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MINIMAL POSITIONS IN A BRANCHING RANDOM WALK

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We consider a branching random walk on the real line, with mean family size greater than 1. Let B_n denote the minimal position of a member of the nth generation. It is known that (under a weak condition) there is a finite constant γ, defined in terms of the distributions specifying the process, such that as $n \to \infty$, we have $B_n = \gamma n + o(n)$ a.s. on the event S of ultimate survival. Our results here show that (under appropriate conditions), on S the random variable B_n is strongly concentrated and the $o(n)$ error term may be replaced by $O(\log n)$.

1. Introduction. We shall consider a branching random walk on the real line, as for example in [4–6]. An initial ancestor, who forms the zeroth generation, is at the origin. She has children, the first generation, whose displacements from the origin correspond to a point process Z on the line. Thus Z is a random locally finite counting measure on the real line. These children then behave like independent copies of the ancestor relative to their initial positions, their children form the second generation and so on. When all displacements are nonnegative we may interpret these displacements as times, and we obtain a Crump–Mode–Jagers age-dependent branching process (see [12]).

Denote the position of an individual v in the process by $B(v)$. Let $Z_n(t)$ denote the number of individuals v in the nth generation with $B(v) \le t$, and let $Z_n = \sup_t Z_n(t)$. [We shall consider only cases when $Z_n(t)$ is a.s. finite for all t, though Z_n may be infinite.] The event S of ultimate survival is the event that $Z_n > 0$ for all positive integers n. Let $m = EZ_1$, the mean number of children of an individual. We shall restrict our attention to the case that $1 < m \le \infty$, so that the event S has positive probability. We are interested in B_n, the infimum of the positions of the members of the nth generation. Note that when we have an age-dependent branching process, then B_n is the first time of a birth in the nth generation. We set $B_n = +\infty$ if $Z_n = 0$: we shall impose conditions that ensure that $B_n > -\infty$ if $Z_n > 0$. Let $z_{n1}, z_{n2}, \dots$ be an enumeration of the positions of the members of the nth generation. Thus $Z_n(t) = |\{r\colon z_{nr} \le t\}|$ and $B_n = \inf_r\{z_{nr}\}$.

Let $F(t) = EZ_1(t)$ and let $\alpha = \inf\{t\colon F(t) > 0\}$. Observe that if α is a finite number rather than $-\infty$, then a simple translation brings us back to the case of age-dependent branching processes. Our first theorem is for the case that

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α is finite, and it concerns the concentration on S of the random variables B_n around the median b_n: we shall see that the sequence B_n is "exponentially tight." In [13, 15] it is shown that if α is finite and $P(S) = 1$, then under a moment condition the sequence B_n is tight, and if displacements are bounded, then an exponential bound as in (1) holds for any $x \geq 0$. [If displacements are bounded, then we may clearly drop the restriction in (1) that $x \leq n$.] The following theorem extends these results.

THEOREM 1. *Let $m > 1$, let α be finite and let b_n be the median of B_n on S, say $b_n = \inf\{t : P(B_n \leq t|S) \geq \frac{1}{2}\}$. Then there exist constants c and $\delta > 0$ such that for each positive integer n and each $0 \leq x \leq n$ we have*

$$P(|B_n - b_n| > x \mid S) < ce^{-\delta x}. \tag{1}$$

Suppose that $m > 1$ and α is finite. Then it follows from Theorem 1 and a Borel–Cantelli lemma that there is a constant c such that a.s. on S, for n sufficiently large we have $|B_n - b_n| \leq c \log n$. Also, observe that $P(B_n = \alpha n)$ is the probability that the embedded Galton–Watson branching process formed by the children with displacement exactly α survives at least to generation n. Hence, if $F(\alpha) > 1$, then by Theorem 1 we have $b_n = \alpha n + O(1)$ (see [3, 13, 15, 24]). Similarly, if $F(\alpha) = 1$, then $P(B_n = \alpha n) = \Omega(1/n)$, and so $b_n = \alpha n + O(\log n)$: under further conditions, it may be shown that $b_n = \alpha n + \Theta(\log \log n)$ [8, 15].

The O, o notation is presumably familiar. Following the practice in computer science, we write also $x_n = \Omega(y_n)$ if $y_n = O(x_n)$, and $x_n = \Theta(y_n)$ if both $x_n = O(y_n)$ and $x_n = \Omega(y_n)$.

