta \varepsilon}{16} \right\} > 0. \qquad \square$$

**8. Reduction to one variable and value function regularity.**  In this section we treat the smoothness of the value function $v$ in NT. (Theorem 6.9 establishes smoothness in SS and SMM.) The homotheticity property allows us to compute as many derivatives of $v$ as we like in the radial direction. We exploit the fact that $v$ is a viscosity solution of an HJB equation involving $\sigma^2 y^2 v_{yy}/2$ to obtain the existence and continuity of $v_{yy}$. Together these computations allow us to show that $v$ is $C^2$ in NT, except possibly along the positive $y$- and $x$-axis. Along the $x$-axis, the term $\sigma^2 y^2 v_{yy}/2$ drops out of the HJB equation. We eventually show, however, that the positive $x$-axis is in SMM (Theorem 11.6), so this is ultimately of no concern. Along the $y$-axis, the radial direction coincides with the $y$-direction. The best result we obtain is that if the positive $y$-axis is in NT, then $v_{yy}$ exists and is continuous there (Theorem 9.1).

To exploit homotheticity, we reduce the problem to one of a single variable. Define $\mathscr{I} = (-(1 - \lambda)/\lambda, 1/\mu)$ and

(8.1) $$u(z) = v(1 - z, z) \qquad \forall z \in \bar{\mathscr{I}}.$$

For all $(x, y) \in \bar{\mathscr{S}} \setminus \{(0, 0)\}$, Proposition 3.3 implies

$$v(x, y) = \begin{cases} (x + y)^p u\left( \dfrac{y}{x + y} \right), & \text{if } p < 1, \, p \neq 0, \\[2mm] \dfrac{1}{\beta} \log(x + y) + u\left( \dfrac{y}{x + y} \right), & \text{if } p = 0. \end{cases}$$

Note that $u$ and $v$ have the same degree of smoothness. For $z \in \mathscr{I}$, we define

$$d_1(z) = r + (\alpha - r)z - \frac{1}{2}\sigma^2(1 - p)z^2,$$

$$d_2(z) = (\alpha - r)z(1 - z) - \sigma^2(1 - p)z^2(1 - z),$$

$$d_3(z) = \frac{1}{2}\sigma^2 z^2(1 - z)^2,$$

$$d_4(z) = \frac{1}{\mu}(1 - \mu z),$$

$$d_5(z) = \frac{1}{\lambda}(1 - \lambda(1 - z)).$$

Direct computation shows that $v$ is a classical solution to the HJB equation (4.6) if and only if $u$ is a classical solution to the second-order, ordinary

differential equation

$$\min\Big\{ \beta\psi(z) - d_1(z)p\psi(z) - d_2(z)\psi'(z) - d_3(z)\psi''(z)$$

(8.2)     $$-\tilde{U}_p(p\psi(z) - z\psi'(z)),$$

$$p\psi(z) + d_4(z)\psi'(z), p\psi(z) - d_5(z)\psi'(z)\Big\} = 0 \quad \text{if } p < 1, p \neq 0,$$

or

$$\min\Big\{ \beta\psi(z) - \frac{1}{\beta}d_1(z) - d_2(z)\psi'(z)$$

(8.3)     $$-d_3(z)\psi''(z) - \tilde{U}_0\Big(\frac{1}{\beta} - z\psi'(z)\Big),$$

$$\frac{1}{\beta} + d_4(z)\psi'(z), \frac{1}{\beta} - d_5(z)\psi'(z)\Big\} = 0 \quad \text{if } p = 0.$$

PROPOSITION 8.1.   *On $\mathscr{I}$, $u$ is a viscosity solution of* (8.2) *if $p \neq 0$ or* (8.3) *if $p = 0$.*

PROOF.   We prove the subsolution property when $p \neq 0$. Let $z_0 \in \mathscr{I}$ and $\psi \in C^2(\mathscr{I})$ be given such that $\psi \geq u$ and $\psi(z_0) = u(z_0)$. Define $\varphi(x, y) = (x + y)^p \psi(y/(x + y))$ for all $(x, y) \in \mathscr{S}$. Then $\varphi \geq v$ on $\mathscr{S}$ and $\varphi(1 - z_0, z_0) = v(1 - z_0, z_0)$. Because $v$ is a viscosity subsolution of (4.6), we have

$$\min\Big\{ \mathscr{L}\varphi - \tilde{U}_p(\varphi_x), -(1 - \mu)\varphi_x + \varphi_y, \varphi_x - (1 - \lambda)\varphi_y\Big\} \leq 0 \quad \text{at } (1 - z_0, z_0).$$

This is equivalent to

(8.4)
$$\min\Big\{ \beta\psi - d_1 p\psi - d_2\psi' - d_3\psi'' - \tilde{U}_p(p\psi - z\psi'),$$

$$p\psi + d_4\psi', p\psi - d_5\psi'\Big\} \leq 0 \quad \text{at } z_0. \qquad \square$$

Because the function $u$ is the function $v$ evaluated along a line segment, $u$ inherits concavity from $v$. The "convex" analysis of a concave function is particularly transparent. See, for example, [43], Problems 3.6.20, 3.6.21 and Solution 3.6.20 for a derivation of the following facts. For fixed $z \in \mathscr{I}$, the difference quotient $[u(z + h) - u(z)]/h$ is a nonincreasing function of $h$, and so the right- and left-derivatives

$$D^{\pm}u(z) \triangleq \lim_{h \to 0\pm} \frac{1}{h}[u(z + h) - u(z)]$$

exist and are finite. Furthermore, $D^+$ and $D^-$ are right- and left-continuous, respectively, are nonincreasing and agree except on a countable set $N$. We have $\partial u(z) = [D^+u(z), D^-u(z)]$ for all $z \in \mathscr{I}$. On $\mathscr{I} \setminus N$, $u$ is differentiable. Because $u'$ is defined almost everywhere and is nonincreasing, its pointwise derivative $u''$ is also defined almost everywhere on $\mathscr{I}$ and is locally inte-

grable. In light of Corollary 7.6, $u$ satisfies (8.2) if $p \neq 0$ or (8.3) if $p = 0$, almost everywhere in the classical sense.

PROPOSITION 8.2. *The function $u$ defined by (8.1) is $C^1$ on $\mathscr{I} \setminus \{0\}$. If $u$ is not also $C^1$ at $0$, then for every $x > 0$,*

$$
(8.5) \qquad v(x, 0) = \begin{cases} \dfrac{1}{p} C_*^{p-1} x^p, & if\, p \neq 0, \\ \dfrac{1}{\beta} \log x + \dfrac{1}{\beta} \log \beta + \dfrac{r - \beta}{\beta^2}, & if\, p = 0, \end{cases}
$$

*where $C_* \triangleq (\beta - rp)/(1 - p)$ is positive by Proposition 3.4. Furthermore, even if $u$ is not $C^1$ at $0$, its one-sided derivatives exist and are limits of its derivatives from the appropriate sides at $0$.*

PROOF. Let $\mathscr{T}$ be the subset of $\mathscr{I}$ on which $u$ is twice differentiable. According to the preceding discussion, $\mathscr{T}$ has full measure. Let $z_0 \in \mathscr{I} \setminus \mathscr{T}$ be given and let $\{z_n^+\}, \{z_n^-\}$ be sequences in $\mathscr{T}$ for which $z_n^- \uparrow z_0$, $z_n^+ \downarrow z_0$. Then

$$
\partial u(z_0) = \left[ D^+(z_0), D^-(z_0) \right] = \left[ \lim_{n \to \infty} u'(z_n^+), \lim_{n \to \infty} u'(z_n^-) \right].
$$

For specificity, we take the case $p \neq 0$. Because $u$ satisfies (8.2) in the classical sense at each $z_n^\pm$, we have

$$
pu(z_0) + d_4(z_0) D^+ u(z_0) \geq 0, \qquad pu(z_0) - d_5(z_0) D^- u(z_0) \geq 0.
$$

Now $d_4$ and $d_5$ are both positive on $\mathscr{I}$, so

$$
(8.6) \qquad pu(z_0) + d_4(z_0)\delta > 0, \qquad pu(z_0) - d_5(z_0)\delta > 0
$$

$$
\forall \delta \in (D^+ u(z_0), D^- u(z_0)).
$$

We assume that $D^+ u(z_0) < D^- u(z_0)$ so there is a $\delta$ as in (8.6), and we argue by contradiction. With such a $\delta$ and with $\varepsilon > 0$, the function $\psi_\varepsilon(z) \triangleq u(z_0) + \delta(z - z_0) - (z - z_0)^2/(2\varepsilon)$ dominates $u$ locally at $z_0$. The viscosity subsolution property of $u$ implies (see Theorem 7.5) that $\psi_\varepsilon$ satisfies (8.4), but in view of (8.6), we must actually have

$$
(8.7) \qquad \beta u(z_0) - d_1(z_0) pu(z_0) - d_2(z_0)\delta + \frac{1}{\varepsilon} d_3(z_0)
$$

$$
- \tilde{U}_p(pu(z_0) - z_0\delta) \leq 0.
$$

If $z_0 \notin \{0, 1\}$, then $d_3(z_0) > 0$ and (8.7) cannot hold for all $\varepsilon > 0$. This shows that $\partial u(z_0)$ is a singleton when $z_0 \in \mathscr{I} \setminus \{0, 1\}$, so $u$ is $C^1$ on this set.

Now suppose $z_0 = 1$. Because $u$ satisfies (8.2) in the classical sense at each $z_n^\pm$, we have

$$
(8.8) \qquad \begin{aligned} & \beta u(z_n^\pm) - d_1(z_n^\pm) pu(z_n^\pm) - d_2(z_n^\pm) u'(z_n^\pm) \\ & - d_3(z_n^\pm) u''(z_n^\pm) - \tilde{U}_p(pu(z_n^\pm) - z_n^\pm u'(z_n^\pm)) \geq 0. \end{aligned}
$$

The local integrability of $u''$ at 1 implies that the sequences $\{z_n^\pm\}$ can be chosen to satisfy

$$\lim_{n \to \infty} |d_3(z_n^\pm) u''(z_n^\pm)| = 0.$$

Passage to the limit in (8.8) results in

(8.9) $\qquad \beta u(1) - d_1(1) p u(1) - \tilde{U}_p(p u(1) - D^\pm u(1)) \geq 0.$

If $D^+ u(1) < D^- u(1)$ and $\delta \in (D^+ u(1), D^- u(1))$, then (8.7) becomes

$$\beta u(1) - d_1(1) p u(1) - \tilde{U}_p(p u(1) - \delta) \leq 0.$$

Letting $\delta \downarrow D^+ u(1)$ and also $\delta \uparrow D^- u(1)$, we obtain equality in (8.9). However, $\tilde{U}_p$ is strictly decreasing, so equality in (8.9) can hold only if $D^+ u(1) = D^- u(1)$. Again we conclude that $\partial u(z_0)$ is a singleton.

Finally, we consider the case $z_0 = 0$. Arguing as before, we obtain the counterpart to (8.9):

$$(\beta - rp) u(0) - \tilde{U}_p(p u(0)) \geq 0.$$

If $D^+ u(0) < D^- u(0)$ and $\delta \in (D^+ u(0), D^- u(0))$, (8.7) yields the reverse inequality. Solving the resulting equation, we obtain

$$v(1, 0) = u(0) = \frac{1}{p} C_*^{p-1}.$$

The first part of (8.5) follows from the homotheticity of $v$. $\square$

COROLLARY 8.3. *The value function $v$ is $C^1$ on $\mathscr{S} \setminus \{(x, 0); x > 0\}$. If $v$ is not also $C^1$ on $\{(x, 0); x > 0\}$, then (8.5) holds. Furthermore, even if $v$ is not $C^1$ on $\{(x, 0); x > 0\}$, the partial derivative $v_x$ is defined and continuous there, and the one-sided partial derivatives*

$$v_y(x, 0 \pm) \triangleq \lim_{h \to 0\pm} \frac{1}{h} [v(x, h) - v(x, 0)], \qquad x > 0,$$

*are defined and satisfy the one-sided continuity conditions*

$$v_y(x, 0 \pm) = \lim_{(\xi, \eta) \to (x, 0\pm)} v_y(\xi, \eta).$$

PROOF. For $p \neq 0$, the last part of the corollary follows from the formulas

$$v_x(x, y) = p(x + y)^{p-1} u\left(\frac{y}{x + y}\right) - y(x + y)^{p-2} u'\left(\frac{y}{x + y}\right),$$

$$v_y(x, y) = p(x + y)^{p-1} u\left(\frac{y}{x + y}\right) + x(x + y)^{p-2} u'\left(\frac{y}{x + y}\right),$$

valid for $(x, y) \in \mathscr{S}$, $x \neq 0$, $y \neq 0$. The case $p = 0$ is similar. $\square$

Recall from Section 6 the construction of three open, convex, nonintersecting cones SS, NT and SMM satisfying

$$\overline{\mathscr{P}} = \overline{SS} \cup \overline{NT} \cup \overline{SMM}.$$

If we denote $\mathscr{P}^* = \mathscr{P} \setminus \{(x,0); \ x > 0\}$, the set on which we know that $v$ is $C^1$, we may now characterize these cones more simply by the formulas

(8.10) $$\overline{SS} = \overline{\{(x,y) \in \mathscr{P}^*; \ -(1-\mu)v_x + v_y = 0\}},$$

(8.11) $$NT \setminus \{(x,0); \ x > 0\} = \{(x,y) \in \mathscr{P}^*; \ -(1-\mu)v_x + v_y > 0,$$
$$v_x - (1-\lambda)v_y > 0\},$$

(8.12) $$\overline{SMM} = \overline{\{(x,y) \in \mathscr{P}^*; \ v_x - (1-\lambda)v_y = 0\}}.$$

COROLLARY 8.4.  NT $\neq \varnothing$.

PROOF.  We cannot have SS $= \mathscr{P}$, for then the formulas in Theorem 6.9 are inconsistent with the boundary condition of Proposition 3.4 on $\partial_2 \mathscr{P}$. Similarly, SMM must be a proper subset of $\mathscr{P}$.

Suppose NT were empty. Then SS and SMM must each be nonempty and these cones would share a half-line boundary $H \triangleq (\partial SS \cap \partial SMM) \setminus \{(0,0)\}$ in $\mathscr{P}$. There are two possibilities: $v$ is $C^1$ on $H$ or $v$ is not $C^1$ on $H$. In the former case, (8.10) and (8.12) imply $v_x = v_y = 0$ on $H$, which contradicts (6.3). In the latter case, we must have $H = \{(x,0); \ x > 0\}$ and $v$ is given by (8.5). Theorem 6.9 and the continuity of $v$ imply

$$v(x,y) = \begin{cases} \dfrac{1}{p} C_*^{p-1} (x + (1-\mu)y)^p, & \forall (x,y) \in \mathscr{P}, \ y \geq 0, \\[3mm] \dfrac{1}{p} C_*^{p-1} \left(x + \dfrac{y}{1-\lambda}\right)^p, & \forall (x,y) \in \mathscr{P}, \ y < 0, \end{cases}$$

if $p \neq 0$, and

$$v(x,y) = \begin{cases} \dfrac{1}{\beta} \log(x + (1-\mu)y) + \dfrac{1}{\beta} \log \beta + \dfrac{r-\beta}{\beta^2}, \\[3mm] \hspace{5cm} \forall (x,y) \in \mathscr{P}, \ y \geq 0, \\[3mm] \dfrac{1}{\beta} \log\left(x + \dfrac{y}{1-\lambda}\right) + \dfrac{1}{\beta} \log \beta + \dfrac{r-\beta}{\beta^2}, \\[3mm] \hspace{5cm} \forall (x,y) \in \mathscr{P}, \ y < 0, \end{cases}$$

if $p = 0$. If $x > 0$, $y > 0$, then $v$ is $C^2$ at $(x,y)$ and so must satisfy the HJB equation (4.6) in the classical sense. However, if $p \neq 0$, we have [see (5.5)]

$$(\mathscr{L}v)(x,y) - \tilde{U}_p(v_x(x,y))$$

$$= C_*^{p-1}(x + (1-\mu)y)^{p-2}(1-\mu)y$$

$$\times \left[\tfrac{1}{2}(1-p)\sigma^2(1-\mu)y - (\alpha - r)(x + (1-\mu)y)\right],$$

which is negative for $y$ sufficiently close to zero. A similar contradiction is obtained when $p = 0$. □

Because NT $\neq \varnothing$, there are numbers $\theta_1 < \theta_2$ in $\bar{\mathscr{F}}$ such that

$$(8.13) \qquad \mathrm{NT} = \left\{ (x, y) \in \mathscr{S}; \; \theta_1 < \frac{y}{x + y} < \theta_2 \right\}.$$

PROPOSITION 8.5. *The function $u$ defined by (8.1) is $C^2$ on $(\theta_1, \theta_2) \setminus \{0, 1\}$ and, on this set, in the classical sense, satisfies the equation*

$$(8.14) \qquad \begin{aligned} \beta u - d_1 p u - d_2 u' - d_3 u'' - \tilde{U}_p( p u - z u' ) &= 0, \quad \textit{if } p \neq 0, \\ \beta u - \frac{1}{\beta} d_1 - d_2 u' - d_3 u'' - \tilde{U}_0 \left( \frac{1}{\beta} - z u' \right) &= 0, \quad \textit{if } p = 0. \end{aligned}$$

PROOF. We consider only the case $p \neq 0$. For this case, the first derivatives of $u$ and $v$ are related by the formulas

$$v_x(1 - z, z) = p u(z) - z u'(z), \qquad v_y(1 - z, z) = p u(z) + (1 - z) u'(z).$$

From (8.11) we have

$$(8.15) \qquad p u(z) + d_4(z) u'(z) > 0, \qquad p u(z) - d_5(z) u'(z) > 0$$
$$\forall z \in (\theta_1, \theta_2) \setminus \{0\}.$$

We first show that $u$ is a viscosity solution of (8.14) on $(\theta_1, \theta_2) \setminus \{0\}$. Let $z_0 \in (\theta_1, \theta_2) \setminus \{0\}$ be given and let $\psi$ be a $C^2$ function on the component of $(\theta_1, \theta_2) \setminus \{0\}$ containing $z_0$, such that $\psi(z_0) = u(z_0)$ and $\psi \geq u$. Because $u$ is a viscosity solution of (8.2), inequality (8.4) must hold. However, $u'(z_0) = \psi'(z_0)$, so from (8.15) we see that

$$(8.16) \qquad \begin{aligned} &\beta \psi(z_0) - d_1(z_0) p \psi(z_0) - d_2(z_0) \psi'(z_0) - d_3(z_0) \psi''(z_0) \\ &\quad - \tilde{U}_p( p \psi(z_0) - z_0 \psi'(z_0)) \leq 0. \end{aligned}$$

This shows that on $(\theta_1, \theta_2) \setminus \{0\}$, $u$ is a viscosity subsolution of (8.14). The proof that $u$ is a viscosity supersolution is easier and is left to the reader.

