The Annals of Applied Probability
1995, Vol. 5, No. 3, 757–767

A PROOF OF DASSIOS' REPRESENTATION OF THE α-QUANTILE OF BROWNIAN MOTION WITH DRIFT

BY P. EMBRECHTS, L. C. G. ROGERS AND M. YOR

ETH-Zentrum, University of Bath and Université Paris VI

An explanation of a remarkable identity in law, due to A. Dassios, concerning the α-quantile of Brownian motion with drift is given with the help of Bertoin's rearrangement of positive and negative excursions for Brownian motion with drift.

1. Introduction. Let $(B_t, t \geq 0)$ be a real-valued Brownian motion (BM) starting from 0. More generally, for $\mu \in \mathbb{R}$, $(B_t + \mu t, t \geq 0)$ is a Brownian motion with drift μ.

Let P^μ denote the law of $(B_t + \mu t, t \geq 0)$ on the canonical space $(C(\mathbb{R}_+, \mathbb{R}), \mathscr{F}_\infty)$, with $(X_t, t \geq 0)$, the process of coordinates, $\mathscr{F}_t = \sigma\{X_s, s \leq t\}$ its family of σ-fields and $\mathscr{F}_\infty = \sigma\{X_s, s \geq 0\}$.

Now, let $0 \leq \alpha \leq 1$, and define

$$M(\alpha, t)(\omega) = \inf\left\{x: \int_0^t ds\, 1_{(X_s(\omega) \leq x)} > \alpha t\right\}.$$

For fixed ω, $M(\alpha, t)(\omega)$ is the α-quantile of the function $s \to X_s(\omega)$, $s \leq t$, considered as a random variable on the probability space $([0, t]; ds/t)$, equipped with the Borel σ-field.

The aim of this paper is to give a proof of the following striking identity in law.

THEOREM 1 (Dassios [6]). *For every $\mu \in \mathbb{R}$, $\alpha \in [0, 1]$, $t \geq 0$, one has*

$$M(\alpha, t) \overset{\text{(law)}}{=} \sup_{s \leq \alpha t} X_s + \inf_{s \leq (1-\alpha)t} X'_s, \tag{1a}$$

where X' is an independent copy of X, considered under P^μ.

To our knowledge, Miura [14] and Akahori [1] were the first authors to investigate the distribution of $M(\alpha, t)$, with the help of the Feynman–Kac formula. Pursuing Akahori's computations, Dassios obtained the remarkable identity in law stated in Theorem 1.

In this paper, we shall avoid using Feynman–Kac computations and we shall give two different proofs of Theorem 1. The first proof, which is

Received July 1994; revised March 1995.
AMS 1991 *subject classifications*. 60J30, 60J20.
Key words and phrases. Brownian motion with drift, α-quantile.

presented in Section 2, is obtained as a consequence of the identity in law

$$\text{(1b)} \qquad \int_0^t ds\, 1_{(X_s \geq 0)} \overset{\text{(law)}}{=} \sup\left\{ s < t \colon \sup_{u \leq s} X_u = X_s \right\},$$

where X denotes Brownian motion with drift μ. The second proof, which is presented in Section 3, is obtained as a consequence of an extension of Bertoin's path decomposition of Brownian motion with drift by rearranging its positive and negative excursions [2]; see also Vallois [15].

We now discuss briefly the interest of Theorem 1 and, more generally, that of the study of $M(\alpha, t)$ in mathematical finance. In recent years, path-dependent (or look-back) options have been developed; in particular, the so-called Asian options involve knowledge of the distribution of $(1/t)\int_0^t ds \exp(B_s + \mu s)$. Although some explicit description of this distribution has been obtained (Geman and Yor [9] and Yor [18]), the results are nonetheless complicated. Knowledge of the distribution of $M(\alpha, t)$, which, for example follows from (1a), provides much simpler opportunities for developing path-dependent options. The related computations of $E[(M(\alpha, t) - k)^+]$ have been done by Miura [14], Akahori [1] and Dassios [6].

There is no need to duplicate such computations here, and we simply refer the interested reader to the above papers.