We shall be more interested in the case when $F(\alpha) < 1$ (and α need not be finite). Following [23, 4] we define functions $\phi(\theta)$ and $\mu(a)$, and the constant γ. For $\theta \geq 0$ let

$$\phi(\theta) = E\left(\sum_r \exp(-\theta z_{1r})\right),$$

where the sum is over the children r of the initial ancestor. Note that $0 \leq \phi(\theta) \leq \infty$. If $\phi(\theta)$ is finite for some $\theta \geq 0$, then for any real a let

$$\mu(a) = \inf\{e^{\theta a}\phi(\theta) : \theta \geq 0\}$$

and define the *time constant* γ by

$$\gamma = \inf\{a : \mu(a) \geq 1\}.$$

Suppose that $m > 1$ and that $\phi(\theta)$ is finite for some $\theta > 0$. Hammersley [19], Kingman [23] and Biggins [4] (see also [21]) have shown that, under these conditions, the constant γ is finite, and $B_n/n \to \gamma$ a.s. on S as $n \to \infty$. Our second theorem is a refinement of this result: it describes the behaviour of B_n more precisely under further conditions (and gives us back the original result by a standard truncation argument). Suppose that $F(\alpha) < 1$. Then (see Lemma 2) it follows that $\mu(\gamma) = 1$ and there is a unique finite $\tau > 0$ such that $e^{\tau\gamma}\phi(\tau) = 1$. We shall assume further that $\phi(\theta)$ is finite in some neighbourhood of τ. For an equivalent condition see (3.5) of [5]. In the case of an age-dependent branching process (that is, $\alpha \geq 0$), these conditions must hold if $F(\alpha) < 1$ and $1 < \phi(\theta) < \infty$ for some $\theta > 0$ (see (2.3) of [23]).

Let us call the special case in which the initial ancestor's children must appear all at a common (random) position a *common* branching random walk: this is a generalisation of a Bellman–Harris age-dependent branching process.

THEOREM 2. *Let $m > 1$, let $F(\alpha) < 1$, let $\tau > 0$ be such that $e^{\tau\gamma}\phi(\tau) = 1$ and suppose that $\phi(\theta)$ is finite in some neighbourhood of τ.*

(a) *There are constants $c_1 > 0$ and $\delta > 0$ such that for each sufficiently large integer n and each $x \geq 0$ we have*

$$P(B_n \leq \gamma n + c_1 \log n - x) < e^{-\delta x}.$$

(b) *Assume that at least one of the following conditions holds:* (i) *α is finite and $E(Z_1(n)^2)$ is polynomially bounded [which is of course true if $E(Z_1^2)$ is finite];* or (ii) *$Z_1 \leq s$ a.s. for some constant s;* or (iii) *we have a common branching random walk with $E(Z_1^2)$ finite. Then there are constants $c_2 > 0$ and $\delta > 0$ such that for each positive integer n and each $0 \leq x \leq n$ we have*

$$P(B_n \geq \gamma n + c_2 \log n + x \mid S) < e^{-\delta x}.$$

It would of course be nice if we could take $c_1 = c_2$. It is interesting to note that Biggins [5] has shown that, under conditions as described, $B_n - \gamma n \to \infty$ a.s. as $n \to \infty$.

Theorem 2 (a) will follow from a first moment argument, much as in [23, 4–7]. Bramson [9] refined the approach elegantly to obtain the "correct" constant c for branching brownian motion—we do not manage to do this here. Our main effort is in proving part (b): we use a second moment argument, following the lead of [9, 16]. The rest of this paper is devoted to proving Theorems 1 and 2. The paper [9] on branching brownian motion may serve as a model for attempts to improve the present work.

2. Proof of Theorem 1. Throughout, we let $p = 1 - q = P(S)$, where S is the event of ultimate survival. The following lemma is in fact true without any restriction on m, but we are interested only in the case when $m > 1$ and thus $q < 1$.

LEMMA 1. *Let $m > 1$. Then for any sufficiently large a and any sufficiently small $\eta > 1$ we have*

$$P(Z_n(an) < \eta^n) = q + o(\eta^{-n}).$$

PROOF. Let $0 < \varepsilon \leq p/3$. For $b > 0$ let $Z^{(b)}$ denote the corresponding Galton–Watson branching process, where we include only individuals with displacements at most b, and let $S^{(b)}$ denote the event that this truncated process survives. We may choose b sufficiently large that $EZ_1^{(b)} = F(b) > 1$ and $P(S^{(b)}) \geq p - \varepsilon$.