We define the continuous function $h: (\theta_1, \theta_2) \setminus \{0\} \to R$ by

$$h(z) = \beta u(z) - d_1(z) p u(z) - d_2(z) u'(z) - \tilde{U}_p( p u(z) - z u'(z))$$

and consider the ordinary differential equation

$$(8.17) \qquad - d_3(z) w_\varepsilon''(z) + h(z) = \varepsilon,$$

where $\varepsilon$ is a real number, not necessarily positive. Let $[a, b]$ be a nondegenerate interval contained in $(\theta_1, \theta_2) \setminus \{0, 1\}$. For $z \in [a, b]$, define

$$\theta_\varepsilon(z) = \int_a^z \int_a^\zeta \frac{h(\nu) - \varepsilon}{d_3(\nu)} \, d\nu \, d\zeta,$$

$$w_\varepsilon(z) = u(a) + \left[ \frac{u(b) - u(a) - \theta_\varepsilon(b)}{b - a} \right] (z - a) + \theta_\varepsilon(z),$$

so $w_\varepsilon$ is a $C^2$ solution to (8.17) on $(a, b)$ and $w_\varepsilon(a) = u(a), w_\varepsilon(b) = u(b)$.

We use the viscosity property of $u$ of compare $w_\varepsilon$ to $u$. Suppose $\varepsilon > 0$ and $u - w_\varepsilon$ has a local maximum at some point $z_0 \in (a, b)$. Then the viscosity subsolution property (8.16) would imply that

$$-d_3(z_0)w_\varepsilon''(z_0) + h(z_0) \leq 0,$$

a contradiction to (8.17). Thus $u - w_\varepsilon$ attains its maximum over $[a, b]$ at the endpoints, and we conclude $u \leq w_\varepsilon$ on $[a, b]$. Letting $\varepsilon \downarrow 0$, we conclude $u \leq w_0$. Letting $\varepsilon$ be negative and using the viscosity supersolution property, we obtain the reverse inequality $u \geq w_0$. Therefore, $u = w_0$, a $C^2$ function on $(a, b)$. □

COROLLARY 8.6. *In the set* $\mathrm{NT} \setminus \{(x, y); \ x = 0 \ or \ y = 0\}$, *$v$ is $C^2$ and satisfies the equation*

$$(\mathscr{L}v) - \tilde{U}_p(v_x) = 0$$

*in the classical sense.*

COROLLARY 8.7. $\mathrm{SS} \neq \varnothing$.

PROOF. The case $0 \leq p < 1$ has already been treated in Remark 6.10. Here we assume $p < 0$.

If SS were empty, then $\partial_2 \mathscr{S}$ would be part of the boundary of NT. In particular, the number $\theta_2$ in (8.13) would be $b = 1/\mu$. Let us consider the behavior of the solution $u$ to the equation (8.14) near $\theta_2$. The nonlinear function $\tilde{U}_p(\tilde{c}) = (1 - p)\tilde{c}^{p/(p-1)}/p$ is Lipschitz on any half-line of the form $[\gamma, \infty)$, where $\gamma > 0$. The argument $pu(z) - zu'(z) = v_x(1 - z, z)$ of this function in (8.14) converges to $\infty$ as $z \uparrow \theta_2$ because $v_x(1 - z, z) \geq pv(1 - z, z)(1 - z + z/(1 - \lambda))^{-1}$ (Corollary 3.7) and $\lim_{z \uparrow b} pv(1 - z, z) = \infty$ (Corollary 5.5). The rest of equation (8.14) is linear and $d_3(\theta_2) \neq 0$. Therefore, $\lim_{z \to \theta_2} u(z)$ exists and is finite, which is impossible, because $u(z) = v(1 - z, z)$. We conclude that $\mathrm{SS} \neq \varnothing$. □

COROLLARY 8.8. $\mathrm{SMM}$ *contains the cone* $G = \{(x, y) \in \mathscr{S}; \ y < 0\}$.

PROOF. Again we only need to consider the case $p < 0$. Just as in the proof of Corollary 8.7, we show that $\mathrm{SMM} \neq \varnothing$. We must rule out the possibility that NT and SMM meet in $G$.

From Theorem 6.9 we have $v(x, y) = (1/p)B^{p-1}(x + y/(1 - \lambda))^p$ for all $(x, y) \in \mathrm{SMM}$. From Proposition 3.4 we see that $B^{p-1}/p \geq C_*^{p-1}/p$, or equivalently $B \geq C_*$. In SMM, $v_x^{p/(p-1)} = Bpv$ and so when $y < 0$, (6.3) and (6.4) imply

$$\beta v(x, y) - \alpha y v_y(x, y) - r x v_x(x, y) - \tilde{U}_p(v_x(x, y))$$

$$(8.18) \quad \begin{aligned} &> \beta v(x, y) - r\big[y v_y(x, y) + x v_x(x, y)\big] - (1 - p)Bv(x, y) \\ &= \big[\beta - rp - (1 - p)B\big]v(x, y) \\ &= (1 - p)(C_* - B)v(x, y) \\ &\geq 0. \end{aligned}$$

If $(x, y) \in \partial \mathrm{NT} \cap \partial \mathrm{SMM} \cap G$, continuity of $v_x$ and $v_y$ at $(x, y)$ implies that (8.18) still holds. For such $(x, y)$, the smoothness and concavity of $v$ imply

$$\lim_{\substack{(\xi, \eta) \to (x, y) \\ (\xi, \eta) \in \mathrm{NT}}} (\mathcal{L}v)(\xi, \eta) - \tilde{U}_p(v_x(\xi, \eta)) > 0,$$

which contradicts Corollary 8.6. We conclude that $\partial \mathrm{NT} \cap \partial \mathrm{SMM} \cap G = \varnothing$. ☐

**9. Construction and verification of the optimal policy.** We saw in the last section [(8.11) and Corollary 8.8] that

$$\mathrm{NT} = \{(x, y) \in \mathcal{S}; y > 0, -(1 - \mu)v_x + v_y > 0, v_x - (1 - \lambda)v_y > 0\}$$

and that $v$ is $C^2$ on $\mathrm{NT} \setminus \{(x, y); x = 0, y > 0\}$ (Corollary 8.6). In this section we show that even if $\{(x, y); x = 0, y > 0\}$ is a subset of $\mathrm{NT}$, $v_{yy}$ is continuous on all of $\mathrm{NT}$. We subsequently construct and verify the optimal policy for the transaction cost problem.

THEOREM 9.1. *The second derivative $v_{yy}$ is defined and continuous on $\mathrm{NT}$ and $v$ is a solution of*

$$(9.1) \qquad\qquad \mathcal{L}v - \tilde{U}_p(v_x) = 0$$

*in the classical sense on $\mathrm{NT}$.*

PROOF. We consider only the case $p \neq 0$. Assume $\{(x, y); x = 0, y > 0\} \subset \mathrm{NT}$, for otherwise there is nothing to prove. Imitating the first part of the proof of Proposition 8.5, we can easily show that $v$ is a viscosity solution of (9.1) on $\mathrm{NT}$. To upgrade the regularity of $v$, we define the continuous function $h: \mathrm{NT} \to R$ by

$$h(x, y) \triangleq \beta v(x, y) - \alpha y v_y(x, y) - rx v_x(x, y) - \tilde{U}_p(v_x(x, y))$$

and consider the differential equation

$$-\tfrac{1}{2}\sigma^2 y^2 w_{yy}^{(\varepsilon)}(x, y) + h(x, y) = \varepsilon,$$

where $\varepsilon$ is a real number, not necessarily positive.

Choose $a > 0$ so that $(x, a|x|) \in \mathrm{NT}$ for all $x \in R$. For $x \in (-1/a, 1/a)$, $y \in (a|x|, 1/(a|x|))$ and $\varepsilon \in R$, define

$$\theta^{(\varepsilon)}(x, y) = \int_{a|x|}^{y} \int_{a|x|}^{\eta} \frac{h(x, \nu) - \varepsilon}{(1/2)\sigma^2 \nu^2} \, d\nu \, d\eta,$$

$$w^{(\varepsilon)}(x, y) = v(x, a|x|) + \left[ \frac{v(x, 1/(a|x|)) - v(x, a|x|) - \theta^{(\varepsilon)}(x, 1/(a|x|))}{1/(a|x|) - a|x|} \right]$$

$$\times (y - a|x|) + \theta^{(\varepsilon)}(x, y).$$

Just as in the proof of Proposition 8.5, we show that $w^{(0)}(x, y) = v(x, y)$, and because $w^{(0)}_{yy}$ is defined and continuous in a neighborhood of $\{(x, y);\ x = 0,\ y > 0\}$, so is $v_{yy}$. $\square$

In the case that $0 < \theta_1 < \theta_2 < 1$, the following theorem is proved by Davis and Norman [14].

THEOREM 9.2.  *Recall the numbers $\theta_1$, $\theta_2$ of (8.13). We have*

$$(9.2) \qquad\qquad 0 \le \theta_1 < \theta_2 < \frac{1}{\mu}.$$

*Furthermore, if $(x_0, y_0) \in \mathscr{S}$, then there is a triple $(C, L, M) \in \mathscr{A}(x_0, y_0)$ such that with $(X, Y)$ defined by (2.3)–(2.6), the following conditions hold almost surely:*

$$(9.3) \qquad\quad if\ (x_0, y_0) \notin \overline{\mathrm{NT}},\ then\ (X(0), Y(0)) \in \partial\mathrm{NT},$$

$$(9.4) \qquad\qquad (X(t), Y(t)) \in \overline{\mathrm{NT}} \qquad \forall t \ge 0,$$

$$(9.5) \qquad L(t) = \int_0^t 1_{\{Y(s)/(X(s)+Y(s)) = \theta_1\}}\, dL(s) \qquad \forall t \ge 0,$$

$$(9.6) \qquad M(t) = \int_0^t 1_{\{Y(s)/(X(s)+Y(s)) = \theta_2\}}\, dM(s) \qquad \forall t \ge 0,$$

$$(9.7) \qquad C(t) = \left[ v_x(X(t), Y(t)) \right]^{1/(p-1)} \qquad \forall t \ge 0.$$

*This triple is optimal, that is,*

$$(9.8) \qquad\qquad v(x_0, y_0) = E \int_0^\infty e^{-\beta t} U_p(C(t))\, dt.$$

The inequality $\theta_1 \ge 0$ follows from Corollary 8.8. The strict inequalities in (9.2) are restatements of Corollaries 8.4 and 8.7.

The proof of Theorem 9.2 proceeds through several lemmas. To begin this process, we recast the claim of existence of $(C, L, M)$ in a manner more consistent with the literature on reflected diffusions. We shall always assume $(x_0, y_0) \in \overline{\mathrm{NT}}$; if this is not the case, an initial jump can cause $(X(0), Y(0))$ to lie on $\partial\mathrm{NT}$, and we restart the construction with initial condition $(X(0), Y(0))$.

Partition the boundary of NT into

$$\partial_1\mathrm{NT} \triangleq \{(x, y);\ y \ge 0,\ -\theta_1 x - (\theta_1 - 1)y = 0\},$$

$$\partial_2\mathrm{NT} \triangleq \{(x, y);\ y > 0,\ \theta_2 x + (\theta_2 - 1)y = 0\}.$$

On $\partial\mathrm{NT}$ define the reflection direction

$$\gamma(x, y) \triangleq \begin{cases} (-1, 1 - \lambda), & \text{if } (x, y) \in \partial_1\mathrm{NT}, \\ (1 - \mu, -1), & \text{if } (x, y) \in \partial_2\mathrm{NT}, \end{cases}$$

and let $\gamma(x, y) \triangleq (\gamma_1(x, y), \gamma_2(x, y))$. The main assertion of Theorem 9.2 is that for $(x_0, y_0) \in \overline{\mathrm{NT}}$, there is solution to the following problem.

SKOROHOD PROBLEM.   Find continuous processes $X, Y, k$ such that $X(0) = x_0, Y(0) = y_0, k(0) = 0$, $k$ is nondecreasing and

$$(9.9) \qquad\qquad (X(t), Y(t)) \in \overline{\mathrm{NT}} \qquad \forall t \geq 0,$$

$$(9.10) \quad dX(t) = \left[ rX(t) - (v_x(X(t), Y(t)))^{1/(p-1)} \right] dt$$
$$+ \gamma_1(X(t), Y(t)) \, dk(t),$$

$$(9.11) \quad dY(t) = \alpha Y(t) \, dt + \sigma Y(t) \, dW(t) + \gamma_2(X(t), Y(t)) \, dk(t),$$

$$(9.12) \qquad k(t) = \int_0^t 1_{\{(X(t), Y(t)) \in \partial \mathrm{NT}\}} \, dk(t).$$

We map (9.9)–(9.12) into (9.4)–(9.7) with the identifications

$$(9.13) \qquad \begin{aligned} L(t) &= \int_0^t 1_{\{(X(t), Y(t)) \in \partial_1 \mathrm{NT}\}} \, dk(t), \\ M(t) &= \int_0^t 1_{\{(X(t), Y(t)) \in \partial_2 \mathrm{NT}\}} \, dk(t). \end{aligned}$$

LEMMA 9.3.   *If $0 < \theta_1 < \theta_2 < 1$, there is a solution to the Skorohod problem.*

PROOF.   When $0 < \theta_1 < \theta_2 < 1$, $\overline{\mathrm{NT}}$ is contained in the open first quadrant, so $v$ is $C^2$ and $v_x$ is locally Lipschitz at every point in $\overline{\mathrm{NT}}$ except the origin. This fact simplifies the proof.

This lemma does not involve the utility and value functions, except for the appearance of $p$ in (9.10). To aid in its proof, we introduce the value function $v_q$ corresponding to utility function $U_q$, where $q \in (0, \bar{p})$ and $\bar{p} \in (0, 1)$ is the solution to (5.6). Proposition 5.1 guarantees the existence of a constant $m_2$ such that

$$(9.14) \qquad v_q(x, y) \leq m_2(x + y)^q \qquad \forall (x, y) \in \bar{\mathscr{S}}.$$

Homotheticity guarantees that with $m_1 = \min_{\theta_1 \leq z \leq \theta_2} v_q(1 - z, z) > 0$, we also have

$$(9.15) \qquad m_1(x + y)^q \leq v_q(x, y) \qquad \forall (x, y) \in \overline{\mathrm{NT}}.$$

For $n \geq 1$, we truncate NT by defining

$$\mathrm{NT}_n = \{ (x, y) \in \mathrm{NT};\ n^{-2/q} < x + y < n^{2/q} \}.$$

The region $\mathrm{NT}_n$ has four corners, but we may modify $\mathrm{NT}_n$ slightly to obtain a larger, bounded, ice-cream-cone–shaped region $\mathscr{O}_n \subset \mathrm{NT}$ which has a smooth (at least $C^2$) boundary and whose closure excludes the origin. We also extend the definition of the reflection direction function $\gamma$ so that it is smooth (at least $C^2$) on $\partial \mathscr{O}_n$. Stroock and Varadhan [62], Tanaka [64], Lions and Sznitman [49] and Dupuis and Ishii [23] all establish existence and unique-

ness for the Skorohod problem on $\mathscr{O}_n$. Let $(X_n, Y_n, k_n)$ be this solution and define

$$\tau_n \triangleq \inf\{t \geq 0; X_n(t) + Y_n(t) = n^{-2/q}\}, \qquad \tau \triangleq \lim_{n \to \infty} \tau_n,$$

$$\rho_n \triangleq \inf\{t \geq 0; X_n(t) + Y_n(t) = n^{2/q}\}, \qquad \rho \triangleq \lim_{n \to \infty} \rho_n.$$

The uniqueness of $(X_n, Y_n, k_n)$ allows us to define $X(t) = X_n(t), Y(t) = Y_n(t), k(t) = k_n(t), 0 \leq t \leq \tau_n \wedge \rho_n, n \geq 1$, and so $(X, Y, k)$ is defined on $[0, \tau \wedge \rho)$.