2. A first proof of Dassios' identity (1a).

2.1. We first note that the identity (1a) may be amplified as follows (we keep the notation from Theorem 1).

PROPOSITION 1. *For every fixed t and α, one has*

$$(M(\alpha, t), X_t) \overset{\text{(law)}}{=} (S_{\alpha t} + I'_{(1-\alpha)t}, X_{\alpha t} + X'_{(1-\alpha)t}),$$

where

$$S_u = \sup_{s \leq u} X_s \quad \text{and} \quad I'_u = \inf_{s \leq u} X'_s.$$

Proposition 1 is an immediate consequence of Theorem 1 and the following lemma.

LEMMA 1. *Let $t \geq 0$ be fixed. Consider two measurable functionals F and G defined on $C([0, t]; \mathbb{R})$. The following two properties are equivalent*:

(i) *For every $\mu \in \mathbb{R}$, $F(X_s,\ s \leq t) \overset{\text{(law)}}{=} G(X_s,\ s \leq t)$, where, on both sides, X is considered under P^μ.*

(ii) *For a given μ (e.g., $\mu = 0$), the identity in law*

$$(F(X_s, s \leq t), X_t) \overset{\text{(law)}}{=} (G(X_s, s \leq t), X_t)$$

holds.

COMMENTS. (1) To be explicit, Lemma 1 provides a proof of Proposition 1, when it is applied to

$$F(X_s, s \le t) = \inf\left\{x: \int_0^t 1_{(X_s \le x)}\, ds > \alpha t\right\}$$

and

$$G(X_s, x \le t) = \sup_{u \le \alpha t} X_u + \inf_{u \le (1-\alpha)t} \{X_{\alpha t+u} - X_{\alpha t}\}$$
$$\equiv \sup_{u \le \alpha t} X_u + \inf_{u \le (1-\alpha)t} X'_u.$$

(2) A moment's thought shows that Lemma 1 follows from the Cameron–Martin relationship:

$$P^{\mu}_{|\mathcal{F}_t} = \exp\left(\mu X_t - \frac{\mu^2 t}{2}\right) \cdot P^0_{|\mathcal{F}_t}.$$

2.2. As announced in the Introduction, our first proof of Theorem 1 rests on the identity (1b), which is interesting in its own right. For convenience, we shall use the following notation:

$$A_t^+ = \int_0^t ds\, 1_{(X_s \ge 0)} \quad \text{and} \quad \theta_t^+ = \sup\left\{s < t: \sup_{u \le s} X_u = X_s\right\}.$$

We now give the following proof.

PROOF OF THE IDENTITY IN LAW (1B). $A_t^+ \overset{\text{(law)}}{=} \theta_t^+$ under P^{μ}.

(i) Using Lemma 1, it is equivalent to show the identity in law

$$(A_t^+, X_t) \overset{\text{(law)}}{=} (\theta_t^+, X_t),$$

for every $t \ge 0$, under P^0. From the scaling property of BM, we may even restrict ourselves to $t = 1$, and we will first show that, for $x \ge 0$,

(2a) $\quad A_1^+ \overset{\text{(law)}}{=} \theta_1^+ \quad$ under the law Π_x of the Brownian bridge $BB_{0 \to x}$,

starting from 0 at time 0, and ending at x at time 1.

Introducing $\tau_x = \inf\{u: X_u = x\}$, we obtain

(2b) $$\theta_1^+ \overset{\text{(law)}}{=} \tau_x + (1 - \tau_x)U,$$

where U denotes a random variable uniformly distributed on [0, 1], independent of τ_x.

The identity (2b) follows easily from the fact that, under Π_0, θ_1^+ is uniformly distributed, a well-known result noted by Vervaat [16] and Biane [4], and that, conditionally on $\tau_x = t$, the process $(X_{\tau_x + u} - x;\ u \le 1 - \tau_x)$ is a Brownian bridge $BB_{0 \to 0}$ over the time interval $[0, 1 - t]$.

