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A NECESSARY AND SUFFICIENT CONDITION FOR ABSENCE OF ARBITRAGE WITH TAME PORTFOLIOS

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We characterize absence of arbitrage with tame portfolios in the case of invertible volatility matrix. As a corollary we get that, under a certain condition, absence of arbitrage with tame portfolios is characterized by the existence of the so-called equivalent martingale measure. Without that condition, the existence of equivalent martingale measure is equivalent to absence of approximate arbitrage. The proofs are probabilistic and are based on a construction of two specific arbitrages. Some examples are provided.

1. Introduction and main results. In this paper we deal with a financial market in which $d+1, d \geq 1$, securities are being traded during the time interval $0 \leq t \leq 1$. All processes that appear throughout without restriction on their time domain will be taken as defined on $0 \leq t \leq 1$. The source of uncertainty in the market is provided by a d-dimensional standard Brownian motion $W(t) = (W_1(t), \dots, W_d(t))$ defined on a complete probability space (Ω, F, P). The term "adapted" will refer throughout to the filtration $\{F_t: 0 \leq t \leq 1\}$, the P-augmentation of the natural filtration of $W(t)$, namely,

$$F_t = \sigma\{W(s): 0 \leq s \leq t\} \vee N, \tag{1.1}$$

where $N = \{A \in F: P(A) = 0\}$. One of the securities traded is called the bond. Its price process $B(t)$ satisfies

$$dB(t) = B(t) r(t)\, dt, \qquad B(0) = 1, \tag{1.2}$$

where $r(t)$ is an adapted process that represents the short-term interest rate. The other d securities are called stocks, and their price processes $\{S_i(t): i = 1, \dots, d\}$ satisfy

$$\begin{aligned} dS_i(t) &= S_i(t)\left[b_i(t)\, dt + \sum_{k=1}^{d} \sigma_{i,k}(t)\, dW_k(t)\right], \\ S_i(0) &= s_i. \end{aligned} \tag{1.3}$$

Here $\sigma(t) = (\sigma_{i,k}(t))_{1 \leq i, k \leq d}$ is a matrix-valued process which is adapted and invertible for all t; $b(t) = (b_i(t))_{1 \leq i \leq d}$ is an adapted vector process. In order

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that (1.2) and (1.3) represent a well-defined process, we impose

$$(1.4)\qquad P\left(\int_0^1\left(|r(s)| + \sum_{i=1}^{d}|b_i(s)| + \sum_{i,k=1}^{d}\sigma_{i,k}^2(s)\right)ds < \infty\right) = 1.$$

The investor chooses how much money to invest at time t in any of the stocks. Let $\pi_i(t)$ represent the amount of money that the investor invests at time t in the ith stock, $i = 1,\dots,d$. The vector $(\pi_i(t)\colon i = 1,\dots,d)$ is called a portfolio. More precisely, we use the following definition.

Definition 1. The R^d-valued process $\pi(t) = (\pi_1(t),\dots,\pi_d(t))$ is called a *portfolio* if it is adapted and satisfies

$$(1.5)\qquad \begin{aligned} P\left(\int_0^1\|\pi'(s)\sigma(s)\|^2\,ds < \infty\right) &= 1,\\ P\left(\int_0^1|\pi'(s)(b(s) - r(s)1_d)|\,ds < \infty\right) &= 1, \end{aligned}$$

where $\|\cdot\|$ stands for the usual Euclidean norm in R^d, a prime stands for transpose and $1_d = (1,1,\dots,1)' \in R^d$.

The purpose of investment in stocks, which are a riskier investment than bonds, is capital gain in excess of what can be achieved by investment in bonds. That is why we are interested in the discounted capital gain process $X(t)$ defined as follows:

Definition 2. The real-valued process

$$(1.6)\qquad X(t) = \int_0^t\pi'(s)B^{-1}(s)\left[\sigma(s)\,dW(s) + (b(s) - r(s)1_d)\,ds\right]$$

is called the *discounted capital gain process* associated with the portfolio π.