We show that $\rho \geq \tau$ almost surely. Fix $t > 0$. Up to time $t \wedge \rho_n \wedge \tau_n$, we may make the identification (9.7) and (9.13) to obtain the initial part of a triple $(C, L, M) \in \mathscr{A}(x_0, y_0)$. According to the optimality equation (4.5) and (9.15)

$$v_q(x_0, y_0) \geq E\left[\int_0^{t \wedge \rho_n \wedge \tau_n} e^{-\beta s} U_q(C(s))\, ds\right.$$

$$(9.16) \qquad \left. + e^{-\beta(t \wedge \rho_n \wedge \tau_n)} v_q\big(X(t \wedge \rho_n \wedge \tau_n), Y(t \wedge \rho_n \wedge \tau_n)\big)\right]$$

$$\geq e^{-\beta t} E\left[1_{\{\rho_n < t \wedge \tau_n\}} v_q\big(X(\rho_n), Y(\rho_n)\big)\right]$$

$$\geq e^{-\beta t} m_1 n^2 P\{\rho_n < t \wedge \tau_n\}.$$

Thus $\sum_1^\infty P\{\rho_n < t \wedge \tau_n\} < \infty$, and the Borel–Cantelli lemma implies that almost surely, $\rho_n \geq t \wedge \tau_n$ for all sufficiently large $n$. Thus, $\rho \geq t \wedge \tau$ almost surely, and because $t$ is arbitrary, $\rho \geq \tau$.

The triple $(X, Y, k)$ is defined on $[0, \tau)$ and on the set $\{\tau < \infty\}$,

$$\lim_{n \to \infty} X(\tau_n) = \lim_{n \to \infty} Y(\tau_n) = 0.$$

Thus

$$(9.17) \qquad \liminf_{t \uparrow \tau} X(t) = \liminf_{t \uparrow \tau} Y(t) = 0 \quad \text{on } \{\tau < \infty\}.$$

We show that (9.17) implies that $(X, Y)$ is trapped at the origin, that is,

$$(9.18) \qquad \limsup_{t \uparrow \tau} X(t) = = \limsup_{t \uparrow \tau} Y(t) = 0 \quad \text{on } \{\tau < \infty\}.$$

If (9.18) fails, then for some $\varepsilon > 0$, we have

$$(9.19) \qquad P\left\{\tau < \infty, \limsup_{t \uparrow \tau}(X(t) + Y(t)) > \varepsilon\right\} > 0.$$

Let $\eta_n = \inf\{t \geq \tau_n; X(t) + Y(t) \geq \varepsilon\}$. Then $\eta_n < \tau$ on the set in (9.19). On the set $\{\tau < \infty\}$, an argument similar to (9.16) shows that for each $t > 0$,

$$v_q\big(X(\tau_n), Y(\tau_n)\big) \geq e^{-\beta t} E\left[1_{\{\tau < \infty,\, \eta_n \leq t\}} v_q\big(X(\eta_n), Y(\eta_n)\big)\right].$$

Using both (9.14) and (9.15), we conclude

$$\frac{m_2}{n^2} \geq e^{-\beta t} m_1 \varepsilon^q P\{\tau < \infty, \eta_n \leq t\}.$$

It follows from the Borel–Cantelli lemma that almost surely on the set $\{\tau < \infty\}$, we have $\eta_n \geq t$ for all large $n$. Because $t$ is arbitrary, this contradicts the condition $\eta_n < \tau$ for all $n > 0$, almost surely on $\{\tau < \infty\}$.

We have already defined $(X, Y, k)$ on $[0, \tau)$. On the set $\{\tau < \infty\}$, we define

$$X(t) = 0, \qquad Y(t) = 0, \qquad k(t) = k(\tau) \qquad \forall t \geq \tau.$$

We interpret $(v_x(0,0))^{1/(p-1)}$ appearing in (9.10) to be zero. This provides a complete solution to the Skorohod problem (9.9)–(9.12). □

REMARK 9.4. If $1 < \theta_1 < \theta_2 < 1/\mu$, the proof of Lemma 9.3 goes through without modification. Difficulties can occur, however, when either the $x$-axis or the $y$-axis is part of the boundary of NT or NT contains the positive $y$-axis. This is because we do not know that $v$ is $C^2$ on these axes, and so we do not know that $v_x$ in (9.10) is Lipschitz.

Thus, we must consider more carefully the cases $\theta_1 = 0$, $\theta_2 = 1$, $\theta_1 = 1$ and $\theta_1 < 1 < \theta_2$. Of these four, $\theta_1 = 0$ and $\theta_2 = 1$ are easy to resolve. In Theorem 11.6, we show that the case $\theta_1 = 0$ cannot occur, but because the proof of that theorem depends on the present argument, we cannot yet eliminate this case from consideration. If $\theta_1 = 0$, the solution to the Skorohod problem can be constructed much as in Lemma 9.3, until $(X, Y)$ arrives at the $x$-axis. Thereafter, $Y$ is held at zero, $L$ and $M$ are held constant and (9.10) can be solved because homotheticity implies that $v(x, 0)$ is a $C^2$ function of $x$. When $\theta_2 = 1$, again the solution to the Skorohod problem can be constructed much as in Lemma 9.3, until $(X, Y)$ arrives at the $y$-axis. Thereafter, $X$ is held at zero and $Y$ is given by

$$dY(t) = \alpha Y(t)\,dt + \sigma Y(t)\,dW(t)$$

$$(9.20) \qquad\qquad -\frac{1}{1-\mu}\left[v_x(0, Y(t))\right]^{1/(p-1)}\,dt.$$

This equation has a solution because homotheticity implies [see (6.5)] $v_x(0, y) = y^{p-1}v_x(0, 1)$, which is a locally Lipschitz function of $y$. Note that $dL = 0$ and

$$(9.21) \qquad dM(t) = \frac{1}{1-\mu}\left[v_x(0, Y(t))\right]^{1/(p-1)}\,dt = \frac{1}{1-\mu}C(t)\,dt.$$

When $\theta_1 < 1 < \theta_2$, we need to describe how $(X, Y)$ passes across the positive $y$-axis. Because no reflection occurs at the $y$-axis and $v_x$ is strictly positive there, (9.10) shows that $(X, Y)$ can approach the $y$-axis only from the right. The issue is to ascertain that after $(X, Y)$ reaches the $y$-axis, (9.10) and (9.11) with $dk = 0$ provide a mechanism for getting off (and moving left).

If $\theta_1 = 1$, the issue is the same. We want to show that if the initial condition is on the positive $y$-axis, then (9.10) and (9.11) with $dk = 0$ provide a mechanism for moving left into the interior of NT.

Let $(x_0, y_0)$ be given with $x_0 = 0$, $y_0 > 0$. For small $t > 0$, set $k(t) = 0$ and solve (9.11) subject to $Y(0) = y_0$. Substitute this $Y(t)$ into (9.10) and rewrite as

$$(9.22) \qquad \dot{\xi}(t) = f(t, \xi(t)), \qquad \xi(0) = 0,$$

where

$$\xi(t) = e^{-rt}X(t), \qquad f(t, \xi) = -e^{-rt}\left[ v_x(e^{rt}\xi, Y(t)) \right]^{1/(p-1)}.$$

We want (9.22) to have a solution for small $t \geq 0$. We know that $f$ is continuous and is nonincreasing in its second variable. The continuity of $f$ is enough to guarantee local existence of a solution (see, e.g., [36]). Uniqueness of this solution can be seen by observing that if $\xi_1$ and $\xi_2$ are two solutions, then monotonicity of $f(t, \cdot)$ implies

$$\frac{d}{dt}\left[ \xi_1(t) - \xi_2(t) \right]^2 = 2\left[ \xi_1(t) - \xi_2(t) \right]\left[ f(t, \xi_1(t)) - f(t, \xi_2(t)) \right] \leq 0,$$

so $\xi_1 \equiv \xi_2$.

We have completed the proof of the Theorem 9.2 except for the assertion of optimality (9.8). We dispatch that with the following lemma.

LEMMA 9.5. *Let $(x_0, y_0) \in \mathcal{S}$ and $(C, L, M) \in \mathcal{A}(x_0, y_0)$ be given such that $(X, Y)$ defined by (2.3)–(2.6) satisfies (9.3)–(9.7). Then $(C, L, M)$ is optimal.*

PROOF. We apply Itô's formula to $e^{-\beta t}v(X(t), Y(t))$ [see (4.7)], but note that $-(1 - \mu)v_x(x, y) + v_y(x, y) = 0$ when $y/(x + y) = \theta_2$, $v_x(x, y) - (1 - \lambda)v_y(x,y) = 0$ when $y/(x + y) = \theta_1$ and $v(X(0), Y(0)) = v(x_0, y_0)$. Finally, Theorem 9.1 implies

$$(\mathcal{L}v)(X(s), Y(s)) + C(s)v_x(X(s), Y(s))$$
$$= \tilde{U}_p(v_x(X(s), Y(s))) + \left[ v_x(X(s), Y(s)) \right]^{p/(p-1)}$$
$$= U_p(C(s)).$$

Therefore, for any almost surely finite stopping time $\tau$,

$$(9.23) \qquad \begin{aligned} v(x_0, y_0) &= e^{-\beta\tau}v(X(\tau), Y(\tau)) + \int_0^\tau e^{-\beta s}U_p(C(s))\,ds \\ &\quad - \sigma\int_0^\tau e^{-\beta s}Y(s)v_y(X(s), Y(s))\,dW(s), \end{aligned}$$

provided that

$$(9.24) \qquad \int_0^\tau \left[ e^{-\beta s}Y(s)v_y(X(s), Y(s)) \right]^2 ds < \infty.$$

We examine the integrand in (9.24). Homotheticity implies [see (6.5)] that

$$v_y(x, y) = (x + y)^{p-1} v_y\left(\frac{x}{x+y}, \frac{y}{x+y}\right),$$

and since $v_y(1 \dashv z, z)$ is bounded from above and away from zero for $\theta_1 \leq z \leq \theta_2$ [(6.3 and Corollary 8.3)], we have the existence of constants $m_3 > 0, m_4 > 0$ such that

$$(9.25) \quad m_3(x + y)^{p-1} \leq v_y(x, y) \leq m_4(x + y)^{p-1} \qquad \forall (x, y) \in \overline{NT}.$$

For later use, we observe that the same augument applies to $v_x$, so we may choose $m_3$ and $m_4$ to also satisfy

$$(9.26) \quad m_3(x + y)^{p-1} \leq v_x(x, y) \leq m_4(x + y)^{p-1} \qquad \forall (x, y) \in \overline{NT}.$$

Finally,

$$(9.27) \qquad\qquad |y| \leq m_5(x + y) \qquad \forall (x, y) \in \overline{NT},$$

where $m_5 = \max\{|\theta_1|, |\theta_2|\}$. Putting this into (9.25), we see there is a constant $m_6$ satisfying

$$(9.28) \qquad\qquad |y v_y(x, y)| \leq m_6(x + y)^p \qquad \forall (x, y) \in \overline{NT}.$$

We now consider various possibilities for $p$. Assume first that $p < 0$ and define for $n \geq 1$,

$$(9.29) \qquad\qquad \tau_n \triangleq \inf\left\{t \geq 0; X(t) + Y(t) \leq \frac{1}{n}\right\}.$$

According to (9.28), the integrand in (9.24) is bounded for $0 \leq s \leq \tau_n$, so replacing $\tau$ by $t \wedge \tau_n$ in (9.23) and taking expectation, we obtain

$$
\begin{aligned}
v(x_0, y_0) &= E e^{-\beta(t \wedge \tau_n)} v\big(X(t \wedge \tau_n), Y(t \wedge \tau_n)\big) \\
(9.30) &\qquad + E \int_0^{t \wedge \tau_n} e^{-\beta s} U_p(C(s))\, ds \\
&\leq E \int_0^{t \wedge \tau_n} e^{-\beta s} U_p(C(s))\, ds.
\end{aligned}
$$

As $n \to \infty$, $\tau_n \uparrow \tau_0$ defined by $\tau_0 \triangleq \inf\{t \geq 0; X(t) = Y(t) = 0\}$. We show that $\tau_0 = \infty$ almost surely. On $\{\tau_0 < \infty\}$, we have

$$\lim_{n \to \infty} \lim_{t \uparrow \tau_n} e^{-\beta(t \wedge \tau_n)} v\big(X(t \wedge \tau_n), Y(t \wedge \tau_n)\big) = e^{-\beta \tau_0} v(0, 0) = -\infty,$$

$$\lim_{n \to \infty} \lim_{t \uparrow \tau_n} \int_0^{t \wedge \tau_n} e^{-\beta s} U_p(C(s))\, ds \leq 0.$$

Replacing $\tau$ by $t \wedge \tau_n$ in (9.23), we conclude from these observations that

$$(9.31) \quad \lim_{n \to \infty} \lim_{t \uparrow \tau_n} \int_0^{t \wedge \tau_n} e^{-\beta s} Y(s) v_y(X(s), Y(s))\, dW(s) = -\infty \quad \text{on } \{\tau_0 < \infty\}.$$

This can only happen if (9.24) is violated when $\tau = \tau_0$, but the violation of (9.24) would imply that the limit in (9.31) did not exist, even in the extended real numbers ([43], Problem 3.4.11 and its solution on page 232). Therefore, $P\{\tau_0 < \infty\} = 0$.

We now let $n \to \infty$ and $t \to \infty$ in (9.30) and use the monotone convergence theorem to conclude

$$v(x_0, y_0) \le E \int_0^\infty e^{-\beta s} U_p(C(s)) \, ds.$$

We have proved the optimality of $(C, L, M)$ when $p < 0$.

We next assume $0 < p < 1$, and begin by showing that

$$(9.32) \qquad E \int_0^t \left[ e^{-\beta s} Y(s) v_y(X(s), Y(s)) \right]^2 ds < \infty \qquad \forall t \ge 0.$$

We note from (9.28) that the integrand in (9.32) is bounded above by $m_6^2[1 + (X(s) + Y(s))^2]$, and so it suffices to prove

$$(9.33) \qquad E \int_0^t [X(s) + Y(s)]^2 \, ds < \infty \qquad \forall t \ge 0.$$

Define $Z(t) = X(t) + Y(t)$. From (2.4), (2.5) and (9.27), we have

$$Z(t) = Z(0) + \int_0^t [(\alpha - r) Y(s) + rZ(s) - C(s)] \, ds$$

$$(9.34) \qquad\qquad + \sigma \int_0^t Y(s) \, dW(s) - \lambda L(t) - \mu M(t)$$

$$\le Z(0) + m_7 \int_0^t Z(s) \, ds + \sigma N(t),$$

where $m_7 = (\alpha - r)m_5 + r$ and $N(t) = \int_0^t Y(s) \, dW(s)$. For each $n \ge 1$, let $\rho_n = \inf\{t \ge 0; \ Z(t) \ge n\}$ and define $Z^*(t) = \max_{0 \le s \le t} Z(s)$, $N^*(t) = \max_{0 \le s \le t} |N(s)|$. From Doob's maximal martingale inequality applied to $|N|$ ([43], Theorem 1.3.8), we have

$$E(N^*(t \wedge \rho_n))^2 \le 4EN^2(t \wedge \rho_n)$$

$$(9.35) \qquad\qquad = 4E \int_0^{t \wedge \rho_n} Y^2(s) \, ds$$

$$\le 4m_5^2 E \int_0^t Z^2(s \wedge \rho_n) \, ds.$$

Inequalities (9.34) and (9.35) and Hölder's inequality yield for some $m_8 > 0$ and every $T > 0$,

$$E(Z^*(t \wedge \rho_n))^2 \le m_8 \left[ Z^2(0) + E \left( \int_0^t Z^*(s \wedge \rho_n) \, ds \right)^2 + E(N^*(t \wedge \rho_n))^2 \right]$$

$$\le m_8 \left[ Z^2(0) + (T + 4m_5^2) \int_0^t E(Z^*(s \wedge \rho_n))^2 \, ds \right]$$

$$\forall t \in [0, T].$$

According to Gronwall's inequality,

$$E\big(Z^*(t \wedge \rho_n)\big)^2 \le m_8 Z^2(0)\exp\big[tm_8(T + 4m_5^2)\big] \qquad \forall t \in [0,T].$$

Let $n \to \infty$ and take $t = T$ to obtain

$$E\big(X(T) + Y(T)\big)^2 \le E\big(Z^*(T)\big)^2 \le m_8 Z^2(0)\exp\big[Tm_8(T+4m_5^2)\big] \qquad \forall T \ge 0.$$

This proves (9.33), and hence (9.32). Because of (9.32), we can replace the stopping time $\tau$ in (9.23) by an arbitrary constant $t > 0$ and then take expectations to obtain

$$(9.36) \qquad v(x_0, y_0) = Ee^{-\beta t}v(X(t), Y(t)) + \int_0^t e^{-\beta s} U_p(C(s))\, ds.$$

Because of (9.26), the consumption process $C$ given by (9.7) satisfies

$$(9.37) \qquad m_4^{1/(p-1)}(X(t) + Y(t)) \le C(t), \qquad t \ge 0.$$

Standing Assumption 2.3 asserts that

$$\int_0^\infty Ee^{-\beta t}(X(t) + Y(t))^p\, dt \le pm_4^{p/(1-p)}E\int_0^\infty e^{-\beta s}U_p(C(t))\, dt < \infty.$$

Therefore, we can find a sequence $t_n \uparrow \infty$ such that

$$(9.38) \qquad \lim_{n \to \infty} Ee^{-\beta t_n}\big(X(t_n) + Y(t_n)\big)^p = 0.$$

However, homotheticity implies that for $(x, y) \in \overline{\mathrm{NT}}$,

$$v(x, y) = (x + y)^p v\left(\frac{x}{x + y}, \frac{y}{x + y}\right) \le (x + y)^p \max_{\theta_1 \le z \le \theta_2} v(1 - z, z),$$

and so (9.38) shows that

$$\lim_{n \to \infty} Ee^{-\beta t_n}v\big(X(t_n), Y(t_n)\big) = 0.$$

Replace $t$ by $t_n$ in (9.36) and let $n \to \infty$ to obtain the optimality condition (9.8).