We now consider the left-hand side of (2a) under Π_x. We write

$$A_1^+ = \int_0^1 ds\, 1_{(\hat{X}_s \le x)},$$

where $(\hat{X}_s = x - X_{1-s},\ 0 \le s \le 1)$ is a $BB_{0 \to x}$. Thus, we deduce (using obvious notation) that

$$A_1^+ \overset{\text{(law)}}{=} \hat{\tau}_x + (1 - \hat{\tau}_x)U, \tag{2c}$$

where (2c) now follows from the fact that, under Π_0, $A_1^- = 1 - A_1^+$ is uniformly distributed on [0, 1], another well-known result due to Lévy [13].

The identity in law (2a) now follows from (2b) and (2c).

(ii) To finish the proof of the identity in law (1b), we now remark that, for the moment, we have obtained the following identity: for every Borel function $f\colon \mathbb{R} \times [0, 1] \to \mathbb{R}_+$,

$$E^0\big[f(X_1, A_1^+)1_{(X_1>0)}\big] = E^0\big[f(X_1, \theta_1^+)1_{(X_1>0)}\big]. \tag{2d$_+$}$$

To end the proof, it now remains to show

$$E^0\big[f(X_1, A_1^+)1_{(X_1<0)}\big] = E^0\big[f(X_1, \theta_1^+)1_{(X_1<0)}\big]. \tag{2d$_-$}$$

Since, under P^0, X and $(-X)$ have the same law, the left-hand side of $(2\text{d})_-$ is equal to

$$\begin{aligned} E^0\big[f(-X_1, A_1^-)1_{(-X_1<0)}\big] &= E^0\big[f(-X_1, 1 - A_1^-)1_{(X_1>0)}\big] \\ &= E^0\big[f(-X_1, 1 - \theta_1^+)1_{(X_1>0)}\big] \quad [\text{by } (2\text{d})_+] \\ &= E^0\Big[f\big(\hat{X}_1, \hat{\theta}_1^+\big)1_{(\hat{X}_1<0)}\Big], \end{aligned}$$

where the hats refer to the Brownian motion $\hat{X}_t = X_{1-t} - X_1$, $t \le 1$, which satisfies, in particular, $\hat{X}_1 = -X_1$. Hence, we have finally proven (2d). □

2.3. We now prove Theorem 1 as a consequence of the identity in law (1b).

For simplicity, we take $t = 1$ and we write $M(\alpha)$ for $M(\alpha, 1)$. We also note $A_u^+(x) = \int_0^u ds\, 1_{(X_s \ge x)}$ and $A_u^-(x) = \int_0^u ds\, 1_{(X_s \le x)}$. We simply write A_u^- for $A_u^-(0)$. Let $x < 0$, and define $T_x = \inf\{t\colon X_t = x\}$. Then we have

$$\begin{aligned} P(M(\alpha) < x) = P(A_1^-(x) > \alpha) &= P\left(\int_{T_x}^1 ds\, 1_{(X_s \le x)} > \alpha\right) \\ &= \int_0^{1-\alpha} P(T_x \in du) P(A_{1-u}^- > \alpha) \end{aligned}$$

$$= \int_0^{1-\alpha} P(T_x \in du) P((1-u) - \theta_{1-u}^+ > \alpha) \quad \text{from (1b)}. \tag{2e}$$

Now we have

$$(\theta_{1-u}^+ < (1-u) - \alpha) = (S_{1-u-\alpha} = S_{1-u}) = \Big(S_{1-u-\alpha} \ge \sup_{1-u-\alpha \le t \le 1-u} X_t\Big).$$

Hence, we find, from (2e), that

$$\text{(2f)}\qquad P(M(\alpha) < x) = P\Big(T_x \le 1-\alpha;\ \sup_{T_x \le v \le 1-\alpha} X_v \ge \sup_{1-\alpha \le t \le 1} X_t\Big).$$

Define $\tilde{X}_u = X_{(1-\alpha)+u} - X_{1-\alpha}$, $u \le \alpha$. Then $\tilde{X}$ is a BM with drift μ, which is independent of $(X_u, u \le 1-\alpha)$. Thus, we have, with obvious notation,

$$\text{(2g)}\qquad P(M(\alpha) < x) = P\Big(\tilde{S}_\alpha \le \sup_{T_x \le v \le 1-\alpha} (X_v - X_{1-\alpha}); T_x \le 1-\alpha\Big).$$