Take $a = b$. Then from a standard result on Galton–Watson branching processes [1] (see also Lemma 4 of [14]), there exists $\eta_1 > 1$ such that

$$P(Z_n(an) \geq \eta_1^n) \geq P(Z_n^{(a)} \geq \eta_1^n) \geq P(S^{(a)}) - \varepsilon \geq p - 2\varepsilon \geq p/3.$$

Now consider the process $\bar{Z}^{(a)}$ of fecund individuals in $Z^{(a)}$ as in [1, 23]. Notice that $E(\bar{Z}_1^{(a)} \mid S) = F(a) > 1$, and thus $P(\bar{Z}_1^{(a)} \geq 2 \mid S) > 0$. It follows by binomial tail bounds [11] that for some $0 < \delta_1 < 1$,

$$P(Z_n(an) < \delta_1 n \mid S) \leq P(\bar{Z}_n^{(a)} < \delta_1 n \mid S) = o(\delta_1^n),$$

and so

$$P(Z_n(an) < \delta_1 n) \leq q + o(\delta_1^n).$$

Hence

$$\begin{aligned} P(Z_{2n}(2an) < \eta_1^n) &\leq P(Z_n(an) < \delta_1 n) + P(Z_n(an) < \eta_1^n)^{\delta_1 n} \\ &\leq q + o(\delta_1^n) + (1 - p/3)^{\delta_1 n}. \end{aligned}$$

Thus there exist $0 < \delta_2 < 1$ and $\eta_2 > 1$ such that

$$P(Z_n(an) < \eta_2^n) \leq q + o(\delta_2^n).$$

Finally note that

$$P(Z_n(an) < \eta_2^n \mid \bar{S}) \geq P(Z_n = 0 \mid \bar{S}) = 1 - o(\delta_3^n)$$

for some $0 < \delta_3 < 1$, and so

$$P(Z_n(an) < \eta_2^n) \geq q + o(\delta_3^n). \qquad \square$$

PROOF OF THEOREM 1. We know by Lemma 1 that there exist constants a, $c > 0$ and $\eta > 1$ such that for each $k = 1, 2, \ldots,$

$$P(Z_k(ak) < \eta^k) < q + c\eta^{-k}.$$

Further we may take $a \geq |\alpha|$. Choose δ with $0 < \delta < 1$ such that $\delta\eta^{1/2a} > 1$. Note that there exists x_0 such that for any $x \geq x_0$, if $k = \lfloor x/2a \rfloor$, then

$$c\eta^{-k} + \exp(-p\delta^x \eta^k) < \delta^x.$$

For $n = 1, 2, \ldots$ and $x > 0$ let

$$b = b(n, x) = \inf\{t \geq 0 : P(B_n \leq t \mid S) \geq \delta^x\}.$$

Thus $P(B_n < b \mid S) \leq \delta^x$ and

$$P(B_n > b) \leq 1 - p\delta^x \leq \exp(-p\delta^x).$$

Now let $x \geq x_0$, let $k = \lfloor x/2a \rfloor$, and note that $b + x + k\alpha \geq ak + b$, and so

$$\begin{aligned} P(B_n > b + x) &\leq P(B_{n+k} > b + x + k\alpha) \\ &\leq P(B_{k+n} > ak + b) \\ &\leq P(Z_k(ak) < \eta^k) + P(B_n > b)^{\eta^k} \\ &\leq q + c\eta^{-k} + \exp(-p\delta^x \eta^k) \\ &< q + \delta^x. \end{aligned}$$

Suppose temporarily that $q > 0$, and recall that for some $0 < \delta_1 < 1$ we have $P(Z_n > 0 \mid \bar{S}) = o(\delta_1^n)$. Then

$$P(B_n > b + x) \geq pP(B_n > b + x \mid S) + q(1 - o(\delta_1^n)).$$

Thus $P(B_n > b + x \mid S) = O(\delta^x) + o(\delta_1^n)$.