Finally, we consider the case $p = 0$. Inequality (9.37) holds and Standing Assumption 2.3 asserts

$$\int_0^\infty Ee^{-\beta t} \max\{\log(X(t) + Y(t)), 0\}\, dt$$

$$\le \frac{|\log m_4|}{\beta} + E\int_0^\infty e^{-\beta t} \max\{\log C(t), 0\}\, dt < \infty.$$

Homotheticity (3.3) implies that for $(xy) \in \overline{\mathrm{NT}}$,

$$v(x, y) \le \frac{1}{\beta} \log(x + y) + \max_{\theta_1 \le z \le \theta_2} v(1 - z, z),$$

and so we can find a sequence $t_n \uparrow \infty$ such that

$$(9.39) \qquad \lim_{n \to \infty} Ee^{-\beta t_n} \max\{v(X(t_n), Y(t_n)), 0\} = 0.$$

The monotone convergence theorem implies

(9.40)
$$\lim_{n \to \infty} E \int_0^{t_n} e^{-\beta t} \max\{\log C(t), 0\} \, dt$$
$$= E \int_0^{\infty} e^{-\beta t} \max\{\log C(t), 0\} \, dt < \infty,$$

(9.41)
$$\lim_{n \to \infty} E \int_0^{t_n} e^{-\beta t} \max\{-\log C(t), 0\} \, dt$$
$$= \int_0^{\infty} e^{-\beta t} \max\{-\log C(t), 0\} \, dt.$$

According to (9.28), condition (9.32) holds, from which we obtain (9.36). In particular,

$$v(x_0, y_0) \le E e^{-\beta t_n} \max\{v(X(t_n), Y(t_n)), 0\} + E \int_0^{t_n} e^{-\beta s} U_p(C(s)) \, ds.$$

Letting $n \to \infty$ and using (9.39)–(9.41), we prove the optimality of $(C, L, M)$.
$\square$

REMARK 9.6. In the proof of Lemma 9.5, in each case considered, we constructed a nondecreasing sequence of almost surely finite stopping times $\{\rho_n\}_{n=1}^{\infty}$ satisfying the equations

(9.42) $\quad v(x_0, y_0) = E e^{-\beta \rho_n} v(X(\rho_n), Y(\rho_n)) + E \int_0^{\rho_n} e^{-\beta s} U_p(C(s)) \, ds,$

(9.43) $$\lim_{n \to \infty} \rho_n = \infty \quad \text{a.s.}$$

In the case $p < 0$, take $\rho_n = n \wedge \tau_n$ with $\tau_n$ defined by (9.29), and then (9.42) follows from (9.30). In the case $0 \le p < 1$, take $\rho_n = n$ and appeal to (9.36). For $p < 0$ and $0 < p < 1$, the monotone convergence theorem and the optimality of $(C, L, M)$ imply

(9.44) $\quad v(x_0, y_0) = E \int_0^{\infty} e^{-\beta s} U_p(C(s)) \, ds = \lim_{n \to \infty} E \int_0^{\rho_n} e^{-\beta s} U_p(C(s)) \, ds,$

and so

(9.45) $$\lim_{n \to \infty} E e^{-\beta \rho_n} v(X(\rho_n), Y(\rho_n)) = 0.$$

If $p = 0$, we may apply the monotone convergence theorem to

$$\int_0^{\rho_n} e^{-\beta s} \max\{\log C(s), 0\} \, ds \quad \text{and} \quad \int_0^{\rho_n} e^{-\beta s} \max\{-\log C(s), 0\} \, ds$$

separately and use Standing Assumption 2.3 to conclude that (9.44) and (9.45) hold.

REMARK 9.7. For $(x_0, y_0) \in \mathscr{S}$, there is a positive probability that the optimal policy $(C, L, M)$ causes reflection at the upper boundary of NT [i.e.,

$P(M(\infty) > 0) > 0]$, and if $\theta_1 > 0$, there is also a positive probability of reflection at the lower boundary of NT [i.e., $P(L(\infty) > 0) > 0]$. Therefore, $v(x_0, y_0)$ is a strictly decreasing function of $\mu$, and if the choice of parameters is such that $\theta_1 > 0$, $v(x_0, y_0)$ is strictly decreasing in $\lambda$ at this set of parameters. We use this observation in the next section to show that $v$ is $C^2$ across any boundary of NT which does not coincide with the positive $y$-axis or the positive $x$-axis.

Reviewing the properties of the value function $v$ which were used to establish Theorem 9.2, we discover the following result.

COROLLARY 9.8. *Suppose $\tilde{v}: \overline{\mathscr{S}} \to R$ is a continuous function possessing the homotheticity property established for $v$ in Proposition 3.3. Suppose that on $\mathscr{S} \setminus \{(x, 0); x > 0\}$, $\tilde{v}$ is $C^1$ and $v_x$ and $v_y$ are locally bounded above and away from zero, even in neighborhoods of points on the positive $x$-axis. Suppose there are numbers $0 \leq \tilde{\theta}_1 < \tilde{\theta}_2 < 1/\mu$ so that, with open convex cones defined by*

$$\widetilde{SS} \triangleq \left\{ (x, y) \in \mathscr{S}; \frac{y}{x + y} > \tilde{\theta}_2 \right\},$$

$$\widetilde{NT} \triangleq \left\{ (x, y) \in \mathscr{S}; \tilde{\theta}_1 < \frac{y}{x + y} < \tilde{\theta}_2 \right\},$$

$$\widetilde{SMM} \triangleq \left\{ (x, y) \in \mathscr{S}; \frac{y}{x + y} < \tilde{\theta}_1 \right\},$$

*we have that $\tilde{v}$ is $C^2$ on $\widetilde{NT}$, except possibly on the $y$-axis, $\tilde{v}_{yy}$ is continuous on all of $\widetilde{NT}$ and*

$$-(1 - \mu)\tilde{v}_x + \tilde{v}_y = 0 \quad on \ \widetilde{SS},$$

$$\tilde{v}_x - (1 - \lambda)\tilde{v}_y = 0 \quad on \ \widetilde{SMM},$$

$$\mathscr{L}\tilde{v} - \tilde{U}_p(\tilde{v}_x) = 0 \quad on \ \widetilde{NT}.$$

*Then, for every $(x_0, y_0) \in \overline{\mathscr{S}}$, there is a triple $(\tilde{C}, \tilde{L}, \tilde{M}) \in \mathscr{A}(x_0, y_0)$ satisfying conditions analogous to (9.3)–(9.7) and*

$$\tilde{v}(x_0, y_0) \leq E \int_0^\infty e^{-\beta t} U_p(\tilde{C}(t)) \, dt.$$

Propositions 5.1 and 5.4 provide upper bounds on $v$ when $0 \leq p < \bar{p}$. We extend these results to the case $p < 0$, and show that the upper bounds are strict. This extension uses the existence of an optimal policy.

PROPOSITION 9.9. *If $p < 1$, $p \neq 0$, $A(p)$ defined by (5.7) is positive and $\gamma$ satisfies (5.2), then*

$$(9.46) \qquad v(x, y) \leq \frac{1}{p} A^{p-1}(p)(x + \gamma y)^p \qquad \forall (x, y) \in \mathscr{S}.$$

*If $p = 0$ and $\gamma$ satisfies (5.2), then*

$$(9.47) \quad v(x, y) \le \frac{1}{\beta} \log(x + \gamma y) + \frac{1}{\beta} \log \beta + \frac{r - \beta}{\beta^2} + \frac{(\alpha - r)^2}{2\beta^2\sigma^2}$$

$$\forall(x, y) \in \mathscr{S}.$$

*Moreover, the preceding inequalities are strict if*

$$(9.48) \quad \sigma^2(1 - p) \ne \alpha - r.$$

PROOF. Let $(x_0, y_0) \in \mathscr{S}$ and $(C, L, M) \in \mathscr{A}(x_0, y_0)$ be given such that $(X, Y)$ defined by (2.3)–(2.6) satisfies (9.3)–(9.7). According to Lemma 9.5, $(C, L, M)$ is optimal. Define $\varphi(x, y)$ to be the right-hand side of (9.46) or (9.47) according to whether $p \ne 0$ or $p = 0$. Then $\varphi$ satisfies (5.3), (5.4), (5.8) and (5.9). With regard to (5.8), we have for $(x, y) \in \mathscr{S}$,

$$(9.49) \quad \begin{aligned} (\mathscr{L}\varphi) - \tilde{U}_p(\varphi_x(x, y)) &= 0 \\ \Leftrightarrow \quad [\sigma^2(1 - p) - (\alpha - r)]\gamma y - (\alpha - r)x &= 0. \end{aligned}$$

Applying Itô's formula to $e^{-\beta t}\varphi(X(t), Y(t))$ [cf. (4.7)], we have for any almost surely finite stopping time $\tau$,

$$(9.50) \quad \begin{aligned} \varphi(x_0, y_0) &\ge e^{-\beta\tau}\varphi(X(\tau), Y(\tau)) + \int_0^\tau e^{-\beta s}\left[\mathscr{L}\varphi - \tilde{U}_p(\varphi_x)\right] ds \\ &\quad + \int_0^\tau e^{-\beta s}U_p(C(s)) \, ds - \sigma\int_0^\tau e^{-\beta s}Y(s)\varphi_y \, dW(s). \end{aligned}$$

For $p \ne 0$, we have

$$\varphi(x, y) = (x + y)^p \varphi\left(\frac{x}{x + y}, \frac{y}{x + y}\right)$$

and thus

$$(x + y)^p \min_{\theta_1 \le z \le \theta_2} \varphi(1 - z, z) \le \varphi(x, y) \le (x + y)^p \max_{\theta_1 \le z \le \theta_2} \varphi(1 - z, z)$$

$$\forall(x, y) \in \overline{\mathrm{NT}}.$$

The homotheticity established in (3.1) implies that $v$ satisfies similar inequalities, and so there are positive constants $n_1$ and $n_2$ such that, if $p \ne 0$,

$$(9.51) \quad n_1|\varphi(x, y)| \le |v(x, y)| \le n_2|\varphi(x, y)| \quad \forall(x, y) \in \overline{\mathrm{NT}}.$$

If $p = 0$, the definition of $\varphi$ and the homotheticity of $v$ imply that for some positive constant $n_3$,

$$(9.52) \quad |\varphi(x, y) - v(x, y)| \le n_3 \quad \forall(x, y) \in \overline{\mathrm{NT}} \setminus \{(0, 0)\}.$$

For all $p < 1$, a similar argument shows that there are positive constants $n_4$ and $n_5$ such that

$$n_4|\varphi_y(x, y)| \le |v_y(x, y)| \le n_5|\varphi_y(x, y)| \quad \forall(x, y) \in \overline{\mathrm{NT}}.$$

Replace $\tau$ in (9.50) by $\rho_n$ from Remark 9.6. It was shown in the proof of Lemma 9.5 that

$$\int_0^{\rho_n} \left[ e^{-\beta s} Y(s) v_y(X(s), Y(s)) \right]^2 ds < \infty$$

and hence this equality holds when $v_y$ is replaced by $\varphi_y$. Taking expectations in (9.50), we obtain

$$\varphi(x_0, y_0) \geq E e^{-\beta \rho_n} \varphi(X(\rho_n^-), Y(\rho_n)) + E \int_0^{\rho_n} e^{-\beta s} \left[ \mathscr{L}\varphi - \tilde{U}_p(\varphi_x) \right] ds$$

$$+ E \int_0^{\rho_n} e^{-\beta s} U_p(C(s)) \, ds.$$

Letting $n \to \infty$ and using (9.43)–(9.45) and either (9.51) or (9.52), we see that

$$\varphi(x_0, y_0) \geq E \int_0^\infty e^{-\beta s} \left[ \mathscr{L}\varphi - \tilde{U}_p(\varphi_x) \right] ds + v(x_0, y_0),$$

and because of (5.8), we have (9.46).

It remains to show that when (9.48) holds,

$$(9.53) \quad E \int_0^\infty e^{-\beta s} \left[ \mathscr{L}\varphi(X(s), Y(s)) - \tilde{U}_p(\varphi_x(X(s), Y(s))) \right] ds > 0.$$

If (9.53) fails, then (9.49) implies that, almost surely,

$$(9.54) \qquad \left[ \sigma^2(1-p) - (\alpha - r) \right] \gamma Y(s) - (\alpha - r) X(s) = 0,$$

$$\text{Lebesgue a.e. } s \geq 0.$$

However, $X(\cdot), Y(\cdot)$ are right-continuous, so (9.54) would in fact hold for every $x \geq 0$. This implies that the process on the left-hand side of (9.54) has zero quadratic variation, which, in light of (9.48), (2.4) and (2.5), can be the case only if $Y(s) = 0$ for all $s \geq 0$. This implies, in turn, that $X(s) = 0$ for all $s \geq 0$. Such a result is inconsistent with the behavior of the optimal policy for initial condition in $\mathscr{S}$, and we conclude that (9.53) holds. □

COROLLARY 9.10. *If $p < 1$, $p \neq 0$, $A(p)$ defined by (5.7) is positive and*

$$(9.55) \qquad\qquad \sigma^2(1-p) = \alpha - r,$$

*then*

$$(9.56) \qquad v(x, y) = \frac{1}{p} A^{p-1}(p)(x + (1-\mu)y)^p$$

$$\forall (x, y) \in \mathscr{S} \text{ with } x \leq 0.$$

*If $p = 0$ and (9.55) holds, then*

$$(9.57) \quad v(x, y) = \frac{1}{\beta} \log(x + (1-\mu)y) + \frac{1}{\beta} \log \beta + \frac{r - \beta}{\beta^2} + \frac{(\alpha - r)}{2\beta^2 \sigma^2}$$

$$\forall (x, y) \in \mathscr{S} \text{ with } x \leq 0.$$

PROOF. We consider only the case $p \neq 0$; the case $p = 0$ is similar. With $x_0 = 0, y_0 > 0$, we construct the triple $(C, L, M) \in \mathscr{A}(x_0, y_0)$ suggested by (9.20) and (9.21). In particular, we set $L \equiv 0, C(t) = A(p)(1 - \mu)Y(t)$ and $dM(t) = (1/(1 - \mu))C(t)\, dt$, so $X(t) \equiv 0, Y(t) = y_0 \exp[(\alpha - A(p) - \sigma^2/2)t + \sigma W(t)], t \geq 0$. Using (9.55) and the identity $-\beta + p(\alpha - A(p) - \sigma^2/2) = -A(p)$, we compute that

$$v(0, y_0) \geq E \int_0^\infty e^{-\beta t} U_p(C(t))\, dt = \frac{1}{p} A^{p-1}(p)(1 - \mu)^p y_0^p.$$

The reverse inequality follows from Proposition 9.9. We have shown that

$$v(0, y) = \frac{1}{p} A^{p-1}(p)(1 - \mu)^p y^p \qquad \forall y > 0.$$

Now suppose $(x_0, y_0) \in \mathscr{S}$ and $x_0 \leq 0$. We can move from $(x_0, y_0)$ to $(0, x_0/(1 - \mu) + y_0)$ by a transaction, so

$$v(x_0, y_0) \geq v\left(0, \frac{x_0}{(1 - \mu)} + y_0\right) = \frac{1}{p} A^{p-1}(p)(x_0 + (1 - \mu)y_0)^p,$$

and again the reverse inequality follows from Proposition 9.9. $\square$

**10. Further regularity of the value function.** Using convex analysis, we showed that the value function $v$ is $C^\infty$ in SS and SMM (Theorem 6.9). Using the theory of viscosity solutions, we subsequently showed that $v$ is $C^2$ in NT $\setminus \{(x, y); x = 0 \text{ or } y = 0\}$ (Corollary 8.6). In this section, we use a control theory argument to see that $v$ must also be $C^2$ across each boundary

$$\partial_1 \text{NT} = \left\{(x, y) \in \mathscr{S}; \frac{y}{x + y} = \theta_1\right\},$$

$$\partial_2 \text{NT} = \left\{(x, y) \in \mathscr{S}; \frac{y}{x + y} = \theta_2\right\},$$

provided the boundary in question does not coincide with the positive $x$-axis or the positive $y$-axis.

THEOREM 10.1. *The partial derivative $v_{yy}$ is continuous across $\partial_2 \text{NT}$, and if $\theta_2 \neq 1$, then $v$ is $C^2$ across $\partial_2 \text{NT}$. If $\theta_1 \neq 0$, then $v_{yy}$ is continuous across $\partial_1 \text{NT}$, and if $\theta_1 \neq 0$ and $\theta_1 \neq 1$, then $v$ is $C^2$ across $\partial_1 \text{NT}$.*

PROOF. Using homotheticity, we again reduce the matter to a function of one variable, but this time we set

$$w(x) = v(x, 1) \qquad \forall x \geq -(1 - \mu),$$

so that for all $(x, y) \in \bar{\mathscr{S}}$ with $y > 0$,

$$v(x, y) = \begin{cases} y^p w\left(\dfrac{x}{y}\right), & \text{if } p < 1, p \neq 0, \\[2ex] \dfrac{1}{\beta} \log y + w\left(\dfrac{x}{y}\right), & \text{if } p = 0. \end{cases}$$

Corollary 8.3 implies that $w$ is $C^1$. In terms of $w$, the HJB equation (4.6) becomes

(10.1)
$$\min\left\{ \beta w(x) + \left( \frac{1}{2}\sigma^2(1-p) - \alpha \right) pw(x) \right.$$
$$+ (\alpha - r - \sigma^2(1-p))w'(x)$$
$$- \frac{1}{2}\sigma^2 x^2 w''(x) - \tilde{U}_p(w'(x)),$$
$$-(x + 1 - \mu)w'(x) + pw(x),$$
$$\left. \left( x + \frac{1}{1-\lambda} \right)w'(x) - pw(x) \right\} = 0, \quad \text{if } p < 1, \, p \neq 0,$$

or

$$\min\left\{ \beta w(x) + \frac{1}{\beta}\left( \frac{1}{2}\sigma^2 - \alpha \right) + (\alpha - r - \sigma^2)w'(x) \right.$$
$$- \frac{1}{2}\sigma^2 x^2 w''(x) - \tilde{U}_0(w'(x)),$$
$$-(x + 1 - \mu)w'(x) + \frac{1}{\beta},$$
$$\left. \left( x + \frac{1}{1-\lambda} \right)w'(x) - \frac{1}{\beta} \right\} = 0, \quad \text{if } p = 0.$$

Considering first the case of regularity of $v$ across $\partial_2 \mathrm{NT}$, we define $x_2 \triangleq (1 - \theta_2)/\theta_2$ and undertake to show that $w''$ is continuous at $x_2$.