To finish the proof of (1a), from (2g) it remains to prove the following identity: for $y \le 0$ and $x < 0$,

$$\text{(2h)}\qquad P\Big(y < \sup_{T_x \le v \le a} (X_v - X_a); T_x < a\Big) = \hat{P}(T_{y+(-x)} \le a),$$

where $\hat{P}$ is the law of BM with drift $(-\mu)$. This follows from the fact that

$$\hat{X}_w \equiv X_{a-w} - X_a, \qquad w \le a,$$

is a BM with drift $(-\mu)$, together with the strong Markov property. Now, from (2h), we have

$$\hat{P}(T_{y+(-x)} \le a) = \hat{P}(y + (-x) \le S_a) = P\Big(y - \inf_{u \le a} X'_u \le x\Big),$$

which, plugged back into (2h) and (2g), proves that the laws of the negative parts of both sides of (1a) are equal.

To finish our proof of (1a), it remains to show that the positive parts of both sides of (1a) are equally distributed. However, the result for positive parts follows from the result for negative parts by changing μ to $-\mu$, since, with obvious notation,

$$M_\mu(\alpha, t) \overset{\text{(law)}}{=} -M_{-\mu}(1-\alpha, t)$$

and a similar identity in law holds for the right-hand side of (1a).

2.4. We end Section 2 with two remarks concerning the identity in law (1b).

REMARK 1. In the case $\mu = 0$, it is well known that

$$\text{(2i)}\qquad \theta_t^+ \overset{\text{(law)}}{=} \sup\{u \le t: X_u = 0\} \quad \Big(\overset{\text{def}}{=} g_t\Big).$$

This identity in law follows from the well-known Lévy representation of reflecting Brownian motion,

$$\text{(2j)}\qquad (S_u - X_u; u \ge 0) \overset{\text{(law)}}{=} (|X_u|, u \ge 0) \quad \text{under } P^0,$$

but neither (2j) nor (2i) extends to the general case $\mu \ne 0$.

REMARK 2. Some particular cases of the identity in law (1b) are explicitly discussed in the literature: Imhof [10, Lemma 1] remarks that, for $\mu < 0$, under P^{μ}, A_{∞}^{+} has the same distribution as θ_{∞}^{+}. Furthermore, it is proven in [10] that this common distribution is also that of $\tau(-Y)$, where $\tau(z) = \inf\{t\colon X_t = z\}$ and Y is exponential with parameter (-2μ) and is independent of X. This identity in law goes back, in fact, to Williams [17]. Another explanation of this result is given by Doney and Grey [8, page 660], who represent X^{+} and X^{-} as two independent processes, after time-changing them, respectively, by the inverses of A^{+} and A^{-}, $Y \equiv \sup_{t \geq 0}(-X_t)$ is exponential, with parameter (-2μ).

3. A second proof of Dassios' identity, using Bertoin's path decomposition.

3.1. An extension of Bertoin's path decomposition [2] obtained for level $a = 0$, to any level $a \geq 0$, will give a deeper understanding of Theorem 1.

Fix some $a \geq 0$, and let $A_t^{+} = \int_0^t ds\ 1_{(X_s \geq -a)}$ and $A_t^{-} \equiv t - A_t^{+} = \int_0^t ds\ 1_{(X_s \leq -a)}$, with respective inverses $(\alpha_u^{\pm},\ u \geq 0)$. Let l denote the local time of X at $(-a)$. We fix $t > 0$ and define

$$Y_u^{\pm} = X(\alpha_u^{\pm}) \pm \tfrac{1}{2} l(\alpha_u^{\pm}).$$

THEOREM 2. *Define*

$$\tilde{X}_u = \begin{cases} Y^{+}(A_t^{+}) - Y^{+}(A_t^{+} - u), & u \leq A_t^{+}, \\ a + Y^{-}(u - A_t^{+}) + Y^{+}(A_t^{+}), & A_t^{+} \leq u \leq t. \end{cases}$$

Then $(\tilde{X}_u, u \leq t) \overset{(\mathrm{law})}{=} (X_u, u \leq t)$.