The economic justification of (1.6) is that the discounted capital gain from holding the kth stock between times t and $t + ds$ is

$$(\#\text{shares owned})(\text{price increase in the stock}) - (\text{potential gain from bond}),$$

namely, $(\pi_k(t)/S_k(t))\,dS_k(t) - (\pi_k(t)/B(t))\,dB(t)$. It follows that the discounted capital gain from holding all stocks between times t and $t + ds$ is $\pi'(t)[\sigma(t)\,dW(t) + (b(t) - r(t)1_d)\,ds]$. By multiplying by $B^{-1}(t)$ we discount this quantity to time $t = 0$, and by integrating from 0 to t we sum up all the gains that were made up to time t. The process $X(t)$ is well-defined because $B^{-1}(t)$ is bounded away from 0 and ∞ a.s., under assumption (1.4).

Many investors are interested in making money without any risk. This leads us to the following definition.

DEFINITION 3. The portfolio π is called an *arbitrage* if its discounted capital gain process satisfies the following three conditions:

(a) there exists $C > -\infty$ such that $P(X(t) > C$ for all $0 \le t \le 1) = 1$;
(b) $P(X(1) \ge 0) = 1$;
(c) $P(X(1) > 0) > 0$.

REMARK. The π that satisfies condition (a) is called a *tame portfolio* [see Karatzas (1993) or Dybvig and Huang (1988) for more details]. This is a restriction that prevents "doubling schemes" and can be interpreted as putting a limit on borrowing.

The purpose of this paper is to study necessary and sufficient conditions for absence of arbitrage, which is a property that we expect the financial market to have. The sufficiency side is more or less understood [see Karatzas (1993)]. The main contribution of this paper is in the necessity of the conditions when the volatility matrix is invertible. To state our results we need to define a few more objects. First we look at the vector-valued process $\theta(t)$ defined as

$$\theta(t) = \sigma^{-1}(t)(b(t) - r(t)1_d); \tag{1.7}$$

$\theta(t)$ is well defined because of the invertibility of $\sigma(t)$, and it is adapted but so far nothing is assumed about its integrability properties. The discounted capital gain process can now be written as

$$X(t) = \int_0^t \pi'(s)B^{-1}(s)\sigma(s)[dW(s) + \theta(s)\,ds]. \tag{1.8}$$

Let $0 \le \tau \le 1$ be a stopping time. With τ we associate δ, defined by

$$\delta = \begin{cases} \inf\left\{t > \tau : \int_\tau^t \|\theta(s)\|^2\,ds = \infty\right\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases}$$

Next we define $Z_\tau(t)$, which plays a crucial role in what follows:

$$Z_\tau(t) = \begin{cases} \exp\left\{-\int_0^t \{\tau \le s\}\theta'(s)\,dW(s) - \int_0^t \{\tau \le s\}\dfrac{\|\theta(s)\|^2}{2}\,ds\right\}, & \text{if } t \le \delta, \\ 0, & \text{if } t > \delta. \end{cases}$$

REMARKS.

(a) The process $Z_\tau(t)$ is well defined by imposing left-continuity at $t = \delta$, so $Z_\tau(\delta) = 0$ when the infimum defining δ is attained.
(b) The set $\{\tau \le s\}$ is identified with its indicator function.

Finally, we look at the following stopping time:

$$\alpha = \begin{cases} \inf\left\{t: \int_t^{t+h} \|\theta(s)\|^2 \, ds = \infty \; \forall \, h \in (0, 1-t]\right\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{1.9}$$

It is easy to see that if $P(\alpha = 1) = 1$, then, for every stopping time $0 \le \tau \le 1$, the process $Z_\tau(t)$ is a continuous nonnegative supermartingale which may take the value 0. If in addition $E(Z_\tau(1)) = 1$, then $Z_\tau(t)$ is actually a martingale.