We now know that (whether $q > 0$ or not) there are constants x_0, c_0, c_1 and $0 < \delta$, $\delta_1 < 1$ such that, for each $n = 1, 2, \ldots$ and all $x \geq x_0$,

$$P(b(n,x) \leq B_n \leq b(n,x) + x \mid S) \geq 1 - c_0\delta^x - c_1\delta_1^n.$$

Let $x_1 \geq x_0$ be such that $c_0\delta^{x_1} \leq \frac{1}{5}$ and let $n_1 > 0$ be such that $c_1\delta_1^{n_1} \leq \frac{1}{5}$. Then for all $n \geq n_1$ and $x \geq x_1$,

$$b(n,x) < b_n < b(n,x) + x,$$

and then

$$\begin{aligned} P(|B_n - b_n| \leq x \mid S) &\geq P(b(n,x) \leq B_n \leq b(n,x) + x \mid S) \\ &\geq 1 - c_0\delta^x - c_1\delta_1^n. \end{aligned}$$

The theorem now follows easily. □

3. Proof of Theorem 2(a).

We shall use two lemmas.

LEMMA 2. *Let $m > 1$, let $F(\alpha) < 1$ and let $\phi(\theta)$ be finite for some $\theta > 0$. Then $\mu(\gamma) = 1$, there exists a unique finite $\tau > 0$ such that $e^{\tau\gamma}\phi(\tau) = 1$, and for any positive integer n and any x,*

$$EZ_n(\gamma n + x) \leq e^{\tau x}.$$

PROOF. For the first two parts of the lemma, see [23, 4]. Note also from these references that for any $\theta \geq 0$ such that $\phi(\theta)$ is finite, $E\left(\sum_r \exp(-\theta z_{nr})\right) = \phi(\theta)^n$. Hence, for any such θ,

$$EZ_n(an) \leq E\left(\sum_r \exp\theta(an - z_{n,r})\right) = (\exp(\theta a)\phi(\theta))^n,$$

and so $EZ_n(an) \le \mu(a)^n$. Hence

$$EZ_n(\gamma n + x) \le \mu\left(\gamma + \frac{x}{n}\right)^n \le e^{\tau x},$$

as required. □

LEMMA 3. *Let $m > 1$, let $F(\alpha) < 1$, let $\tau > 0$ be such that $e^{\tau\gamma}\phi(\tau) = 1$ and suppose that $\phi(\theta)$ is finite in some neighbourhood of τ. Then there exist constants $0 < c_1 \le c_2$ such that for any $b \ge 0$ and for any positive integer n,*

$$c_1 n^{-1/2} \le EZ_n(\gamma n + b) \le c_2 n^{-1/2}(1+b)\, e^{\tau b}.$$

The preceding lemma is quite loose, but it is in suitable form for our present purposes.

PROOF. We may argue as in [2, 4, 5, 23] to show the following. There is a random variable X with mean 0, variance $\sigma > 0$ and finite moment generating function in a neighbourhood of the origin, such that we have

$$EZ_n(\gamma n + b) = \int 1_{\{x \ge -b/\sigma n^{1/2}\}} \exp(-\tau\sigma n^{1/2} x)\, dG_n(x),$$

where G_n is the distribution function of $\sigma^{-1} n^{-1/2}$ times the sum of n independent random variables each distributed like X. We may now use, for example, the Berry–Esséen central limit theorem (see, e.g., [17]). □

PROOF OF THEOREM 2(a). Let $0 < c < 1/2\tau$. We shall see that we can take $\delta = \tau/2$. If $0 \le x \le 2c \log n$, then by replacing x by $x/2$ and using Markov's inequality and then Lemma 3, we find

$$\begin{aligned} P(B_n \le \gamma n + c\log n - x) &\le E\, Z_n(\gamma n + c \log n - x/2) \\ &\le c'(n^{-(1/2-\tau c)} \log n)\, e^{-\tau x/2}, \end{aligned}$$

for some constant c'. If $x \ge 2c\log n$, then by Lemma 2,

$$P(B_n \le \gamma n + c\log n - x) \le \exp(-(x - c\log n)) \le \exp(-\tau x/2),$$

which completes the proof. □

4. Leading sequences. In order to be able to apply the second moment method in the next section we shall focus on "leading" individuals. Call a vector $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$ *leading* if for each $j = 1, \ldots, n-1$,

$$\sum_{i=1}^{j} x_i \ge \frac{j}{n} \sum_{i=1}^{n} x_i.$$

LEMMA 4. *Let the random variables $X_1, X_2, \ldots, X_n$ be exchangeable. Then*

$$P((X_1, \ldots, X_n) \text{ is leading}) \geq \frac{1}{n}.$$

PROOF. Given a vector $\mathbf{z} = (z_1, \ldots, z_n) \in R^n$, let $\mathbf{z}^{(k)}$ denote the cyclically shifted vector $(z_k, z_{k+1}, \ldots, z_n, z_1, \ldots, z_{k-1})$. Now fix any vector $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$. To prove the lemma it suffices to show that at least one of the n vectors $\mathbf{x}^{(k)}$ is leading.