PROOF. (1) The case $a = 0$ is Bertoin's result.

(2) The basic idea behind the proof is that for X a standard Brownian motion and T an independent exponential variable, we can decompose $(X_u,\ u \leq T)$, conditional on $(\inf_{t \leq T} X_t < -a)$ by splitting this process into $(X_u,\ u \leq H_{-a})$, and the independent piece $(X_u' \equiv a + X_{H_{-a}+u},\ u \leq T - H_{-a})$, where $H_{-a} = \inf\{t\colon X_t = -a\}$, and then use Bertoin's result on the (conditioned) process $(X_t')_{t \leq T - H_{-a}}$.

(3) Here are the details: Let $(X_t)_{t \geq 0}$ be a Brownian motion independent of the exponential random variable T with mean λ^{-1}. We shall provide two decompositions of the path $(X_t)_{t \leq T}$ which correspond to the path decomposition we are concerned with.

As a building block, we take $((Y_t)_{t \leq \tau};\ \xi)$, where $\xi \sim \exp(\sqrt{2\lambda})$ and, given ξ, $(Y_t)_{t \leq \tau}$ is a Brownian motion with drift $\theta \equiv \sqrt{2\lambda}$, run until $\tau \equiv \inf\{t\colon Y_t = \xi\}$. Let $((Y_t')_{t \leq \tau'};\ \xi')$ be an independent copy.

(i) *The process*

$$X_t' = \begin{cases} Y_t, & t \leq \tau, \\ Y'_{\tau + \tau' - t}, & \tau \leq t \leq \tau + \tau', \end{cases}$$

is identical in law to $(X_t)_{t \leq T}$. Indeed, it is clear that $\sup_{u \leq T} X_u \sim \exp(\theta)$ and, given that $H_a < T$, $(X_u)_{u \leq H_a}$ is distributed as a Brownian motion with drift θ, run until it hits a. The independence of the pre- and postmaximum pieces of path is a general excursion-theoretic result, and the law of the postmaximum piece comes from duality.

(i′) Let us introduce a process $(Y_t'')_{t \le \tau''}$ which is a Brownian motion with drift θ, run until it hits $a > 0$ and independent of Y and Y'. *Conditional on* $(\xi' > a)$, we have

$$(Y_t')_{t \le \tau'} \overset{\text{(law)}}{=} (\tilde{Y}_t)_{t \le \tilde{\tau}}$$

where

$$\tilde{Y}_t = \begin{cases} Y_t, & t \le \tau, \\ Y''_{(t-\tau)} + Y_\tau, & \tau \le t \le \tau'', \end{cases}$$

which induces a corresponding decomposition of X':

$$X_t' = \begin{cases} Y_t, & t \le \tau, \\ Y''_{(\tau+\tau''-t)} + \xi - a, & \tau \le t \le \tau + \tau'', \\ Y'_{(\tau+\tau'+\tau''-t)} + \xi - \xi' - a, & \tau + \tau'' \le t \le \tau + \tau'' + \tau'. \end{cases}$$

(ii) Coming back to our idea in (2) above, we decompose $(X_t)_{t \le T}$, conditional on $\{\inf_{u \le T} X_u < -a\}$ by splitting it at $H_{-a} \equiv \inf\{t\colon X_t = -a\}$. The piece of path $(X_t)_{t \le H_{-a}}$ is identical in law to $(-Y_t'')_{t \le \tau''}$; the piece of path $(a + X_{H_{-a}+t})_{t \le T - H_{-a}}$ is independent and identical in law of $(X_t)_{t \le T}$, unconditioned.