If $p \neq 0$, then we know from Theorem 6.9 that for some $A > 0$,

$$w(x) = \frac{1}{p}A^{p-1}(x + 1 - \mu)^p \qquad \forall x \in (-(1-\mu), x_2],$$

and, in particular, $w''(x_2 -) \triangleq \lim_{x \uparrow x_2} w''(x)$ is defined. From (10.1) we see also that

$$\beta w(x_2) + \left( \tfrac{1}{2}\sigma^2(1-p) - \alpha \right) pw(x_2)$$
$$+ (\alpha - r - \sigma^2(1-p))w'(x_2) - \tilde{U}_p(w'(x_2))$$
$$\geq \tfrac{1}{2}\sigma^2 x_2^2 w''(x_2 -).$$

For $x \in (x_2, (1 - \theta_1)/\theta_1)$, we have

$$\beta w(x) + \left( \tfrac{1}{2}\sigma^2(1-p) - \alpha \right) pw(x)$$
$$+ (\alpha - r - \sigma^2(1-p))w'(x) - \tilde{U}_p(w'(x))$$
$$= \tfrac{1}{2}\sigma^2 x^2 w''(x),$$

and letting $x \downarrow x_2$, we see that $\lim_{x \downarrow x_2} x^2 w''(x)$ exists and

$$(10.2) \qquad x_2^2 w''(x_2 -) \leq \lim_{x \downarrow x_2} x^2 w''(x).$$

There are two cases to consider. Assume first that $\theta_2 = 1$, so $x_2 = 0$. We must prove continuity of $v_{yy}(x, y)$ at $x = 0$, $y > 0$, and it suffices to prove continuity of $v_{yy}(x, 1)$ at $x = 0$. However,

$$v_{yy}(x, 1) = -(1 - p)pw(x) + 2(1 - p)xw'(x) + x^2 w''(x).$$

From (10.2) with $x_2 = 0$, we have $\lim_{x \downarrow 0} x^2 w''(x) \geq 0$, and the reverse inequality follows from the concavity of $w$, inherited from the concavity of $v$. Thus, $\lim_{x \downarrow 0} x^2 w''(x) = 0$. Because $w''(0 -)$ exists, we also have $\lim_{x \uparrow 0} x^2 w''(x) = 0$, and continuity of $v_{yy}(x, 1)$ at $x = 0$ is proved.

Now assume $\theta_2 \neq 1$, so $x_2 \neq 0$. We must prove that $v$ is $C^2$ at $x = x_2$, $y > 0$, and it suffices to prove continuity of $w''$ at $x_2$. From (10.2) we have

$$(10.3) \qquad w''(x_2 -) \leq w''(x_2 +).$$

Let us assume $w''(x_2 -) < w''(x_2 +)$ and work toward a contradiction. Under this assumption, let $\varepsilon_0 > 0$ be such that

$$[x_2, x_2 + \varepsilon_0] \subset [x_2, (1 - \theta_1)/\theta_1) \setminus \{0\}.$$

For $\varepsilon \in [0, \varepsilon_0]$, define $\mu_\varepsilon$ and $A_\varepsilon$ by

$$(10.4) \qquad (x_2 + \varepsilon + 1 - \mu_\varepsilon) = \frac{pw(x_2 + \varepsilon)}{w'(x_2 + \varepsilon)},$$

$$(10.5) \qquad A_\varepsilon^{p-1} = \frac{pw(x_2 + \varepsilon)}{(x_2 + \varepsilon + 1 - \mu_\varepsilon)^p}.$$

The mappings $\varepsilon \mapsto \mu_\varepsilon$ and $\varepsilon \mapsto A_\varepsilon$ are $C^1$, and $\mu_0 = \mu$. Differentiation of (10.4) yields

$$1 - \left.\frac{\partial \mu}{\partial \varepsilon}\right|_{\varepsilon=0} = p - \frac{pw(x_2)w''(x_2 +)}{\left[w'(x_2)\right]^2}$$

$$< p - \frac{pw(x_2)w''(x_2 -)}{\left[w'(x_2)\right]^2}$$

$$= 1.$$

Therefore, $\partial \mu / \partial \varepsilon|_{\varepsilon=0} > 0$, so for sufficiently small positive $\tilde{\varepsilon}$, we have $\mu_{\tilde{\varepsilon}} > \mu$.

If $p = 0$, then Theorem 6.9 gives

$$w(x) = \frac{1}{\beta} \log(x + 1 - \mu) + A \qquad \forall x \in (-(1 - \mu), x_2]$$

for some $A \in R$. We proceed as before, except that in place of (10.4) and (10.5), we now define $\mu_\varepsilon$ and $A_\varepsilon$ by

$$(x_2 + \varepsilon + 1 - \mu_\varepsilon) = \frac{1}{\beta w'(x_2 + \varepsilon)},$$

$$A_\varepsilon = w(x_2 + \varepsilon) - \frac{1}{\beta} \log(x_2 + \varepsilon + 1 - \mu_\varepsilon).$$

We come to the same conclusion: Under the assumption $w''(x_2 - ) < w''(x_2 + )$, there is a small positive $\tilde{\varepsilon}$ such that $\mu_{\tilde{\varepsilon}} > \mu$.

We complete the proof only for the case $p \neq 0$; the argument for $p = 0$ is fully analogous. Define $\tilde{w} \colon [-(1 - \mu_{\tilde{\varepsilon}}), \infty) \to R$ and $\tilde{v} \colon \mathscr{S} \to R$ by

$$\tilde{w}(x) = \begin{cases} \dfrac{1}{p} A_{\tilde{\varepsilon}}^{p-1}(x + 1 - \mu_{\tilde{\varepsilon}})^p, & \text{if } -(1 - \mu_{\tilde{\varepsilon}}) \leq x < x_2 + \tilde{\varepsilon}, \\ w(x), & \text{if } x \geq x_2 + \tilde{\varepsilon}, \end{cases}$$

$$\tilde{v}(x, y) = \begin{cases} y^p \tilde{w}\left(\dfrac{x}{y}\right), & \text{if } (x, y) \in \bar{\mathscr{S}}, y > 0, \\ v(x, y), & \text{if } (x, y) \in \bar{\mathscr{S}}, y \leq 0. \end{cases}$$

We further define $\tilde{\theta}_2 = 1/(1 + x_2 + \tilde{\varepsilon})$, $\tilde{\theta}_1 = \theta_1$. From (10.4) and (10.5), we see that $\tilde{w}$ is $C^1$, so $\tilde{v}$ is continuous on the closure of

$$\tilde{\mathscr{S}} = \left\{ (x, y); \, x + \frac{y}{1 - \lambda} > 0, \, x + (1 - \mu_{\tilde{\varepsilon}})y > 0 \right\}$$

and $\tilde{v}$ is $C^1$ on $\tilde{\mathscr{S}} \backslash \{(x, 0); x > 0\}$. Indeed, $\tilde{v}, \tilde{\theta}_1, \tilde{\theta}_2$ satisfy all the hypotheses of Corollary 9.8, but with $\mu_{\tilde{\varepsilon}}$ replacing $\mu$ in that corollary. Thus, there is a triple $(\tilde{C}, \tilde{L}, \tilde{M})$ which is feasible for the initial condition $(x_2 + \tilde{\varepsilon}, 1)$ in the problem with $\mu_{\tilde{\varepsilon}}$ replacing $\mu$ such that

$$v(x_2 + \tilde{\varepsilon}, 1) = w(x_2 + \tilde{\varepsilon}) = \tilde{w}(x_2 + \tilde{\varepsilon}) = \tilde{v}(x_2 + \tilde{\varepsilon}, 1)$$

$$\leq E \int_0^\infty e^{-\beta t} U_p(\tilde{C}(t)) \, dt.$$

This means that at $(x_2 + \tilde{\varepsilon}, 1)$, the value function in the problem with $\mu_{\tilde{\varepsilon}}$ replacing $\mu$ dominates $v(x_2 + \tilde{\varepsilon}, 1)$, the value function in the original problem. Remark 9.7 contradicts this conclusion, and so equality must hold in (10.3). This concludes the proof that $v$ is $C^2$ across $\partial_2 \mathrm{NT}$ if $\theta_2 \neq 1$.

Now assume $\theta_1 \neq 1$ and $\theta_1 \neq 0$, so that $\theta_1 > 0$. The proof that $v$ is $C^2$ across $\partial_1 \mathrm{NT}$ is similar to the proof for $\partial_2 \mathrm{NT}$. We highlight the differences for the case $p \neq 0$. Define $x_1 = (1 - \theta_1)/\theta_1$. In place of (10.3), one has $w''(x_1 + ) \leq w''(x_1 - )$. Assume strict inequality holds. For small $\varepsilon > 0$, define

$\lambda_\varepsilon$, $A_\varepsilon$ by

$$\left( x_1 - \varepsilon + \frac{1}{1 - \lambda_\varepsilon} \right) = \frac{pw(x_1 - \varepsilon)}{w'(x_1 - \varepsilon)},$$

$$A_\varepsilon^{p-1} = \frac{pw(x_1 - \varepsilon)}{\left(x_1 - \varepsilon + 1/(1 - \lambda_\varepsilon)\right)^p}$$

and verify that $\partial \lambda_\varepsilon / \partial \varepsilon |_{\varepsilon=0} > 0$. Choose $\tilde{\varepsilon} > 0$ so that $\lambda_{\tilde{\varepsilon}} > \lambda$ and proceed as before. □

COROLLARY 10.2. *The value function $v$ is $C^2$ in $\mathscr{S} \backslash \{(x, y); \ x = 0 \ \text{or} \ y = 0\}$.*

PROOF. Combine Theorem 6.9, Corollary 8.6 and Theorem 10.1. □

COROLLARY 10.3. *The functions $v$, $v_x$, $v_y$ and $v_{yy}$ are continuous on $\mathscr{S} \backslash \{(x, 0); \ x > 0\}$ and satisfy the HJB equation (4.6) in the classical sense on this set. In particular, for $(x, y) \in \mathscr{S}$ with $y \neq 0$,*

$$(10.6) \qquad \frac{y}{x + y} \leq \theta_1 \quad \Leftrightarrow \quad v_x(x, y) - (1 - \lambda)v_y(x, y) = 0,$$

$$(10.7) \qquad \frac{y}{x + y} \geq \theta_2 \quad \Leftrightarrow \quad -(1 - \mu)v_x(x, y) + v_y(x, y) = 0,$$

$$(10.8) \quad \theta_1 \leq \frac{y}{x + y} \leq \theta_2 \quad \Rightarrow \quad (\mathscr{L}v)(x, y) - \tilde{U}_p(v_x(x, y)) = 0.$$

PROOF. The regularity assertions are contained in Corollary 8.3 and Theorem 10.1. The HJB equation holds in the classical sense on $\mathscr{S} \backslash \{(x, y); x = 0 \ \text{or} \ y = 0\}$ because of Corollary 7.6, Theorem 7.7 and Corollary 10.2. For $(x, y) \in \mathscr{S}$ with $x = 0$, we obtain the HJB equation from continuity. Implications (10.6)–(10.8) follow from the HJB equation (4.6) and (8.10)–(8.12). □

**11. Location of the free boundaries.** The no transaction region NT = $\{(x, y) \in \mathscr{S}; \ \theta_1 < y/(x + y) < \theta_2\}$ is characterized by the numbers $\theta_1$ and $\theta_2$ satisfying (9.2): $0 \leq \theta_1 < \theta_2 < 1/\mu$. In this section we provide bounds on $\theta_1$ and $\theta_2$ and show that $\theta_1 > 0$.

LEMMA 11.1. *If $p < 1$, $p \neq 0$, then there is a positive constant $A$ satisfying $A/p < C_* /p$ such that*

$$(11.1) \qquad v(x, y) = \frac{1}{p} A^{p-1} \left( x + (1 - \mu) y \right)^p \qquad \forall (x, y) \in \text{SS}.$$

*If $p = 0$, then there is a constant $A > (\log \beta)/\beta + (r - \beta)/\beta^2$ such that*

$$(11.2) \qquad v(x, y) = \frac{1}{\beta} \log(x + (1 - \mu)y) + A \qquad \forall (x, y) \in \text{SS}.$$

PROOF. The formulas (11.1) and (11.2) are from Theorem 6.9, and the nonstrict inequalities $A/p \leq C_*/p$ if $p \neq 0$, $A \geq (\log \beta)/\beta + (r - \beta)/\beta^2$ if $p = 0$, follow from Proposition 3.4.

We show that when $p \neq 0$, we cannot have $A = C_*$; the argument for $p = 0$ is similar. Assume $A = C_*$, so that for all $(x, y) \in \mathrm{SS}$,

$$(11.3) \qquad v(x, y) = \frac{1}{p} C_*^{p-1} (x + (1 - \mu)y)^p.$$

We show that this equality would then hold for all $(x, y) \in \mathscr{S}$ with $y \geq 0$. Let $(x, y) \in \mathscr{S}$ be given and choose $\gamma > 0$ large enough that

$$(x_0, y_0) \triangleq (x - (1 - \mu)\gamma, y + \gamma) \in \mathrm{SS}.$$

Then $(x, y)$ can be reached from $(x_0, y_0)$ by a transaction, so according to Proposition 3.5 and our assumption,

$$\frac{1}{p} C_*^{p-1} (x + (1 - \mu)y)^p = v(x_0, y_0) \geq v(x, y).$$

The reverse inequality follows from Proposition 3.4. Thus, if (11.3) holds on SS, then it holds on $\{(x, y) \in \mathscr{S}; \ y \geq 0\}$. This would imply that SS contains the latter set, or equivalently, that $\theta_2 \leq 0$, which contradicts (9.2). $\square$

THEOREM 11.2. *For all $p < 1$ we have*

$$(11.4) \qquad \theta_2 < \frac{\alpha - r}{\frac{1}{2}(1 - \mu)\sigma^2(1 - p) + \mu(\alpha - r)}.$$

*If $\sigma^2(1 - p) \neq \alpha - r$ and*

$$A(p) \triangleq \frac{\beta - rp}{1 - p} - \frac{p(\alpha - r)^2}{2\sigma^2(1 - p)^2}$$

*is positive, then*

$$(11.5) \qquad \theta_2 > \frac{\alpha - r}{(1 - \mu)\sigma^2(1 - p) + \mu(\alpha - r)};$$

*if $\sigma^2(1 - p) = \alpha - r$ and $A(p) > 0$ holds, then $\theta_2 = 1$. If $A(p)$ is positive and $\sigma^2(1 - p) > \lambda(\alpha - r)$, then*

$$(11.6) \qquad \theta_1 < \frac{(1 - \lambda)(\alpha - r)}{\sigma^2(1 - p) - \lambda(\alpha - r)}.$$

PROOF. We derive the preceding bounds under the assumption $p \neq 0$; the case $p = 0$ is similar.

To obtain (11.4), we use (11.1) and the inequality $A/p < C_*/p$ to compute, just as in (5.17), that

$$(\mathscr{L}v)(x, y) - \tilde{U}_p(x, y)$$

$$= A^{p-1}(x + (1 - \mu)y)^p \left[ \frac{1-p}{p}(C_* - A) + \frac{1}{2}\sigma^2(1-p) \right.$$

$$\left. \times \frac{(1-\mu)^2 y^2}{(x + (1-\mu)y)^2} - \frac{(\alpha - r)(1-\mu)y}{(x + (1-\mu)y)} \right]$$

$$> A^{p-1}(x + (1-\mu)y)^{p-1}(1-\mu)y$$

$$\times \left[ \frac{1}{2}\sigma^2(1-p)\frac{(1-\mu)y}{x + (1-\mu)y} - (\alpha - r) \right]$$

for all $(x, y) \in$ SS. We set $x = 1/\theta_2 - 1$ and $y = 1$ so $y/(x + y) = \theta_2$ and (10.8) implies

$$\frac{1}{2}\sigma^2(1-p)\frac{1-\mu}{1/\theta_2 - \mu} - (\alpha - r) < 0.$$

This is equivalent to (11.4).

To obtain (11.5), recall from Proposition 9.9 that if $\sigma^2(1-p) \neq \alpha - r$, then the constant $A$ in (11.1) satisfies $A/p > A(p)/p$. We compute

$$0 \le (\mathscr{L}v)(x, y) - \tilde{U}_p(x, y)$$

$$= A^{p-1}(x + (1-\mu)y)^p \left[ \frac{\beta - rp}{p} - \frac{(\alpha - r)^2}{2\sigma^2(1-p)} - \frac{1-p}{p}A \right.$$

$$\left. + \frac{1}{2\sigma^2(1-p)}\left( \sigma^2(1-p)\frac{(1-\mu)y}{x + (1-\mu)y} - (\alpha - r) \right)^2 \right]$$

$$< A^{p-1}(x + (1-\mu)y)^p \frac{1}{2\sigma^2(1-p)}$$

$$\times \left[ \sigma^2(1-p)\frac{(1-\mu)y}{x + (1-\mu)y} - (\alpha - r) \right]^2$$

for all $(x, y) \in$ SS. This shows that if $(x, y) \in$ SS, then the expression

$$(11.7) \qquad \sigma^2(1-p)\frac{(1-\mu)y}{x + (1-\mu)y} - (\alpha - r)$$

cannot be zero. This expression approaches $+\infty$ at $\partial_2 \mathscr{S}$ (the left-hand boundary of SS), so it must be positive everywhere in SS. Neither can the expression be zero when $y/(x + y) = \theta_2, y > 0$ (the right-hand boundary) of SS, for

if it were, $\mathscr{L}v - \tilde{U}_p$ would be negative at points in SS near this boundary. The positivity of (11.7) when $x = 1/\theta_2 - 1$ and $y = 1$ gives us (11.5).