Now we can state the main result of the paper.

THEOREM 1. *There is no arbitrage if and only if* $P(\alpha = 1) = 1$ *and* $E(Z_r(1)) = 1$ *for every constant* $0 \le r \le 1$.

We next look at the condition

$$P\left(\int_0^t \|\theta(s)\|^2 \, ds < \infty, 0 \le t < 1\right) = 1. \tag{1.10}$$

We get the following corollary, which shows that some time there is no arbitrage even when there is a positive probability that θ explodes toward $t = 1$ (see also Example 2 in Section 4).

COROLLARY 1. *Assume* (1.10). *Then there is no arbitrage if and only if* $E(Z_0(1)) = 1$.

Next look at the process $W^*(t)$ defined as

$$W^*(t) = W(t) + \int_0^t \theta(s) \, ds;$$

$W^*(t)$ is well defined whenever $\theta(s)$ is integrable a.s. The following condition is a special case of condition (1.10):

$$P\left(\int_0^1 \|\theta(s)\|^2 \, ds < \infty\right) = 1. \tag{1.11}$$

The following is an important corollary of Theorem 1.

COROLLARY 2. *Assume* (1.11). *Then there is no arbitrage if and only if there exists a probability measure* Q *equivalent to* P *such that* $W^*(t)$ *is a* Q*-Brownian motion.*

The measure Q is known as an *equivalent martingale measure*, and it is well understood that its existence guarantees absence of arbitrage [see, e.g., Duffie (1992), Chapter 6]. Corollary 2 states that the converse is true, in the context of tame portfolios, if we assume (1.11).

Next we describe what are all the possible arbitrages, if there are any, under condition (1.11). We get the following corollary.

COROLLARY 3. *Assume* (1.11). *There is an arbitrage τ whose capital gain process X satisfies $P(X(t) > -1,\ 0 \le t \le 1) = 1$, if and only if there exists φ, an adapted, R^d-valued process that satisfies $P(\int_0^1 \|\varphi(s)\|^2\, ds < \infty) = 1$, so that*

$$\pi(t) = \frac{Y(t)}{Z_0(t)} B(t)(\sigma^{-1}(t))'(\varphi(t) + \theta(t)),$$

where the process $Y(t)$, defined by

$$Y(t) = \exp\left\{ \int_0^t \varphi'(s)\, dW(s) - \int_0^t \frac{\|\varphi(s)\|^2}{2}\, ds \right\},$$

satisfies $P(Y(1) \ge Z_0(1)) = 1$ and $P(Y(1) > Z_0(1)) > 0$. The capital gain process of π is $X(t) = Y(t)/Z_0(t) - 1$.

A question that arises naturally from Corollary 2 is how to characterize the existence of an equivalent martingale measure, in the general case, when (1.11) is not assumed. For that purpose we need the following definition.

DEFINITION 4. A sequence $\{\pi_n:\ n = 1, 2, \dots\}$ of portfolios is called an *approximate arbitrage* if its respective sequence of discounted capital gains X_n satisfies the following three conditions:

(a) there exists $C > -\infty$ such that $P(X_n(t) > C$ for all $0 \le t \le 1) = 1$, for every n;
(b) $P(X_n(1) \ge 0) \to 1$;
(c) there exists $\delta > 0$ such that $P(X_n(1) > \delta) > \delta$, for all n.

This looks like a natural generalization of the arbitrage concept. A related definition appears in Duffie (1992). Now we can state the next theorem.