We now use Bertoin's result on the (conditioned) process

$$X_t' \equiv a + X_{H_{-a}+t}, \qquad t \le T - H_{-a} \equiv T'.$$

More precisely, with $A_t^{\pm} \overset{\text{def}}{=} \int_0^t 1_{(\pm X_s > 0)}\, ds$ and their respective inverses $\alpha^{\pm}$, we generate processes

$$Z_u^{\pm} = X'(\alpha_u^{\pm}) \pm \tfrac{1}{2} l(\alpha_u^{\pm}),$$

with respective lifetimes $A^{\pm}(T')$. Now, according to Bertoin's result,

$$(Z^+(A_{T'}^+) - Z^+(A_{T'}^+ - t))_{t \le A_{T'}^+}$$

is the premaximum piece of path of a Brownian path fragment of duration T', and so has the law of $(Y_t)_{t \le \tau}$.

Similarly, $(Z^-(A_{T'}^- - t) - Z^-(A_{T'}^-))_{t \le A_{T'}^-}$ is the reversal of the postmaximum path, so again has the law of $(Y_t)_{t \le \tau}$. Comparing with the decomposition (i′) gives the result. □

3.2. We now deduce a number of consequences from Theorem 2. First, we list some a.s. identities, which, thanks to Theorem 2, can be turned into identities in law:

(i) $Y^+(A_t^+) = (X_t \vee (-a)) + \frac{1}{2} l_t$;
(ii) $Y^-(A_t^-) = (X_t \wedge (-a)) - \frac{1}{2} l_t$;

(iii) $A_t^+ = \tilde{\theta}_t^+(a) \equiv \sup\{u < t: \tilde{S}_u - \tilde{X}_u \le a\}$;

(iv) $\tilde{\theta}_t^+(a) - \tilde{\theta}_t^+ = T_{-a}$, on $(T_{-a} < t)$;

(v) $\tilde{X}_t = X_t$ [this follows from (i) + (ii)];

(vi) $\tilde{S}_t = (X_t + a)^+ + \frac{1}{2}l_t$, on $(T_{-a} < t)$, and $\tilde{S}_t = (X_t - (I_t \vee (-a)))^+ + \frac{1}{2}l_t$, in general;

(vii) $\tilde{S}_t - \tilde{X}_t = (X_t + a)^+ - (I_t \vee (-a)) + \frac{1}{2}l_t$.

Now we have the following corollary.

COROLLARY 2.1. *For every $a \ge 0$, we let*

$$A_t^+ \equiv A_t^+(a) = \int_0^t ds\, 1_{(X_s > -a)}$$

and

$$\theta_t^+(a) = \sup\{u < t: S_u - X_u \le a\}.$$

Then

$$A_t^+(a) \overset{\text{(law)}}{=} \theta_t^+(a),$$

where both variables are considered under P^μ.

This corollary follows immediately from (iii) above and extends the result of Proposition 1 to all $a \ge 0$.

COROLLARY 2.2 (Karatzas–Shreve trivariate identity [11]). *Here, we denote $A_t^+ \equiv A_t^+(0)$. The identity in law*

$$\left(X_t^+ + \tfrac{1}{2}l_t, X_t^- + \tfrac{1}{2}l_t, A_t^+\right) \overset{\text{(law)}}{=} \left(S_t, S_t - X_t, \theta_t^+\right)$$

holds, where both variables are considered under P^μ.

PROOF. This identity in law follows from (iii), (v) and (vii), where we take $a = 0$. Hence, we obtain

$$A_t^+ = \tilde{\theta}_t^+, \qquad \tilde{X}_t = X_t \quad \text{and} \quad \tilde{S}_t - \tilde{X}_t^+ = \tfrac{1}{2}l_t.$$

It remains to use Theorem 2 to conclude the proof. □

COMMENT. An explicit formula for the density of the trivariate distribution featured in Corollary 2.2. is proven in detail in the Karatzas and Shreve book [11].

In their research paper [12], Karatzas and Shreve explain the identity in law via a Sparre–Andersen type pathwise decomposition. Bertoin and Pitman [3] discuss a number of closely related path decompositions which transform the Brownian bridge into the Brownian meander and/or the Brownian excursion. The main difference between Bertoin–Pitman [3] and our paper is that in [3] the authors are interested, in particular, in path transformations of the Brownian bridge, whereas in the present paper we are interested in pathwise transformations of Brownian motion $(X_u,\ u \le t)$, which keep the end value X_t fixed. This allows us, thanks to Lemma 1, to obtain results which are valid for Brownian motion with any constant drift and to deduce some of the results of Bertoin–Pitman by conditioning on $X_t = 0$.