If $\sigma^2(1 - p) = \alpha - r$ and $A(p) > 0$, the foregoing argument yields

$$\theta_2 \geq \frac{\alpha - r}{(1 - \mu)(1 - p)\sigma^2 + \mu(\alpha - r)} = 1.$$

From Corollary 9.10 and the equivalence (10.7), we see that $\theta_2 \leq 1$.

We turn to (11.6). Recall from Theorem 6.9 that

$$v(x, y) = \frac{1}{p}B^{p-1}\left(x + \frac{y}{1 - \lambda}\right)^p$$

for some $B > 0$ and all $(x, y) \in \mathrm{SMM}$. If $\sigma^2(1 - p) = \alpha - r$, then (11.6) becomes $\theta_1 < 1$, which follows from (9.2) and $\theta_2 = 1$. Assume, therefore, that $\sigma^2(1 - p) \neq \alpha - r$. Then Proposition 9.9 implies $B/p > A(p)/p$. We compute as before:

$$0 \leq (\mathscr{L}v)(x, y) - \tilde{U}_p(x, y)$$

$$= B^{p-1}\left(x + \frac{y}{1 - \lambda}\right)^p\left[\frac{\beta - rp}{p} - \frac{(\alpha - r)^2}{2\sigma^2(1 - p)} - \frac{1 - p}{p}B\right.$$

$$\left. + \frac{1}{2\sigma^2(1 - p)}\left(\sigma^2(1 - p)\frac{y/(1 - \lambda)}{x + y/(1 - \lambda)} - (\alpha - r)\right)^2\right]$$

$$< B^{p-1}\left(x + \frac{y}{1 - \lambda}\right)^p\frac{1}{2\sigma^2(1 - p)}\left[\sigma^2(1 - p)\frac{y/(1 - \lambda)}{x + y/(1 - \lambda)} - (\alpha - r)\right]^2$$

for all $(x, y) \in \mathrm{SMM}$. Arguing as before, we see that the expression

$$\sigma^2(1 - p)\frac{y/(1 - \lambda)}{x + y/(1 - \lambda)} - (\alpha - r)$$

must be negative in SMM and also at the boundary where $y/(x + y) = \theta_1$. Taking $x = 1/\theta_1 - 1$, $y = 1$ and assuming $\sigma^2(1 - p) > \lambda(\alpha - r)$, we obtain (11.6). □

REMARK 11.3. In the problem with no transaction costs ($\mu = \lambda = 0$), which should not be formulated as a singular stochastic control problem, the optimal portfolio keeps the proportion of wealth invested in the stock at

$$(11.8) \qquad\qquad \theta_* \triangleq \frac{\alpha - r}{(1 - p)\sigma^2}$$

(see, e.g., [14]). This is the problem originally solved by Merton [54], and the solution is obtained under the assumption

$$(11.9) \qquad\qquad A(p) \triangleq \frac{\beta - rp}{1 - p} - \frac{p(\alpha - r)^2}{2\sigma^2(1 - p)^2} > 0.$$

We refer to $\{(x, y) \in \mathscr{S};\ y/(x + y) = \theta^*\}$ as the *Merton line*. Note that a necessary and sufficient condition for the Merton line to be in the first quadrant is

(11.10) $$(1 - p)\sigma^2 > \alpha - r.$$

Davis and Norman [14] assume (11.9) and (11.10) for their key Theorem 5.1, and they ultimately show that the Merton line lies in the no-transaction region NT, that is,

(11.11) $$\theta_1 < \theta_* < \theta_2.$$

We have recovered this result: (11.11) follows immediately from (11.10), (11.5) and (11.6). We have also discovered that when the Merton line coincides with the positive $y$-axis, it coincides with $\partial_2 \mathrm{NT}$, that is, $(1 - p)\sigma^2 = \alpha - r \Rightarrow \theta_* = \theta_2 = 1$. This resolves part of the conjecture following Theorem 7.1 in [14]. We conjecture that if $(1 - p)\sigma^2 < \alpha - r$, then $\theta_* > \theta_2$. The bounds in Theorem 11.2 do not support the proof of this, but they allow us to construct examples to illustrate it. For instance, if

$$\frac{1}{2}\left(\frac{1}{\mu} + 1\right)(1 - p)\sigma^2 < \alpha - r,$$

then (11.4) and (11.5) imply

$$1 < \theta_2 < \frac{\alpha - r}{\frac{1}{2}(1 - \mu)(1 - p)\sigma^2 + \mu(\alpha - r)} < \theta_*.$$

For small values of $\sigma$, the Merton line is close to the line $\{(x, y):\ x + y = 0,\ y > 0\}$ whereas $\partial_2 \mathrm{NT} \triangleq \{(x, y) \in \mathscr{S};\ y/(x + y) = \theta_2\}$ is close to $\partial_2 \mathscr{S} = \{(x, y) \in \mathscr{S};\ x + (1 - \mu)y = 0,\ y > 0\}$. This disappearance of the region SS as $\sigma$ approaches zero is consistent with the results obtained in [61] for the model with $\sigma = 0$.

REMARK 11.4. We conjecture that the bounds (11.5) and (11.6) are valid even without the assumption of positivity of $A(p)$. Note that these bounds do not involve $\beta$, but $A(p)$ does.

We next show that regardless of the parameters of the model, we have $\theta_1 > 0$. In other words, it is always advantageous to hold some of the stock. We first characterize the condition $\theta_1 = 0$ in terms of (8.5), and then contradict this characterization. A consequence of Theorem 11.6 is that the value function is smooth on the positive $x$-axis, because this half-line is contained in SMM (see Theorem 6.9).

LEMMA 11.5. *If $\theta_1 = 0$, then (8.5) holds.*

PROOF. Assume $\theta_1 = 0$, let $x_0 > 0$, $y_0 = 0$ be given and let $(C, L, M) \in \mathscr{A}(x_0, y_0)$ be optimal for these initial conditions. According to Remark 9.4, $L \equiv 0$ and $M \equiv 0$, so $C$ is the optimal solution for the deterministic control

problem in which the possibility of investment in the stock is ignored. It is easily verified that (8.5) provides the value function for this problem. □

THEOREM 11.6.  *We have $\theta_1 > 0$, that is, the positive $x$-axis is contained in the region* SMM.

PROOF.  We assume $\theta_1 = 0$, so (8.5) holds, and obtain a contradiction. We proceed under the assumption $p \neq 0$; the case $p = 0$ similar.

From (8.5) and Theorem 6.9, we have

$$v(x, y) = \frac{1}{p}C_*^{p-1}\left(x + \frac{y}{1-\lambda}\right)^p \qquad \forall x > 0, \ -(1-\lambda)x < y \leq 0.$$

Let $u$ be defined by (8.1), so

$$u(z) = \frac{1}{p}C_*^{p-1}\left(\frac{1-\lambda+\lambda z}{1-\lambda}\right)^p \qquad \forall z \in (-(1-\lambda)/\lambda, 0].$$

In particular, $u'(0^-) = \lambda C_*^{p-1}/(1-\lambda)$. We define $\varphi(z) \triangleq (1/p)C_*^{p-1}(1-\mu z)^p$ and $k(z) \triangleq u(z) - \varphi(z)$ for all $z \in (-(1-\lambda)/\lambda, 1/\mu]$. From Proposition 3.4 we have $u(z) \geq \varphi(z)$ for all $z \in [0, 1/\mu)$, and thus $k(z) \geq 0$ for $0 \leq z < 1/\mu$ and $k(0) = 0$. From these facts and the concavity of $u$, we have

$$0 \leq k'(0+) = u'(0+) - \varphi'(0) \leq u'(0-) - \varphi'(0)$$

(11.12)
$$= \left(\frac{\lambda}{1-\lambda} + \mu\right)C_*^{p-1} < \infty.$$

Now $\theta_2 > \theta_1 = 0$ and $u$ satisfies (8.14) on $(0, \theta_2 \wedge 1)$, that is,

(11.13)      $f(z) + g(z) + h(z) = 0 \qquad \forall z \in (0, \theta_2 \wedge 1),$

where

$$f(z) \triangleq \beta\varphi(z) - d_1(z)p\varphi(z) - d_2(z)\varphi'(z) - d_3(z)\varphi''(z)$$
$$- \tilde{U}_p(p\varphi(z) - z\varphi'(z))$$
$$= (1-\mu)C_*^{p-1}\left[-(\alpha-r)(1-\mu z)^{p-1}z\right.$$
$$\left. + \frac{1}{2}\sigma^2(1-p)(1-\mu)(1-\mu z)^{p-2}z^2\right],$$

$$g(z) \triangleq \beta k(z) - d_1(z)pk(z) - d_2(z)k'(z) - d_3(z)k''(z),$$

$$h(z) \triangleq \tilde{U}_p(p\varphi(z) - z\varphi'(z)) - \tilde{U}_p(p\varphi(z) - z\varphi'(z) + pk(z) - zk'(z))$$
$$= \frac{1-p}{p}C_*^{p-1}(1-\mu z)^p$$
$$- \frac{1-p}{p}\left[C_*^{p-1}(1-\mu_z)^{p-1} + pk(z) - zk'(z)\right]^{p/(p-1)}.$$

Note that $f(0+) = 0$ and $h(0+) = 0$, and because of (11.13), we must also have

$$(11.14) \qquad 0 = g(0+) = -\tfrac{1}{2}\sigma^2 \lim_{z \downarrow 0} z^2 k''(z).$$

For $0 < z < \theta_2 \wedge 1$, the derivatives $f'(z)$ and $h'(z)$ are defined, so $g'(z)$ must also be defined and

$$(11.15) \qquad f'(z) + g'(z) + h'(z) = 0 \qquad \forall z \in (0, \theta_2 \wedge 1).$$

The existence of $g'(z)$ implies the existence of $k'''(z)$ for $0 < z < \theta_2 \wedge 1$.

We undertake to let $z \downarrow 0$ in (11.15). We have easily that $\lim_{z \downarrow 0} f'(z) = f'(0) = -(1 - \mu)(\alpha - r)C_*^{p-1}$ and, consequently,

$$(11.16) \qquad \lim_{z \downarrow 0} \big[ g'(z) + h'(z) \big] = (1 - \mu)(\alpha - r)C_*^{p-1}.$$

Direct computation utilizing (11.12) and (11.14) reveals that

$$(11.17) \qquad \begin{aligned} \lim_{z \downarrow 0} \big[ g'(z) + h'(z) \big] &= -(\alpha - r)k'(0+) \\ &\quad - \lim_{z \downarrow 0} \big[ Azk''(z) + Bz^2 k'''(z) \big], \end{aligned}$$

where $A \triangleq C_* + \sigma^2 + \alpha - r$ and $B \triangleq \sigma^2/2$. We show that the limit on the right-hand side is nonnegative. If it were negative, then for some $z_2 > 0$ and $\varepsilon > 0$ sufficiently small, we would have

$$Ak''(\zeta) + B\zeta k'''(\zeta) \le -\frac{\varepsilon}{\zeta} \qquad \forall \zeta \in (0, z_2].$$

Let $z_1 \in (0, z_2)$ be given and integrate the preceding inequality from $z_1$ to $z_2$ to obtain

$$(A - B)\big[ k'(z_2) - k'(z_1) \big] + B\big[ z_2 k''(z_2) - z_1 k''(z_1) \big] \le \varepsilon \log(z_1/z_2).$$

Because $\lim_{z_1 \downarrow 0} \varepsilon \log(z_1/z_2) = -\infty$ and $B > 0$, we have $\lim_{z_1 \downarrow 0} z_1 k''(z_1) = \infty$. Thus, for sufficiently small $z_1 > 0$, the inequality $k''(\zeta) \ge 1/\zeta$ holds for all $\zeta \in (0, z_1]$. Integrating from $z_0 \in (0, z_1)$ to $z_1$, we obtain $k'(z_1) - k'(z_0) \ge \log(z_1/z_0)$, which implies $\lim_{z_0 \downarrow 0} k'(z_0) = -\infty$. This violates (11.12), and we conclude that $\lim_{z \downarrow 0}[ Azk''(z) + Bz^2 k'''(z)] \ge 0$. Omitting this term in (11.17), we obtain the inequality

$$\lim_{z \downarrow 0}[ g'(z) + h'(z) ] \le -(\alpha - r)k'(0+) \le 0,$$

which is inconsistent with (11.16). $\square$

## 12. Finiteness of the value function.

As noted in Section 2, Standing Assumption 2.3 can fail when $0 < p < 1$. Its validity is equivalent to the

finiteness of the value function. Proposition 5.1 shows that

$$(12.1) \qquad A(p) \triangleq \frac{\beta - rp}{1 - p} - \frac{p(\alpha - r)^2}{2\sigma^2(1 - p)^2} > 0$$

is a sufficient condition for a finite value function. This condition, which does not involve the transaction costs $\lambda$ and $\mu$, also guarantees a finite value function in the problem without transaction costs.

If $\sigma = 0$, so that the stock is really another money market, an investor who incurs no transaction costs can arbitrage the difference between $\alpha$ and $r$. Thus, the value function is infinite if $\sigma = 0$ and $\lambda = \mu = 0$, and this is reflected by the fact that with $\beta$, $p$, $r$ and $\alpha$ held constant, (12.1) is violated by small $\sigma$. In fact, if $\sigma$ is small, then $\lambda$ and $\mu$ become important and (12.1) is no longer a useful sufficient condition for value function finiteness.

In [61], Remark 13.1, a necessary and sufficient condition for finiteness of the value function $V$ in the problem with $\beta > 0, 0 < p < 1, 0 < r < \alpha, \sigma = 0, 0 < \lambda < 1, 0 < \mu < 1$ was found to be

$$(12.2) \qquad \beta - \alpha p > \frac{p(\alpha - r)(1 - \lambda)(1 - \mu)}{1 - (1 - \lambda)(1 - \mu)}.$$

[Note that $\lambda$ in [61] is $\lambda/(1 - \lambda)$ in this paper.] The problem in this paper with the same parameters $\beta$, $p$, $r$, $\alpha$, $\lambda$ and $\mu$, but with $\sigma > 0$, has a value function $v$ dominated by $V$, as we show shortly. Thus, (12.2) is another sufficient condition for Standing Assumption 2.3, a condition involving $\lambda$ and $\mu$, but not $\sigma$.

To prove the claim that $V$ dominates $v$, we recall from [61], Theorem 3.1, that $V$ is nonnegative, continuous and concave on $\overline{\mathscr{S}}, C^1$ on $\mathscr{S}$ and satisfies the HJB equation

$$(12.3) \qquad \begin{aligned} \min\Big\{ &\beta V - \alpha y V_y - rx V_x - \tilde{U}_p(V_x), \\ &-(1 - \mu)V_x + V_y, V_x - (1 - \lambda)V_y \Big\} = 0 \quad \text{on } \mathscr{S}. \end{aligned}$$

We want to apply Itô's formula to $V$, and for that we need the following lemma.

LEMMA 12.1. *Let $\mathscr{O}$ be an open subset of $R^n$ and let $\varphi \colon [0, \infty) \times \overline{\mathscr{O}} \to R$ be continuous on $[0, \infty) \times \overline{\mathscr{O}}, C^1$ on $[0, \infty) \times \mathscr{O}$ and, for each $t \geq 0$, let $\varphi(t, \cdot)$ be convex on $\mathscr{O}$. Let $M$ be an $n$-dimensional vector of continuous local martingales with $M(0) = 0$ and let $F$ be an $n$-dimensional vector of RCLL, finite-variation processes with $F(0) = 0$. Let $X(0) \in \overline{\mathscr{O}}$ be given and define the semimartingale*

$$X(t) = X(0) + M(t) + F(t), \qquad t \geq 0.$$

*For $k \geq 1$ and $x \in R^n$, set $B_k(x) \triangleq \{y \in R^n; \|y - x\| \leq 1/k\}$ and $\mathscr{O}_k \triangleq \{x \in \mathscr{O}; \overline{B_k(x)} \subset \mathscr{O}\}$. Define $\rho_k \triangleq \inf\{t \geq 0; X(t) \notin \mathscr{O}_k\}$. For any bounded stopping time*

$\tau$, *we have the Itô inequality*

$$\varphi(\tau \wedge \rho_k, X(\tau \wedge \rho_k))$$

$$\geq \varphi(0, X(0)) + \int_0^{t \wedge \rho_k} \varphi(s, X(s)) \, ds$$

(12.4)
$$+ \int_0^{\tau \wedge \rho_k} D\varphi(s, X(s-)) \, dM(s) + \int_0^{\tau \wedge \rho_k} D\varphi(s, X(s-)) \, dF(s)$$

$$+ \sum_{0 < s \leq \tau \wedge \rho_k} [\varphi(s, X(s)) - \varphi(s, X(s-))$$

$$- D\varphi(s, X(s-))(F(s) - F(s-))].$$

PROOF.   If $\varphi(t, \cdot)$ were $C^2$, (12.4) would be a special case of Itô's formula for semimartingales (e.g., [57] and [59]) and the fact that convexity of $\varphi(s, \cdot)$ implies

$$\int_0^{\tau \wedge \rho_k} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \varphi(s, X(s-)) \, d\langle M_i, M_j \rangle(s) \geq 0.$$

In the present circumstances, we mollify $\varphi$ to obtain a $C^2$ function to which Itô's formula can be applied.