THEOREM 2. *There is no approximate arbitrage if and only if there exists a probability measure Q equivalent to P such that $W^*(t)$ is a Q-Brownian motion.*

Now we describe the organization of the paper. In Section 2 we prove two lemmas that will be useful: Lemma 1 gives an alternative condition to the one that appears in the statement of Theorem 1; Lemma 2 constructs an interesting arbitrage, one of the two needed for the proof of Theorem 1. In an example, which precedes Lemma 2, we deal with a special case, for which the construction of the arbitrage is simpler than that of Lemma 2. In Section 3 we prove Theorem 1, its three corollaries and Theorem 2. In Section 4 we give five simple examples that we have encountered while preparing this work. In the first example (1.11) holds and there is an arbitrage. In the second example there is no arbitrage, but there exists an approximate arbitrage. The third, a minor modification of the second, shows that even when $P(\alpha = 1) = 1$ and $E(Z_0(1)) = 1$ both hold there can be an arbitrage. The fourth example shows that the condition $E(Z_r(1)) = 1$ for every $0 \le r \le 1$, by itself, does not prevent an arbitrage opportunity. The fifth example shows how to use Corollary 3 to construct an arbitrage.

Background. In this paper we deal with a specific model of financial markets, a model that we have learned from Karatzas (1993). Other models are possible: the trading can be restricted to a discrete time, other processes can describe the prices of financial assets, other restrictions can be imposed on the portfolios and so on. In the last 15 years there has been a considerable amount of work done on the arbitrage problem in some of the possible models. The idea of equivalent martingale measure appears first in Harrison and Kreps (1979). A characterization of absence of arbitrage as equivalent to the existence of a martingale measure, in the case where Ω is finite, was achieved by Harrison and Pliska (1981) and was generalized by several authors. The concept of approximate arbitrage was created because, in general, it turns out that absence of arbitrage is not equivalent to the existence of martingale measures. There are several different definitions of this concept (sometimes called free lunch) in the literature, but they all mean roughly the same thing. Stricker (1990) assumed that the asset price processes are in L^p, $1 \leq p < \infty$; Delbaen (1992) assumed that the asset price processes are bounded; and Fritelli and Lakner (1994) have essentially no mathematical restrictions on the asset prices. All these results proved that, with a definition of approximate arbitrage which is appropriate to each of these models, the equivalent martingale measure characterizes absence of approximate arbitrage. These results should be compared to our Theorem 2. For example, when we adapt the result of Delbaen (1992) to our setup, we get a type of approximate arbitrage which is different from ours, whose absence is equivalent to the existence of a martingale measure. We refer the reader to Frittelli and Lakner (1994), whose introduction contains a review of the recent history of the subject which is more extensive than ours and whose reference list gives a fairly complete account of the literature.

In this paper the main result is Theorem 1, which characterizes absence of arbitrage in our model. This result is more delicate than results which characterize absence of approximate arbitrage for our model. Accordingly, Theorem 2 is an easy consequence of Theorem 1. We believe that our main result is new. Furthermore, we have used only probabilistic techniques, in contrast to the proofs of the results on approximate arbitrage mentioned before, which use the Hahn–Banach theorem and other methods from functional analysis. Our approach looks to us more natural to mathematical finance, in the sense that we show how to take advantage of an arbitrage opportunity if one exists.

After submission of this paper, one of the referees pointed out that a recent paper by Delbaen and Schachermayer (1995) has some overlap with our paper. We were unaware of the existence of this paper. The same referee also remarked that another recent paper by the same authors [Delbaen and Schachermayer (1994)] contains a result similar to Theorem 2.

REMARK. In some of the formulas we identify sets with their indicator functions. Recall also that a process that is defined without restriction on its time domain is taken as defined as $0 \leq t \leq 1$.

2. Two lemmas. The following lemma gives a condition equivalent to the one that appears in the statement of Theorem 1. The proof of Theorem 1 will actually characterize absence of arbitrage in terms of the equivalent condition.

LEMMA 1. *We have* $P(\alpha = 1) = 1$ *and* $E(Z_r(1)) = 1$, *for every* $0 \le r \le 1$, *if and only if* $E(Z_\tau(1)) = 1$ *for every stopping time* $0 \le \tau \le 1$.