Let $s = \sum_{i=1}^n x_i$ and let $\mathbf{y} = (y_1, \ldots, y_n)$, where $y_i = x_i - s/n$. Note that $\sum_{i=1}^n y_i = 0$. Now $\mathbf{x}^{(k)}$ is leading if and only if $\mathbf{y}^{(k)}$ satisfies $\sum_{i=1}^j (\mathbf{y}^{(k)})_i \geq 0$ for each $j = 1, \ldots, n$; and it may be seen that this happens if and only if k minimises over all $j \in \{1, \ldots, n\}$ the sum $\sum_{i=1}^{j-1} y_i$ (see [18], Section XII.6 Lemma 2, and [16]). □

Given an individual $v = v_n$ in the nth generation, let v_j denote the ancestor in generation j of v_n and let the corresponding *positions sequence* be $(B(v_1), B(v_2), \ldots, B(v_n))$ and *displacements sequence* be $(B(v_1), B(v_2) - B(v_1), \ldots, B(v_n) - B(v_{n-1}))$. Call v *leading* if its displacements sequence is leading, that is, if the positions sequence satisfies $B(v_j) \geq (j/n)B(v)$ for each $j = 1, \ldots, n-1$. Let $Z_n^*(t)$ denote the number of leading individuals v in generation n with $B(v) \leq t$. Thus $Z_n^*(t) \leq Z_n(t)$: our plan is to give a lower bound for $Z_n^*(t)$.

LEMMA 5. *For any positive integer n and any real t,*

$$E(Z_n^*(t)) \geq \frac{1}{n} E(Z_n(t)).$$

PROOF. Fix a positive integer n and a real t. Let N be a positive integer. Consider the altered branching random walk in which individuals are sterilised so that they produce at most N offspring. Then enough further dummy offspring are created with displacement $+\infty$ to make up to exactly N. Let us use the notation $\hat{Z}_n(t)$ and so on to refer to this new process. Now there are exactly N^n individuals in generation n. Pick one uniformly at random, say v_n. Then its displacements sequence $(X_1, \ldots, X_n)$ consists of independent and identically distributed random variables. Hence by the last lemma,

$$\begin{aligned} E(Z_n^*(t)) &\geq E(\hat{Z}_n^*(t)) \\ &= N^n P((X_1, \ldots, X_n) \text{ is leading}, \sum_{i=1}^n X_i \leq t) \\ &\geq \frac{1}{n} N^n P\Big(\sum_{i=1}^n X_i \leq t\Big) \\ &= \frac{1}{n} E(\hat{Z}_n(t)). \end{aligned}$$

Now let $N \to \infty$. $\square$

It is easy to see that equality holds in Lemma 4 if, say, the joint distribution is continuous. Thus if displacements are continuously distributed, then we also have equality in Lemma 5 [noting that $E\hat{Z}_n^*(t) \to EZ_n^*(t)$ as $N \to \infty$].

5. Proof of Theorem 2(b). The following lemma is a basic tool in the study of random graphs, where its use is called the second moment method. It follows easily from the Cauchy–Schwarz inequality.

LEMMA 6. *If the random variable X has finite mean $EX \geq 0$, then*

$$P(X > 0) \geq (EX)^2/E(X^2).$$

To use this inequality for $Z_n^*(t)$ we need an upper bound on the second moment. The next three lemmas handle the three cases in Theorem 2(b).