For $k \geq 1$, let $\zeta_k \colon R^n \to [0, \infty)$ be a $C^\infty$ function satisfying $\int_{R^n} \zeta_k(x) \, dx = 1$ and having support in $B_{1/k}(0) \triangleq \{x \in R^n; \|x\| < 1/k\}$. Define $\varphi_k \colon [0, \infty) \times \overline{\mathcal{O}} \to R$ by

$$\varphi_k(t, x) = \int_{B_{1/k}(x)} \varphi(t, y) \zeta_k(y - x) \, dy = \int_{B_{1/k}(0)} \varphi(t, x + z) \zeta_k(z) \, dz.$$

Then $\varphi_k$ is continuous and $C^1$ on $[0, \infty) \times \mathcal{O}_k$, $\varphi_k(t, \cdot)$ is $C^\infty$ on $\mathcal{O}_k$ and

$$D\varphi_k(t, x) = \int_{B_{1/k}(0)} D\varphi(t, x + z) \zeta_k(z) \, dz \qquad \forall t \geq 0.$$

We see then that $\varphi_k \to \varphi$ and $D\varphi_k \to D\varphi$ uniformly on compact subsets of $[0, \infty) \times \mathcal{O}$.

We show that $\varphi_k(t, \cdot)$ is convex on $\mathcal{O}_k$. From Taylor's theorem we have for $x \in \mathcal{O}_k$, $y \in R^n$ and $h > 0$ sufficiently small that

(12.5)
$$\varphi_k(t, x + hy) + \varphi_k(t, x - hy) - 2\varphi_k(t, x)$$
$$= h^2 y \cdot D^2 \varphi_k(t, x) y + o(h^2).$$

On the other hand,

$$\varphi_k(t, x + hy) + \varphi_k(t, x - hy) - 2\varphi_k(t, x)$$

(12.6)
$$= \int_{B_k(0)} [\varphi(t, x + hy + z) + \varphi(t, x - hy + z)$$

$$- 2\varphi(t, x + z)] \zeta_k(z) \, dz$$

$$\geq 0,$$

because the convexity of $\varphi(t, \cdot)$ implies nonnegativity of the integrand. Equating (12.5) and (12.6), dividing by $h^2$ and letting $h \downarrow 0$, we see that $D^2 \varphi_k(t, \cdot)$ is positive semidefinite on $\mathscr{O}_k$.

For any bounded stopping time $\tau$, we have from Itô's formula for $C^2$ functions that for $m \geq k \geq 1$,

$$
\varphi_m(\tau \wedge \rho_k, X(\tau \wedge \rho_k))
$$

$$
\geq \varphi_m(0, X(0)) + \int_0^{\tau \wedge \rho_k} \frac{\partial}{\partial t} \varphi_m(s, X(s-)) \, ds
$$

$$
+ \int_0^{\tau \wedge \rho_k} D\varphi_m(s, X(s-)) \, dM(s) + \int_0^{\tau \wedge \rho_k} D\varphi_m(s, X(s-)) \, dF(s)
$$

$$
+ \sum_{0 < s \leq \tau \wedge \rho_k} \left[ \varphi_m(s, X(s)) - \varphi_m(s, X(s-)) \right.
$$

$$
\left. - D\varphi_m(s, X(s-))(F(s) - F(s-)) \right].
$$

Letting $m \to \infty$, one can show by standard arguments that (12.4) holds. $\square$

THEOREM 12.2. *Assume* (12.2). *Then the value function $V$ for the problem with $\sigma = 0$ (and $\beta > 0, 0 < p < 1, 0 < r < \alpha, 0 < \lambda < 1$ and $0 < \mu < 1$) is finite and dominates the value function $v$ for the problem with $\sigma > 0$ (and the same vales for $\beta$, $p$, $r$, $\alpha$, $\lambda$ and $\mu$).*

PROOF. Repeat the proof of Proposition 5.1, but with $V$ in place of $\varphi$ in that proof and using (12.3) and (12.4) in place of (4.7), (5.3), (5.4) and (5.8). $\square$

REMARK 12.3. Another sufficient condition for Standing Assumption 2.3 when $0 < p < 1$, obtained by Fleming and Soner [26], Theorem 8.7.2, is that $\beta - pd_1(z) \geq 0$ for all $z \in \mathscr{I}$ (see the beginning of our Section 8 for definitions).

## APPENDIX

### Sensitivity of the indirect utility to transaction costs in a consumption-based model (S. E. Shreve).

*A. Comparison.* For the sensitivity analysis of this appendix, we need sharp upper and lower bounds on the value function $v$ for small but positive values of the transaction cost parameters $\lambda$ and $\mu$. We obtain these bounds by construction of super-and subsolutions to the one-dimensional HJB equation (8.2). Comparison of supersolutions and subsolutions, based on the maximum principle, is a key part of the theory of viscosity solutions of second-order partial differential equations ([47], [38], [37]). In the present context, we are dealing with nearly $C^2$ functions, which enables us to obtain comparisons without appeal to the full power of the viscosity solution machin-

ery. This section develops the comparison result for our problem and in the next section, the super- and subsolutions are constructed.

In this Appendix we assume

(A.1) $$0 < p < 1, \quad \lambda > 0, \quad \mu > 0.$$

It is convenient to rewrite (8.2) in terms of the *Merton proportion* $\theta_*$ of (11.8) and $A(p)$ of (11.9). We assume throughout that (11.9) holds, that is, that $A(p)$ is strictly positive. In terms of $\theta_*$ and $A(p)$, the functions $d_1, d_2, d_3$ of Section 8 satisfy

$$\beta - pd_1(z) = (1 - p)A(p) + \tfrac{1}{2}p(1 - p)\sigma^2(z - \theta_*)^2,$$

$$d_2(z) = -(1 - p)\sigma^2 z(1 - z)(z - \theta_*),$$

$$d_3(z) = \tfrac{1}{2}\sigma^2 z^2(1 - z)^2.$$

We rewrite (8.2) as

(A.2)
$$\min\{\mathscr{D}\psi(z) - \tilde{U}(p\psi(z) - z\psi'(z)),$$
$$\mu p\psi(z) + (1 - \mu z)\psi'(z),$$
$$\lambda p\psi(z) - (1 - \lambda + \lambda z)\psi'(z)\} = 0,$$

where

(A.3)   $\mathscr{D}\psi(z) \triangleq (\beta - d_1(z)p)\psi(z) - d_2(z)\psi'(z) - d_3(z)\psi''(z).$

We shall consider this equation together with the boundary conditions

(A.4) $$\psi\left(-\frac{1 - \lambda}{\lambda}\right) = 0, \quad \psi\left(\frac{1}{\mu}\right) = 0.$$

We shall be considering functions $\psi\colon \overline{\mathscr{I}} \mapsto R$ which are a class $C^1$ on the open interval $\mathscr{I} \triangleq (-(1 - \lambda)/\lambda, 1/\mu)$, are continuous on the closed interval $\overline{\mathscr{I}}$ and are of class $C^2$ on the open interval $\mathscr{I}$ except at possibly finitely many points. One of these points may be 1, at which place we make no assumption on the existence of one-sided second derivatives. At any of the finitely many points $z_0 \neq 1$ in $\mathscr{I}$ where $\psi''$ is not defined, we assume that the one-sided second derivatives $\lim_{h \downarrow 0}[\psi'(z_0 + h) - \psi'(z_0)]/h$ and $\lim_{h \uparrow 0}[\psi'(z_0 + h) - \psi'(z_0)]/h$ exist and equal the respective one-sided limits $\psi''(z_0 +) \triangleq \lim_{z \downarrow z_0} \psi''(z)$ and $\psi''(z_0 -) \triangleq \lim_{z \uparrow z_0} \psi''(z)$. For such a function $\psi$ and for $z \in \mathscr{I}$, we have the integration-by-parts formula

$$\int_1^z (\zeta - 1)^2 \psi''(\zeta)\, d\zeta = (z - 1)\left[(z - 1)\psi'(z) - 2\int_1^z \frac{\zeta - 1}{z - 1}\psi'(\zeta)\, d\zeta\right]$$
$$= o(z - 1),$$

and hence

(A.5) $$\liminf_{z \to 1}|(z - 1)^2 \psi''(z)| = 0.$$

DEFINITION A.1.   A function $\psi$ as described above is said to be a *supersolution* to (A.2) with boundary conditions (A.4) if

(A.6)
$$\min\{\mathscr{D}\psi(z) - \tilde{U}(p\psi(z) - z\psi'(z)),$$
$$\mu p\psi(z) + (1 - \mu z)\psi'(z),$$
$$\lambda p\psi(z) - (1 - \lambda + \lambda z)\psi'(z)\} \geq 0$$

and

(A.7)
$$\psi\left(-\frac{1-\lambda}{\lambda}\right) \geq 0, \qquad \psi\left(\frac{1}{\mu}\right) \geq 0,$$

where (A.6) is to hold at all $z \in \mathscr{I}$ where $\psi''(z)$ is defined. We say that $\psi$ is a *subsolution* to (A.2) with boundary condition (A.4) if $\psi$ satisfies (A.6) and (A.7) with the inequalities reversed.

REMARK A.2.   Letting $z \to 1$ along an appropriate subsequence in (A.6) and using (A.5), we see that a supersolution satisfies

(A.8)
$$\min\{(\beta - pd_1(1))\psi(1) - d_2(1)\psi'(1) - \tilde{U}(p\psi(1) - \psi'(1)),$$
$$\mu p\psi(1) + (1 - \mu)\psi'(1), \lambda p\psi(1) - \psi'(1)\} \geq 0.$$

At any $z_0 \neq 1$ where $\psi''$ is not defined, a supersolution satisfies

(A.9)
$$\min\{(\beta - pd_1(z_0))\psi(z_0) - d_2(z_0)\psi'(z_0) - d_3(z_0)\psi''(z_0\pm)$$
$$-\tilde{U}(p\psi(z_0) - z_0\psi'(z_0)),$$
$$\mu p\psi(z_0) + (1 - \mu z_0)\psi'(z_0),$$
$$\lambda p\psi(z_0) - (1 - \lambda + \lambda z_0)\psi'(z_0)\} \geq 0.$$

A subsolution satisfies (A.8) and (A.9) with the inequalities reversed.

COMPARISON THEOREM A.3.   *Assume (A.1) and $A(p) > 0$. If $\psi_1$ is a supersolution to (A.2), (A.4) and $\psi_2$ is a subsolution, then $\psi_1 \geq \psi_2$. In particular, any supersolution majorizes the function $u$ defined by (8.1), and any subsolution minorizes $u$.*

PROOF.   We first argue that $u$ is both a supersolution and a subsolution to (A.2) and (A.4). The boundary behavior (A.4) for $u$ was established in Proposition 3.4. The function $u$ is of class $C^2$ at 0 because of Theorems 6.9 and 11.6; $u$ is of class $C^1$ elsewhere in $\mathscr{I}$ because of Proposition 8.2; $u$ is of class $C^2$ except possibly at 1 because of Corollary 10.2; and $u$ is a classical solution of (A.2) because of Proposition 8.1 and Corollary 7.6. Thus, the second assertion in the theorem will follow from the first.

We now prove the first assertion. Let $\psi_1$ and $\psi_2$ be as described, and assume that $\psi_1 \geq \psi_2$ does not hold. Then $\psi_1 - \psi_2$ attains its minimum at

some point $z_0 \in \mathcal{I}$, where we necessarily have

$$\psi_1(z_0) < \psi_2(z_0), \qquad \psi_1'(z_0) = \psi_2'(z_0).$$

Consequently,

(A.10)    $0 \leq \mu p \psi_1(z_0) + (1 - \mu z_0) \psi_1'(z_0) < \mu p \psi_2(z_0) + (1 - \mu z_0) \psi_2'(z_0),$

$$(A.11) \quad \begin{aligned} 0 &\leq \lambda p \psi_1(z_0) - (1 - \lambda + \lambda z_0) \psi_1'(z_0) \\ &< \lambda p \psi_2(z_0) - (1 - \lambda + \lambda z_0) \psi_2'(z_0). \end{aligned}$$

If $z_0 = 1$, we may also use the facts that $\beta - pd_1(z) > 0$ and $\tilde{U}$ is decreasing to obtain

$$(A.12) \quad \begin{aligned} 0 &\leq (\beta - pd_1(z_0)) \psi_1(z_0) - d_2(z_0) \psi_1'(z_0) \\ &\quad - \tilde{U}(p\psi_1(z_0) - z_0 \psi_1'(z_0)) \\ &< (\beta - pd_1(z_0)) \psi_2(z_0) - d_2(z_0) \psi_2'(z_0) \\ &\quad - \tilde{U}(p\psi_2(z_0) - z_0 \psi_2'(z_0)). \end{aligned}$$

Taken together, (A.10)–(A.12) contradict the assertion that $\psi_2$ is a subsolution. If $z_0$ is a point where $\psi_1''(z_0)$ and $\psi_2''(z_0)$ are defined, then $\psi_1''(z_0) \geq \psi_2''(z_0)$ because $\psi_1 - \psi_2$ has a minimum at $z_0$. Therefore,

$$(A.13) \quad \begin{aligned} 0 &\leq \mathscr{D}\psi_1(z_0) - \tilde{U}(p\psi_1(z_0) - z_0 \psi_1'(z_0)) \\ &< \mathscr{D}\psi_2(z_0) - \tilde{U}(p\psi_2(z_0) - z_0 \psi_2'(z_0)), \end{aligned}$$

and again the subsolution property for $\psi_2$ is contradicted. If $\psi_1''(z_0)$ or $\psi_2''(z_0)$ is not defined, we still have $\psi_1''(z_0 \pm) \geq \psi_2''(z_0 \pm)$, (A.13) holds with $\psi_i''(z_0)$ replaced by $\psi_i''(z_0 \pm)$, $i = 1, 2$, and the subsolution property for $\psi_2$ is violated. $\square$

*B. Construction of a subsolution and a supersolution.* In order to establish the dependence of the function $u$ on the parameters $\lambda$ and $\mu$, we construct a supersolution and a subsolution to (A.2) and (A.4). We shall prove the following result.

THEOREM B.1. *Assume* (A.1), $A(p) > 0$ *and* $0 < \theta_* < 1$. *Fix* $\eta \in (0, \infty)$ *and let* $\mu$ *and* $\lambda$ *be related by the equation* $\mu = \eta \lambda$. *There exist constants* $k_2 \geq k_1 > 0$, *depending on* $\eta$, $p$, $\theta_*$ *and* $A(p)$, *but not depending on* $\lambda$, *such that for all* $\lambda > 0$ *sufficiently small,*

$$(B.1) \quad \frac{1}{p} A^{p-1}(p) - k_2 \lambda^{2/3} \leq u(\theta_*) \leq \frac{1}{p} A^{p-1}(p) - k_1 \lambda^{2/3}.$$

For $q > 0$, we shall use the symbol $O(\lambda^q)$ to denote a function of $\lambda$ with the property that $0 < k_1 \lambda^q \leq O(\lambda^q) \leq k_2 \lambda^q < \infty$ for all sufficiently small $\lambda > 0$. By contrast, $o(\lambda^q)$ is a function satisfying $\lim_{\lambda \downarrow 0} \lambda^{-q} |o(\lambda^q)| = 0$. Thus (B.1) may be restated as $u(\theta_*) = (1/p) A^{p-1}(p) - O(\lambda^{2/3})$.

REMARK B.2. Before proving Theorem B.1, we discuss its consequences for the *liquidity premium* associated with transaction costs. With $\eta$ fixed as in the theorem, let us indicate the dependence of the function $u$ on both $\alpha$, the mean rate of return of the stock, and on $\lambda$, by writing $u(z; \alpha, \lambda)$. We denote also the dependence of $A(p)$ and $\theta_*$ on $\alpha$ by writing $A(p; \alpha)$ and $\theta_*(\alpha)$. For $\lambda > 0$, the liquidity premium is defined to be that positive number $\rho(\lambda)$ for which

$$u(\theta_*(\alpha + \rho(\lambda)); \alpha + \rho(\lambda), \lambda) = u(\theta_*(\alpha); \alpha, 0).$$

Note in this connection that $u(z; \alpha, 0) = (1/p)A^{p-1}(p; \alpha)$ (Remark 5.2).

A careful examination of the proof of Theorem B.1 shows that it can be extended to imply the equality

(B.2)
$$
\begin{aligned}
u(\theta_*(\alpha + \rho(\lambda)); &\alpha + \rho(\lambda), \lambda) \\
&= u(\theta_*(\alpha + \rho(\lambda)); \alpha + \rho(\lambda), 0) - O(\lambda^{2/3}).
\end{aligned}
$$

From the definition of $A(p; \alpha)$ we have

(B.3)
$$
\begin{aligned}
u(\theta_*(\alpha + \rho(\lambda)); &\alpha + \rho(\lambda), 0) - u(\theta_*(\alpha); \alpha, 0) \\
&= \frac{p}{2\sigma^2(1 - p)^2}\left[2(\alpha - r)\rho(\lambda) + \rho^2(\lambda)\right].
\end{aligned}
$$

Adding (B.2) and (B.3), we conclude that $\rho(\lambda) = O(\lambda^{2/3})$. In particular, $\lim_{\lambda \downarrow 0}(\rho(\lambda))/\lambda = \infty$.