PROOF. Sufficiency is trivial. If $P(\alpha < 1) > 0$, then $E(Z_\alpha(1)) < 1$ and this contradicts the assumption, since α is a stopping time. Since constants are stopping times we are done.

Now we prove necessity. Suppose there exists a stopping time $0 \le \tau \le 1$ such that $E(Z_\tau(1)) < 1$. Let

$$\tau_n = \frac{k}{n} \quad \text{if } \tau \in \left(\frac{k-1}{n}, \frac{k}{n}\right], k = 1, 2, \dots, n, n = 1, 2 \dots . \tag{2.1}$$

It is well known that $\{\tau_n: n = 1, 2, \dots\}$ is a sequence of stopping times and $\tau_n \downarrow \tau$ a.s. Also, $E(Z_\tau(\tau_n)) \to 1$ since $Z_\tau(\tau) = 1$ and $\{Z_\tau(t)\}$ is a supermartingale which is continuous a.s., because $P(\alpha = 1) = 1$. We claim that there exists n such that $E(Z_{\tau_n}(1)) < 1$. Otherwise $E(Z_{\tau_n}(1)) = 1$ for every n, and since $E(Z_{\tau_n}(1) | F_{\tau_n}) \le Z_{\tau_n}(\tau_n) = 1$ a.s., we must have $E(Z_{\tau_n}(1)|F_{\tau_n}) = 1$ a.s., for every n. This will create a contradiction:

$$\begin{aligned} 1 &> E(Z_\tau(1)) \\ &= E\big(Z_\tau(\tau_n) Z_{\tau_n}(1)\big) \\ &= E\big(Z_\tau(\tau_n) E(Z_{\tau_n}(1) | F_{\tau_n})\big) \\ &= E(Z\tau(\tau_n)) \to 1. \end{aligned}$$

So the claim is proved. Let us, therefore, assume that, for some n,

$$E\big(Z_{\tau_n}(1)\big) < 1. \tag{2.2}$$

Since

$$\begin{aligned} E\big(Z_{\tau_n}(1)\big) &= E\left[\sum_{k=1}^{n} Z_{k/n}(1) \left\{\tau_n = \frac{k}{n}\right\}\right] \\ &= E\left[\sum_{k=1}^{n} E\big(Z_{k/n}(1) | F_{k/n}\big) \left\{\tau_n = \frac{k}{n}\right\}\right] \end{aligned}$$

and $1 = E[\sum_{k=1}^{n} \{\tau_n = k/n\}]$, it follows that

$$E\left[\sum_{k=1}^{n} \big[1 - E\big(Z_{k/n}(1) | F_{k/n}\big)\big] \left\{\tau_n = \frac{k}{n}\right\}\right] > 0. \tag{2.3}$$

So there exists $k \in \{1, \dots, n\}$ for which $P(E(Z_{k/n}(1)|F_{k/n}) < 1) > 0$. For this k we have $E(Z_{k/n}(1)) < 1$. This is a contradiction to the assumption of the lemma. □

In the next lemma we will assume

$$P(\alpha < 1) > 0. \tag{2.4}$$

The goal is to show that if (2.4) holds, then there is an arbitrage. The following special case is considerably simpler than the general case but, nonetheless, will expose some of the ideas that are involved in the proof of Lemma 2.

EXAMPLE. We assume now that $\theta(t)$ is deterministic.