4. Some remarks on Akahori's generalized arcsine formula. As a first step in his computation of the distribution of $M(\alpha, t)$ under P^{μ}, Akahori [1] derived the following formula for the distribution of $A^-(t) = \int_0^t ds\, 1_{(X_s \le 0)}$, under P^{μ}, for $\mu \ge 0$:

$$P^{\mu}(A^-(t) \in ds) = \frac{ds}{2}\left(2\mu + \int_{t-s}^{\infty} \frac{\exp(-\mu^2 u/2)\, du}{\sqrt{2\pi u^3}}\right)\left(\int_s^{\infty} \frac{\exp(-\mu^2 v/2)\, dv}{\sqrt{2\pi v^3}}\right) \tag{4a}$$

from the time-honored Feynman–Kac method. [In the case $\mu \le 0$, we use the symmetry of the distribution of Brownian motion, which gives $P^{\mu}(A^-(t) \in ds) = P^{-\mu}(t - A^-(t) \in ds)$.]

An interpretation, and an explanation of (4a) in terms of "plain" Brownian motion, that is, Brownian motion without drift, may be given as follows:

(a) We first remark that, from the Cameron–Martin–Girsanov theorem and the well-known fact that under P^0, $(1/t)A^-(t)$ is arcsin distributed on [0, 1], we have

$$E^0\left[\exp\left(\mu X_t - \frac{\mu^2 t}{2}\right)\middle| A^-(t) = s\right] = \pi\sqrt{s(t-s)}\,\frac{P^{\mu}(A^-(t) \in ds)}{ds}. \tag{4b}$$

(b) Combining formulae (4a) and (4b), we obtain

$$E^0[\exp(\mu X_t) | A^-(t) = s] = H_+(\mu\sqrt{t-s})H_-(\mu\sqrt{s}) \quad \text{for } \mu \ge 0, \tag{4c}$$

where

$$H_+(\nu) = \sqrt{2\pi}\,\nu \exp\left(\frac{\nu^2}{2}\right) + \frac{1}{2}\int_0^{\infty} \frac{dx}{(1+x)^{3/2}} \exp\left(-\frac{\nu^2 x}{2}\right),$$

$$H_-(\nu) = \frac{1}{2}\int_0^{\infty} \frac{dy}{(1+y)^{3/2}} \exp\left(-\frac{\nu^2 y}{2}\right).$$

Next we state the following elementary lemma.

LEMMA 2. *Let $\nu \ge 0$ and denote by m_1 a Rayleigh variable, that is,*

$$P(m_1 \in d\rho) = d\rho\, \rho \exp\left(-\frac{\rho^2}{2}\right), \qquad \rho \ge 0.$$

Then, one has $H_+(\nu) = E[\exp(\nu m_1)]$ and $H_-(\nu) = E[\exp(-\nu m_1)]$.

The proof of Lemma 2 is left to the reader.

We immediately deduce from (4c) the following:

PROPOSITION 3. *Under P^0, conditionally on $A^-(t) = s$, one has*

$$X_t \overset{(\text{law})}{=} \sqrt{t-s}\, m_1 - \sqrt{s}\, m_1', \tag{4d}$$

where m_1 and m_1' are two independent Rayleigh variables.

We now explain and complete the identity in law (4d). We write

$$X_t = \left(X_t^+ + \tfrac{1}{2}l_t\right) - \left(X_t^- + \tfrac{1}{2}l_t\right),$$

and, from Corollary 2.2, we see that the identity in law (4d) may be reinterpreted as follows: conditionally on $\theta_t^+ = u$, the law of the variable

$$X_t \equiv S_t - (S_t - X_t) \quad \text{is that of} \quad \sqrt{u}\, m_1 - \sqrt{(t-u)}\, m_1'.$$

This follows from Denisov's path decomposition of Brownian motion $(X_u, u \le t)$ before and after θ_t^+, in terms of two independent Brownian meanders ([7]; see also Biane and Yor [5]); in particular, we deduce from Denisov's result that

$$m_1 \overset{\text{def}}{=} \frac{1}{\sqrt{\theta_t^+}} S_t, \qquad m_1' \overset{\text{def}}{=} \frac{S_t - X_t}{\sqrt{t + \theta_t^+}} \quad \text{and} \quad \theta_t^+ \quad \text{are independent},$$

m_1 and m_1' are Rayleigh variables and θ_t^+ is arc-sine distributed on $[0, t]$.