LEMMA 7. *Suppose that α is finite, let a be any real and let $b = a + |\alpha|$. Then for any positive integer n,*

$$E(Z_n^*(an)^2) \leq EZ_n^*(an) + E(Z_1(bn)^2)E(Z_{n-1}(an-\alpha))\sum_{j=0}^{n-1} E(Z_j(aj)).$$

PROOF. Let A denote the random set of leading individuals v in generation n with $B(v) \leq an$, so that $Z_n^*(an) = |A|$. Let $D(v)$ denote the displacement of a possible individual v from its parent. In the following sums the index j will run from 0 to $n-1$, w will denote a possible individual in generation j, x and y will denote possible offspring of w with the position of x at most that of y and u and v will denote possible descendants in generation n of x and y, respectively. We have

$$\begin{aligned} &E(|A|^2) - E(|A|) \\ &\quad = 2\sum_j \sum_w \sum_x \sum_u \sum_y \sum_v P(u \in A,\ v \in A) \\ &\quad = 2\sum_j \sum_w \sum_x \sum_u \sum_y P(u \in A,\ D(y) \leq bn) \\ &\qquad \times \sum_v P(v \in A \mid u \in A,\ D(y) \leq bn). \end{aligned}$$

To see this last equation, observe that if the descendant v of y is in A, then we must have

$$D(y) \leq B(v) - \alpha(n-1) \leq an - \alpha(n-1) \leq bn.$$

Now let v be a possible descendant of y, as in the last sum in the preceding equation, and suppose that $v \in A$. Then $B(y) \geq ((j+1)/n)B(v)$ and so the total displacement $B(v) - B(y)$ of v from y satisfies

$$B(v) - B(y) \le \frac{n-j-1}{n} B(v) \le (n-j-1)a.$$

Thus we may bound the last sum as

$$\sum_v P(v \in A \mid u \in A,\ D(y) \le bn) \le E(Z_{n-j-1}(a(n-j-1))).$$

Hence $E(|A|^2) - E(|A|)$ is at most

$$2\sum_j E(Z_{n-j-1}(a(n-j-1))) \sum_w \sum_x \sum_u \sum_y P(u \in A,\ D(y) \le bn).$$

Now let W denote the number of children of w with displacement from w at most bn, so that W has the same distribution as $Z_1(bn)$. Also suppose for convenience that x and y run over $1, 2, \dots$ with $x < y$. Then the foregoing bound may be written as

$$(2)\qquad 2\sum_j E(Z_{n-j-1}(a(n-j-1))) \sum_w \sum_{y=2}^{\infty} P(W \ge y) \sum_{x=1}^{y-1} \sum_u P(u \in A \mid W \ge y).$$

Let $f(w)$ denote the expected number of descendants u of w in generation $n-1$ with $B(u) \le an - \alpha$. Then for each j, w, y, x we have

$$\sum_u P(u \in A \mid W \ge y) \le f(w),$$

because the displacement of x from w is at least α. Hence

$$\begin{aligned}
&E(|A|^2) - E(|A|)\\
&\quad \le 2\sum_j E(Z_{n-j-1}(a(n-j-1))) \sum_w \sum_{y=2}^{\infty} (y-1) P(W \ge y) f(w)\\
&\quad = 2\sum_j E(Z_{n-j-1}(a(n-j-1))) \sum_{y=2}^{\infty} (y-1) P(W \ge y) E Z_{n-1}(an - \alpha)\\
&\quad \le \sum_j E(Z_j(aj))\ E(Z_1(bn)^2) E Z_{n-1}(an - \alpha),
\end{aligned}$$

because

$$2\sum_{y=2}^{\infty} (y-1) P(W \ge y) \le E(W^2) = E(Z_1(bn)^2) \qquad (\le \infty). \qquad \square$$

LEMMA 8. *Assume that* $Z_1 \le s$ *a.s. for some constant* s. *Then for any positive integer* n *and any* a,

$$E(Z_n^*(an)^2) \le E Z_n^*(an)\left(1 + (s-1) \sum_{j=0}^{n-1} E Z_j(aj)\right).$$

PROOF. We may assume that each individual has s possible children, presented in a random order. We use notation as in the proof of the last lemma, except that we take x and y distinct but in either order. Then we see as before that

$$\begin{aligned}
&E(|A|^2) - E(|A|) \\
&= \sum_j \sum_w \sum_x \sum_u \sum_y P(u \in A) \sum_v P(v \in A \mid u \in A) \\
&\le \sum_j E(Z_{n-j-1}(a(n-j-1))) \sum_w \sum_x \sum_u \sum_y P(u \in A) \\
&\le (s-1) \sum_j E(Z_{n-j-1}(a(n-j-1))) \sum_w \sum_x \sum_u P(u \in A) \\
&= (s-1) \sum_j E(Z_{n-j-1}(a(n-j-1))) E|A|
\end{aligned}$$

and the lemma follows. □

Given a common branching random walk, let D denote a typical displacement of (all) the children of a given individual. Observe that $\phi(\theta) = EZ_1\, E(e^{-\theta D})$ for any $\theta \ge 0$.