To make things simple, we take $d = 1$, $r = 0$ and $\sigma = 1$. We assume

$$\int_0^t \theta^2(s)\,ds = \infty, \qquad 1 \geq t > 0,$$

$$\int_t^1 \theta^2(s)\,ds < \infty, \qquad 1 \geq t > 0.$$

Our goal is to find an arbitrage. Let

$$\tau(t) = \frac{\left(\int_t^1 \theta^2(s)\,ds + 1\right)^{-3}}{27}, \qquad 0 < t \leq 1,$$

$$\tau(0) = 0.$$

Obviously $\tau(t)$ is an increasing function and is continuous at $t = 0$. Define

$$\beta(t) = \sqrt{\frac{d\tau(t)}{dt}}\,\operatorname{sign}(\theta(t)), \qquad 1 \geq t > 0.$$

Some elementary calculations show that

$$\int_0^t \beta(s)\theta(s)\,ds = \tau(t)^{1/3},$$

$$\int_0^t \beta^2(s)\,ds = \tau(t).$$

Define the following stopping time:

$$\gamma = \begin{cases} \inf\left\{t > 0: \int_0^t \beta(s)\,dW(s) = -\dfrac{\tau(t)^{1/3}}{2}\right\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases}$$

Observe that $P(\gamma > 0) = 1$. The reason is that, by a time-change argument,

$$\int_0^t \beta(s)\,dW(s) = W^*(\tau(t)),$$

where W^* is a standard Brownian motion. So

$$\frac{\int_0^t \beta(s)\,dW(s)}{\tau(t)^{1/3}} \to 0 \quad \text{as } t \to 0, \text{ a.s.}$$

Finally, let

$$\pi(t) = \beta(t)\{t \leq \gamma\}.$$

The capital gain $X(t)$, associated with the portfolio π, satisfies the following a.s.:

$$X(t) \geq \frac{\tau(t \wedge \gamma)^{1/3}}{2}, \qquad 0 \leq t \leq 1.$$

It follows that π is an arbitrage. This example cannot be generalized immediately to handle random θ, because the arbitrage that was constructed here will not be adapted in that case.

LEMMA 2. *Assume* (2.4). *Then there exists an arbitrage.*

PROOF. Define for every $j = 1, \dots, d$,

$$\alpha_j = \begin{cases} \inf\left\{t: \int_t^{t+h} \theta_j^2(s)\, ds = \infty \,\forall\, h \in (0, 1-t]\right\}, \\ 1, \quad \text{if no such } t \text{ exists,} \end{cases}$$

where $\theta_j(t)$ is the jth coordinate of $\theta(t)$. Since $\alpha = \min\{\alpha_1, \dots, \alpha_d\}$ we conclude that there exists $j \in \{1, \dots, d\}$ such that

$$P(\alpha_j < 1) > 0.$$

We will assume first that $P(\alpha_j = 0) = 1$, namely, we assume that

$$P\left(\int_0^t \theta_j^2(s)\, ds = \infty, 0 \le t \le 1\right) = 1. \tag{2.5}$$

We start the construction of an arbitrage by selecting two sequences of constants $t_k \downarrow 0$ and $c_k \uparrow \infty$ such that

$$\sum_{k=1}^{\infty} P\left(\int_{t_k}^{t_{k-1}} \left(\theta_j^2(s) \wedge c_k\right) ds \le 1\right) < \infty. \tag{2.6}$$

This can be done as follows: Let $t_0 = c_0 = 1$. After selecting $\{t_i, c_i: i = 0, \dots, k-1\}$, we observe that

$$P\left(\int_t^{t_{k-1}} \left(\theta_j^2(s) \wedge \frac{1}{t}\right) ds \le 1\right) \to 0, \qquad t \to 0,$$

as follows from

$$P\left(\int_t^{t_{k-1}} \left(\theta_j^2(s) \wedge \frac{1}{t}\right) ds \to \infty, t \to 0\right) = 1,$$

So we select $0 < t_k < t_{k-1}/2$ so that

$$P\left(\int_{t_k}^{t_{k-1}} \left(\theta_j^2(s) \wedge \frac{1}{t_k}\right) ds \le 1\right) \le \frac{1}{k^2}, \tag{2.7}$$

and we take $c_k = 1/t_k$. Next we introduce a sequence of stopping times,

$$\tau_k = \begin{cases} \inf\left\{t_k < t \le t_{k-1}: \int_{t_k}^{t} \left(\theta_j^2(s) \wedge c_k\right) ds = 1\right\}, \\ t_{k-1}, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{2.8}$$