REFERENCES

[1] AKAHORI, J. (1995). Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.* **5** 383–388.

[2] BERTOIN, J. (1991). Décomposition du mouvement brownien avec dérive en un minimum local par juxtaposition de ses excursions positives et négatives. *Séminaire de Probabilités XXV. Lecture Notes in Math.* **1485** 330–344. Springer, Berlin.

[3] BERTOIN, J. and PITMAN, J. W. (1994). Path transformations connecting Brownian bridge, excursion and meander. *Bull. Sci. Math.* **118** 147–166.

[4] BIANE, PH. (1986). Relations entre pont brownien et excursion normalisée du mouvement brownien. *Ann. Inst. H. Poincaré* **22** 1–7.

[5] BIANE, PH. and YOR, M. (1988). Quelques précisions sur le méandre brownien. *Bull. Sci. Math. Sér. 2* **112** 101–109.

[6] DASSIOS, A. (1995). The distribution of the quantiles of a Brownian motion with drift and the pricing of path-dependent options. *Ann. Appl. Probab.* **5** 389–398.

[7] DENISOV, I. V. (1983). A random walk and a Wiener process near a maximum. *Teor. Veroyatnost. i Primenen.* **28** 821–824. [English translation in *Theory Probab. Appl.* (1985) 713–716.]

[8] DONEY, R. A. and GREY, D. R. (1989). Some remarks on Brownian motion with drift. *J. Appl. Probab.* **26** 659–663.

[9] GEMAN, H. and YOR, M. (1993). Bessel processes, Asian options and perpetuities. *Math. Finance* **3** 349–375.

[10] IMHOF, J. P. (1986). On the time spent above a level by Brownian motion with negative drift. *Adv. in Appl. Probab.* **18** 1017–1018.

[11] KARATZAS, I. and SHREVE, M. (1987). *Brownian Motion and Stochastic Calculus.* Springer, Berlin.

[12] KARATZAS, I. and SHREVE, M. (1987). A decomposition of the Brownian path. *Statist. Probab. Lett.* **5** 87–94.

[13] LÉVY, P. (1939). Sur certains processus stochastiques homogènes. *Compositio Math.* **7** 283–339.

[14] MIURA, R. (1992). A note on look-back options based on order statistics. *Hitotsubashi Journal of Commerce Management* **27** 15–28.

[15] VALLOIS, P. (1992). Amplitude du mouvement brownien, et juxtaposition des excursions positives et négatives. *Séminaire de Probabilités XXVI. Lecture Notes in Math.* **1526** 361–373. Springer, New York.

[16] VERVAAT, W. (1979). A relation between Brownian bridge and Brownian excursion. *Ann. Probab.* **7** 141–149.

[17] WILLIAMS, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions, I. *Proc. London Math. Soc.* (*3*) **28** 738–768.

[18] YOR, M. (1992). On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.* **24** 509–531.

[19] YOR, M. (1995). The distribution of Brownian quantiles. *J. Appl. Probab.* To appear.

P. EMBRECHTS
DEPARTMENT OF MATHEMATICS
ETH-ZENTRUM
CH-8092 ZÜRICH
SWITZERLAND

L. C. G. ROGERS
SCHOOL OF MATHEMATICS
UNIVERSITY OF BATH
BATH BA2 AY
UNITED KINGDOM

M. YOR
LABORATOIRE DE PROBABILITÉS
UNIVERSITÉ PARIS VI
4 PLACE JUSSIEU
TOUR 56, 3ÈME ETAGE
F.75252 PARIS
FRANCE