LEMMA 9. *Suppose that we have a common branching random walk. Let $m > 1$, let $F(\alpha) < 1$, let $\phi(\theta)$ be finite for some $\theta > 0$ and assume that $E\, Z_1^2$ is finite. Then for any positive integer n,*

$$E(Z_n^*(\gamma n)^2) \le EZ_n^*(\gamma n) + EZ_1^2\, e^{\tau\gamma}\, Ee^{-\tau D} \sum_{j=0}^{n-1} E(Z_j(\gamma j)).$$

PROOF. Minor changes in the early parts of the proof of Lemma 7 lead to the inequality (2) given there, except that now $a = \gamma$ and W denotes the total number of children of w. Let $f(w, x, d)$ denote the expected number of descendants u of the possible child x of w such that $B(u) \le \gamma n$, given that w has at least x children and the displacement of child x from w is d. Observe that this does not depend on x, and let $f(w) = Ef(w, x, D)$. Thus for each j, w, y, x we have

$$\sum_u P(u \in A \mid W \ge y) \le \sum_u P(B(u) \le \gamma n \mid W \ge y) = f(w).$$

So we have

$$\begin{aligned}
&E(|A|^2) - E(|A|) \\
&\le 2 \sum_j E(Z_{n-j-1}(\gamma(n-j-1))) \sum_w \sum_{y=2}^{\infty} P(W \ge y)(y-1)\, f(w) \\
&\le EZ_1^2 \sum_j E(Z_{n-j-1}(\gamma(n-j-1))) \sum_w f(w),
\end{aligned}$$

as in the proof of Lemma 7. However,

$$\sum_w f(w) = E\sum_w f(w,1,D) = EZ_{n-1}(\gamma n - D) \le Ee^{\tau(\gamma - D)}$$

by Lemma 2, and this completes the proof. □

PROOF OF THEOREM 2(b). If case (i) holds, let $j \ge \frac{1}{2}$ be such that

$$E(Z_1((\gamma + |\alpha|)n)^2) = O(n^j).$$

Then in each case, by Lemmas 3, 7, 8 and 9 we have

$$E(Z_n^*(\gamma n)^2) = O(n^j).$$

Hence by Lemmas 6, 5 and 3,

$$\begin{aligned} P(Z_n(\gamma n) > 0) &\ge P(Z_n^*(\gamma n) > 0) \\ &\ge \frac{(EZ_n^*(\gamma n))^2}{E(Z_n^*(\gamma n)^2)} \\ &= \Omega\left(\frac{n^{-2}(EZ_n(\gamma n))^2}{n^j}\right) \\ &= \Omega(n^{-(j+3)}). \end{aligned}$$

Thus

$$P(B_n > \gamma n) \le 1 - c_1 n^{-(j+3)} \le \exp(-c_1 n^{-(j+3)})$$

for some constant $c_1 > 0$. We may now argue much as in the proof of Theorem 1. Let $a > |\gamma|$ and $\eta > 1$ be as in Lemma 1. Let the constant c_2 be $2a(j+4)/(\log \eta)$ and let $k = \lfloor x/2a \rfloor$. Note that $\gamma n + x \ge ak + \gamma(n-k)$ and that $n^{-(j+3)}\eta^k \ge n/\eta$ if $x \ge c_2 \log n$. Hence

$$\begin{aligned} P(B_n \ge \gamma n + x) &\le P(Z_k(ak) < \eta^k) + P(B_{n-k} > \gamma(n-k))^{\eta^k} \\ &\le q + \eta^{-k} + \exp(-c_1 n^{-(j+3)}\eta^k) \\ &\le q + \eta^{-k} + \exp(-(c_1/\eta)n), \end{aligned}$$

if $x \ge c_2 \log n$. However,

$$P(B_n \ge \gamma n + x \mid \bar{S}) \ge P(Z_n = 0 \mid \bar{S}) = 1 - o(e^{-\delta_1 n})$$

for some $\delta_1 > 0$. It follows that there exist constants c_3 and $\delta_2 > 0$ such that for $x \ge c_2 \log n$,

$$P(B_n \ge \gamma n + x \mid S) \le c_3(e^{-\delta_2 x} + e^{-\delta_2 n}),$$

and part (b) of the theorem follows. □

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