From the Borel–Cantelli lemma it follows that $\tau_k < t_{k-1}$ for all k large enough a.s. Now fix $\lambda \in (1, 2)$ and set

$$\begin{aligned} \beta(t) &= \sum_{k=1}^{\infty} \{t_k \le t \le \tau_k\} k^{-\lambda}, \\ \hat{\theta}(t) &= \left(|\theta_j(t)| \wedge \sqrt{c_k}\right)\{t_k \le t \le t_{k-1}\}. \end{aligned} \tag{2.9}$$

We observe that

$$\lim_{t\to 0}\frac{\int_0^t \beta(s)\hat{\theta}^2(s)\,ds}{\left(\int_0^t \beta^2(s)\hat{\theta}^2(s)\,ds\right)^{1/3}} = \lim_{n\to\infty}\frac{\sum_{k=n+1}^{\infty}k^{-\lambda}}{\left(\sum_{k=n+1}^{\infty}k^{-2\lambda}\right)^{1/3}}$$

$$= \lim_{n\to\infty}\frac{(2\lambda-1)^{1/3}}{\lambda-1}n^{(2-\lambda)/3} = \infty \quad \text{a.s.}$$

Now set

$$\hat{\pi}(t) = \begin{cases} \dfrac{\beta(t)\hat{\theta}^2(t)}{\theta_j(t)}, & \text{if } \theta_j(t) \neq 0, \\ 0, & \text{if } \theta_j(t) = 0. \end{cases} \tag{2.10}$$

Since

$$\int_0^t \hat{\pi}(s)\,dW_j(s) = W^*\left(\int_0^t \hat{\pi}^2(s)\,ds\right),$$

where W^* is a standard Brownian motion [Karatzas and Shreve (1987), page 174], it follows that

$$\lim_{t\to 0}\frac{\int_0^t \hat{\pi}(s)\,dW_j(s)}{\left(\int_0^t \hat{\pi}^2(s)\,ds\right)^{1/3}} = 0 \quad \text{a.s.}$$

Also, it follows from (2.10) that

$$\int_0^t \hat{\pi}^2(s)\,ds \le \int_0^t \beta^2(s)\hat{\theta}^2(s)\,ds.$$

So we have

$$\lim_{t\to 0}\frac{\int_0^t \hat{\pi}(s)\,dW_j(s) + \int_0^t \hat{\pi}(s)\theta_j(s)\,ds}{\left(\int_0^t \hat{\pi}^2(s)\,ds\right)^{1/3}} = \infty \quad \text{a.s.} \tag{2.11}$$

Let

$$\tau = \begin{cases} \inf\left\{t > 0\colon \displaystyle\int_0^t \hat{\pi}(s)\,dW_j(s) \right. \\ \qquad \left. + \displaystyle\int_0^t \hat{\pi}(s)\theta_j(s)\,ds = \left(\int_0^t \hat{\pi}^2(s)\,ds\right)^{1/3}\right\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{2.12}$$

Let $\overline{\pi}(t)$ be an R^d-valued process whose jth coordinate is $\hat{\pi}(t)$ and whose other coordinates are identically 0. Finally, we define an arbitrage

$$\pi(t) = B(t)\left(\sigma^{-1}(t)\right)'\overline{\pi}(t)\{t \le \tau\}. \tag{2.13}$$

Observe that the discounted capital gain $X(t)$ associated with this arbitrage satisfies

$$\begin{aligned} P(X(1) > 0) &= 1, \\ P(X(t) \ge 0, 0 \le t \le 1) &= 1. \end{aligned} \tag{2.14}$$