PROOF OF THEOREM B.1. We observe first that the convex function $g : [0, A^{p-1}(p)/2] \mapsto [0, \infty)$ defined by $g(x) \triangleq (1 - p)(A^{p-1}(p) - x)^{1/(p-1)}$ has derivative $g'(x) = (A^{p-1}(p) - x)^{(2-p)/(p-1)}$ bounded below by $g'(0) = A^{2-p}(p)$ and above by $g'(A^{p-1}(p)/2) = 2^{(2-p)/(1-p)}A^{2-p}(p)$. The mean value theorem implies that for $0 \le x \le A^{p-1}(p)/2$ [recall (2.9)],

(B.4)
$$
\begin{aligned}
\tilde{U}(A^{p-1}(p) - x) &= \frac{1}{p}(A^{p-1}(p) - x)g(x) \\
&= \frac{1}{p}(A^{p-1}(p) - x)[(1 - p)A(p) + xg'(\xi(x))],
\end{aligned}
$$

where $\xi(x)$ takes values in $[0, A^{p-1}(p)/2]$ and

(B.5)
$$A^{2-p}(p) \le g'(\xi(x)) \le 2^{(2-p)/(1-p)}A^{2-p}(p).$$

*Step* 1. *Choice of the constants and the variables.* We define

$$m \triangleq \frac{(1 \wedge \eta)pA^{p-1}(p)}{4}, \qquad M \triangleq \frac{(1 \vee \eta)pA^{p-1}}{\theta_* \wedge (1 - \theta_*)}.$$

For the supersolution construction, we choose positive constants $\gamma_1$ and $\gamma_2$ to satisfy

(B.6)
$$\gamma_1 > 2^{(2-p)/(1-p)}\frac{A(p)}{\sigma^2\theta_*^2(1 - \theta_*)^2},$$

$$\gamma_2 < \left[2^{(3-2p)/(p-1)}p(1 - p)\sigma^2 m^2 A^{p-2}(p)\gamma_1\right]^{1/3}.$$

For the subsolution construction, we instead choose

(B.7)   $0 < \gamma_1 < 16A(p)/\sigma^2, \qquad \gamma_2 > \left[\dfrac{8p(1-p)\sigma^2 M^2 A^{p-1}(p)\gamma_1}{16A(p) - \gamma_1\sigma^2}\right]^{1/3}.$

We simplify notation by defining the variable $\varepsilon \triangleq \gamma_2\lambda^{2/3}/\gamma_1$.
For fixed $\lambda$, consider the quadratic functions of $\delta$:

$$f_1(\delta) \triangleq (2-p)\gamma_2\lambda\delta^2 - 2\gamma_2(1 - \lambda + \lambda\theta_*)\delta + p(A^{p-1}(p) - \varepsilon)\lambda^{1/3},$$

$$f_2(\delta) \triangleq (2-p)\gamma_2\eta\lambda\delta^2 - 2\gamma_2(1 - \eta\lambda\theta_*)\delta + \eta p(A^{p-1}(p) - \varepsilon)\lambda^{1/3}.$$

We have for sufficiently small $\lambda > 0$,

$$f_1\left(\frac{p(A^{p-1}(p) - \varepsilon)\lambda^{1/3}}{2\gamma_2(1 - \lambda + \lambda\theta_*)}\right) > 0, \qquad f_1\left(\frac{p(A^{p-1}(p) - \varepsilon)\lambda^{1/3}}{\gamma_2(1 - \lambda + \lambda\theta_*)}\right) < 0.$$

There is a number $\delta_1$ between these two arguments satisfying $f_1(\delta_1) = 0$. In particular, $\delta_1$ depends on $\lambda$ and satisfies

(B.8)                  $\dfrac{m}{\gamma_2}\lambda^{1/3} \le \delta_1 \le \dfrac{M}{\gamma_2}\lambda^{1/3},$

provided that $\lambda$ is small enough to ensure that $\varepsilon \le A^{p-1}(p)/2$. Similarly, if $\lambda$ is small enough that $\eta\lambda \le 1$ and $\varepsilon \le A^{p-1}(p)/2$, then there is a number $\delta_2$, depending on $\lambda$, which satisfies $f_2(\delta_2) = 0$ and

(B.9)                  $\dfrac{m}{\gamma_2}\lambda^{1/3} \le \delta_2 \le \dfrac{M}{\gamma_2}\lambda^{1/3}.$

We choose $\lambda$ small enough to ensure that $z_1 \triangleq \theta_* - \delta_1$ and $z_2 \triangleq \theta_* + \delta_2$ both lie in $(0, 1)$.

*Step 2. Definition of the super/subsolutions.* We define the continuous function $w : \mathscr{I} \mapsto [0, \infty)$ by

(B.10)   $w(z) = \begin{cases} \dfrac{1}{p}\left(A^{p-1}(p) - \varepsilon - \gamma_1\varepsilon\delta_1^2\right)\left(\dfrac{1 - \lambda + \lambda z}{1 - \lambda + \lambda z_1}\right)^p, \\[2mm] \qquad\qquad\qquad\qquad -\dfrac{1-\lambda}{\lambda} \le z \le z_1, \\[3mm] \dfrac{1}{p}\left(A^{p-1}(p) - \varepsilon - \gamma_1\varepsilon(z - \theta_*)^2\right), \\[2mm] \qquad\qquad\qquad\qquad z_1 \le z \le z_2, \\[3mm] \dfrac{1}{p}\left(A^{p-1}(p) - \varepsilon - \gamma_1\varepsilon\delta_2^2\right)\left(\dfrac{1 - \mu z}{1 - \mu z_2}\right)^p, \\[2mm] \qquad\qquad\qquad\qquad z_2 \le z \le \dfrac{1}{\mu}, \end{cases}$

where $\lambda$ is chosen small enough to ensure that $\varepsilon + \gamma_1 \varepsilon \delta_i^2 \le A^{p-1}(p)/2$, $i = 1, 2$. We have the derivative formula

$$(B.11) \qquad w'(z) = \begin{cases} \dfrac{\lambda p}{1 - \lambda + \lambda z} w(z), & -\dfrac{1 - \lambda}{\lambda} < z \le z_1, \\[2mm] -\dfrac{2\gamma_1 \varepsilon}{p}(z - \theta_*), & z_1 \le z \le z_2, \\[2mm] \dfrac{-\mu p}{1 - \mu z} w(z), & z_2 \le z < \dfrac{1}{\mu}. \end{cases}$$

The equations $f_1(\delta_1) = 0$ and $f_2(\delta_2) = 0$ guarantee that $w'$ is defined and continuous at $z_1$ and $z_2$. Finally,

$$(B.12) \quad w''(z) = \begin{cases} -\dfrac{\lambda^2 p(1 - p)}{(1 - \lambda + \lambda z)^2} w(z), & -\dfrac{1 - \lambda}{\lambda} < z < z_1, \\[2mm] -\dfrac{2\gamma_1 \varepsilon}{p}, & z_1 < z < z_2, \\[2mm] -\dfrac{\mu^2 p(1 - p)}{(1 - \mu z)^2} w(z), & z_2 < z < \dfrac{1}{\mu}. \end{cases}$$

The function $w$ is $C^2$ on $\mathscr{I}$ except at $z_1$ and $z_2$, and at these two points, the one-sided second derivatives exist and are the appropriate one-sided limits of the second derivative.

*Step 3. Comparison.* Note that

$$w(\theta_*) = \frac{1}{p}\big(A^{p-1}(p) - \varepsilon\big) = \frac{1}{p}A^{p-1}(p) - \frac{\gamma_2}{\gamma_1}\lambda^{2/3}.$$

Theorem B.1 will follow from Comparison Theorem A.1 once we show that $w$ is a supersolution to (A.2) and (A.4) under condition (B.6) and $w$ is a subsolution under condition (B.7). Now $w$ was constructed to satisfy (A.4), so it remains only to examine the inequality versions of (A.2) in the three intervals $(-(1 - \lambda)/\lambda, z_1)$, $(z_1, z_2)$ and $(z_2, 1/\mu)$.

*Step* 3(i) $[-(1 - \lambda)/\lambda < z < z_1]$. We have $\lambda p w(z) - (1 - \lambda + \lambda z)w'(z) = 0$, which gives us the *subsolution inequality*

$$(B.13) \qquad \begin{aligned} \min\big\{ &\mathscr{D}w(z) - \bar{U}(pw(z) - zw'(z)), \\ &\mu p w(z) + (1 - \mu z)w'(z), \\ &\lambda p w(z) - (1 - \lambda + \lambda z)w'(z)\big\} \le 0, \end{aligned}$$

in the interval $(-(1 - \lambda)/\lambda, z_1)$. To verify the *supersolution inequality*

$$(\text{B.14}) \quad \min\{\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z)), \mu pw(z)$$
$$+ (1 - \mu z)w'(z), \lambda pw(z) - (1 - \lambda + \lambda z)w'(z)\} \geq 0,$$

we note that both $w$ and $w'$ are positive in $(-(1 - \lambda)/\lambda, z_1)$, so $\mu pw(z) + (1 - \mu z)w'(z) \geq 0$.

It remains only to show that under (B.6),

$$(\text{B.15}) \quad \mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z)) \geq 0, \quad -\frac{1 - \lambda}{\lambda} < z < z_1.$$

In this interval,

$$pw(z) - zw'(z) = \left(A^{p-1}(p) - \varepsilon - \gamma_1 \varepsilon \delta_1^2\right)$$
$$\times \left(\frac{1 - \lambda}{1 - \lambda + \lambda z_1}\right)\left(\frac{1 - \lambda + \lambda z}{1 - \lambda + \lambda z_1}\right)^{p-1},$$

and (B.14) and (B.15) imply

$$\tilde{U}(pw(z) - zw'(z))$$
$$= \left(\frac{1 - \lambda}{1 - \lambda + \lambda z_1}\right)^{p/(p-1)}\left(\frac{1 - \lambda + \lambda z}{1 - \lambda + \lambda z_1}\right)^{p}\tilde{U}\left(A^{p-1}(p) - \varepsilon - \gamma_1 \varepsilon \delta_1^2\right)$$
$$\leq \left(\frac{1 - \lambda}{1 - \lambda + \lambda z_1}\right)^{p(p-1)} w(z)\big[(1 - p)A(p)$$
$$+ 2^{(2-p)/(1-p)}A^{2-p}(p)\left(\varepsilon + \gamma_1 \varepsilon \delta_1^2\right)\big].$$

Now $-d_3(z)w''(z) \geq 0$, and if $-(1 - \lambda)/\lambda < z \leq 0$, then $-d_2(z)w'(z) \geq 0$ also. Thus, $-(1 - \lambda)/\lambda < z \leq 0$ implies

$$\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z))$$
$$\geq \left\{(1 - p)A(p)\left[1 - \left(\frac{1 - \lambda}{1 - \lambda + \lambda z_1}\right)^{p/(p-1)}\right] + \frac{1}{2}p(1 - p)\sigma^2\delta_1^2\right.$$
$$\left. - \left(\frac{1 - \lambda}{1 - \lambda + \lambda z_1}\right)^{p/(p-1)} 2^{(2-p)/(1-p)}A^{2-p}(p)\frac{\gamma_2}{\gamma_1}\lambda^{2/3}(1 + \gamma_1\delta_1^2)\right\}w(z)$$
$$\geq \left\{\left(\frac{1}{2}p(1 - p)\sigma^2\frac{m^2}{\gamma_2^2} - 2^{(2-p)/(1-p)}A^{2-p}(p)\frac{\gamma_2}{\gamma_1}\right)\lambda^{2/3} + o(\lambda^{2/3})\right\}w(z),$$

which is nonnegative for small $\lambda > 0$ because of the second part of (B.6). For $0 < z < z_1$, we have

$$\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z))$$

$$\geq \left\{(1-p)A(p)\left[1 - \left(\frac{1-\lambda}{1-\lambda+\lambda z_1}\right)^{p/(p-1)}\right] + \frac{1}{2}p(1-p)\sigma^2\delta_1^2\right.$$

$$-\frac{1}{4}(1-p)\sigma^2\theta_*\frac{\lambda p}{1-\lambda} - \left(\frac{1-\lambda}{1-\lambda+\lambda z_1}\right)^{p/(p-1)}$$

$$\left. \times 2^{(2-p)/(1-p)}A^{2-p}(p)\frac{\gamma_2}{\gamma_1}\lambda^{2/3}\left(1 + \gamma_1\delta_1^2\right)\right\}w(z),$$

and we conclude as before.

*Step* 3(ii) ($z_2 < z < 1/\mu$). This is completely analogous to the argument in Step 3(i).

*Step* 3(iii) ($z_1 < z < z_2$). We claim that in this interval, $\mu pw(z) + (1 - \mu z)w'(z) \geq 0$ and $\lambda pw(z) - (1 - \lambda + \lambda z)w'(z) \geq 0$ for all $\lambda > 0$ sufficiently small. We establish the latter inequality; the proof of the former uses the relation $\mu = \eta\lambda$ and is completely analogous.

Direct computation reveals that

$$\lambda pw(z) - (1 - \lambda + \lambda z)w'(z)$$

$$= \frac{\gamma_1\varepsilon\lambda}{p}\left[(2-p)z^2 + 2\left(p\theta_* - 1 - \theta_* + \frac{1}{\lambda}\right)z - 2\theta_*\frac{1-\lambda}{\lambda} - p\theta_*^2\right]$$

$$+ \lambda\left(A^{p-1}(p) - \varepsilon\right),$$

and this quadratic function in $z$ is minimized at $z = (-p\theta_* + 1 + \theta_* - 1/\lambda)/(2-p)$, which is negative for small $\lambda > 0$. Therefore,

$$\lambda pw(z) - (1 - \lambda + \lambda z)w'(z) \geq \lambda pw(z_1) - (1 - \lambda + \lambda z_1 w'(z_1))$$

$$= 0 \qquad \forall z \in (z_1, z_2).$$

It follows from these considerations that $w$ is a supersolution if and only if

(B.16)      $\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z)) \geq 0, \qquad z_1 < z < z_2,$

and $w$ is a subsolution if and only if the reverse inequality holds.

For $z_1 < z < z_2$, we have $pw(z) - zw'(z) = A^{p-1}(p) - x$, where $x \triangleq \varepsilon + \gamma_1\varepsilon(z - \theta_*)^2 - (2\gamma_1\varepsilon/p)z(z - \theta_*) = \varepsilon + o(\lambda^{2/3})$. It follows from (B.4) and (B.5) that

$$\tilde{U}(pw(z) - zw'(z))$$

$$= \frac{1}{p}\left(A^{p-1}(p) - x\right)\left[(1-p)A(p) + xg'(\xi(x))\right]$$

$$= (1-p)A(p)w(z) + \frac{1}{p}A^{p-1}(p)\varepsilon g'(\xi(x)) + o(\lambda^{2/3}).$$

Therefore,

$$
\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z))
$$

$$
= \frac{1}{2}(1-p)\sigma^2(z-\theta_*)^2\big(A^{p-1}(p) - \varepsilon - \gamma_1\varepsilon(z-\theta_*)^2\big)
$$

$$
- \frac{2\gamma_1\varepsilon}{p}(1-p)\sigma^2 z(1-z)(z-\theta_*)^2 + \frac{\gamma_1\varepsilon}{p}\sigma^2 z^2(1-z)^2
$$

(B.17)
$$
- \frac{1}{p}A^{p-1}(p)\varepsilon g'(\xi(x)) + o(\lambda^{2/3})
$$

$$
= \frac{1}{2}(1-p)\sigma^2(z-\theta_*)^2 A^{p-1}(p) + \frac{\gamma_2}{p}\sigma^2 z^2(1-z)^2\lambda^{2/3}
$$

$$
- \frac{\gamma_2}{p\gamma_1}A^{p-1}(p)g'(\xi(x))\lambda^{2/3} + o(\lambda^{2/3}).
$$

We see from (A.8), (A.9) and (A.5) that

$$
\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z))
$$

$$
\leq \left[\frac{1}{2}(1-p)\sigma^2 A^{p-1}(p)\frac{M^2}{\gamma_2^2} + \frac{\gamma_2\sigma^2}{16p} - \frac{\gamma_2}{p\gamma_1}A(p)\right]\lambda^{2/3} + o(\lambda^{2/3}).
$$

Under condition (B.7), this expression is negative for all sufficiently small $\lambda > 0$, and thus $w$ is a subsolution to (A.2) and (A.4).

For the final step of the proof, we assume (B.6) and establish (B.16). Note that for $z_1 < z < z_2$, we have

$$
z(1-z) = [z - \theta_* + \theta_*][(1-\theta_*) - (z - \theta_*)] = \theta_*(1-\theta_*) + o(1).
$$

Therefore, (B.17) and (B.5) imply

$$
\mathscr{D}w(z) - \tilde{U}(pw(z) - zw'(z))
$$

$$
\geq \frac{\gamma_2}{p}\left[\sigma^2\theta_*^2(1-\theta_*)^2 - \frac{1}{\gamma_1}2^{(2-p)/(1-p)}A(p)\right]\lambda^{2/3} + o(\lambda^{2/3}),
$$

which is positive for sufficiently small $\lambda > 0$. □

REMARK B3. We have shown that the value function decreases like $\lambda^{2/3}$ near $\lambda = 0$. The definition of $z_1$ and $z_2$ used in the construction of $w$ in the proof of Theorem B.1 suggests that the width of the no-transaction interval $\theta_1 - \theta_2$ should be $O(\lambda^{1/3})$ [see (8.13) for the definition of $\theta_1$ and $\theta_2$]. This is the order of the width of the no-transaction interval obtained in [25] for their transaction cost problem without intermediate consumption. It is also consistent with the numerical results of [7] and [14], both of which found a rapid opening of the no-transaction interval as $\lambda$ increases from zero.

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