Next we drop assumption (2.5). Now we assume

$$P(\alpha_j < 1) > 0. \tag{2.15}$$

Let $P_\omega(\cdot) = P(\cdot|F_{\alpha_j})(\omega)$ be the regular conditional probability of P given F_{α_j}. We clearly have, for $\omega \in \{\alpha_j < 1\}$,

$$P_\omega\left(\int_{\alpha_j(\omega)}^t \theta_j^2(s)\,ds = \infty,\ \alpha_j(\omega) < t \le 1\right) = 1.$$

Also, under P_ω, the process $\{W_j(\alpha_j(\omega) + h) - W_j(\alpha_j(\omega))\colon 0 \le h \le 1 - \alpha(\omega)\}$ is a standard Brownian motion. So, for every $\omega \in \{\alpha_j < 1\}$, we can construct a process π_ω that satisfies $\pi_\omega(t) = 0$, $0 \le t \le \alpha_j(\omega)$, and the capital gain associated with it X_{π_ω} satisfies

$$\begin{aligned} P_\omega\big(X_{\pi_\omega}(1) > 0\big) &= 1, \\ P_\omega\big(X_{\pi_\omega}(t) \ge 0, 0 \le t \le 1\big) &= 1. \end{aligned} \tag{2.16}$$

Now define

$$\pi(\omega, t) = \begin{cases} 0, & 0 \le t \le \alpha_j(\omega), \\ \pi_\omega(\omega, t), & \alpha_j(\omega) < t \le 1. \end{cases}$$

In order to make sure that π is adapted, we need to construct the processes $\{\pi_\omega\}$ with some care. We select constants $r_k \downarrow 0$ such that $t_k = (\alpha_j + r_k) \wedge 1$, $c_k = 1/r_k$ will satisfy the following a.s. on $\{\alpha_j < 1\}$:

$$\sum_{k=1}^{\infty} P_\omega\left(\int_{t_k}^{t_{k-1}} \big(\theta_j^2(s) \wedge c_k\big)\,ds \le 1\right) < \infty. \tag{2.17}$$

This can be done as follows: after selecting $\{t_i, c_i\colon i = 0, \dots, k-1\}$, we observe that, for $\omega \in \{\alpha_j < 1\}$,

$$P_\omega\left(\int_{(\alpha_j + r)\wedge 1}^{t_{k-1}} \left(\theta_j^2(s) \wedge \frac{1}{r}\right) ds \le 1\right) \to 0, \qquad r \to 0.$$

So we select $0 < r_k < r_{k-1}/2$ so that

$$P\left[\{\alpha_j < 1\} \cap \left\{P_\omega\left(\int_{(\alpha_j + r_k)\wedge 1}^{t_{k-1}} \left(\theta_j^2(s) \wedge \frac{1}{r_k}\right) ds \le 1\right) \ge \frac{1}{k^2}\right\}\right] \le \frac{1}{k^2}.$$

From the Borel–Cantelli lemma we get (2.17). The construction of π_ω continues now as in the first part of the proof, with only one change: $\alpha_j(\omega)$, takes the role of 0. This shows that we can assume that π is adapted. From (2.16) we get

$$\begin{aligned} P(X(1) > 0) &= E\big(P_\omega\big(X_{\pi_\omega}(1) > 0\big)\big) \\ &= P(\alpha_j < 1) \\ &> 0, \\ P(X(t) \ge 0, 0 \le t \le 1) &= E\big(P_\omega\big(X_{\pi_\omega}(t) \ge 0, 0 \le t \le 1\big)\big) \\ &= 1. \end{aligned} \tag{2.18}$$

So π is an arbitrage. □

3. Proofs of the theorems. For convenience, we repeat the theorems and corollaries here.

THEOREM 1. *There is no arbitrage if and only if* $P(\alpha = 1) = 1$ *and* $E(Z_r(1)) = 1$ *for every constant* $0 \le r \le 1$.

PROOF. (Sufficiency.) From Lemma 1 we get $E(Z_\tau(1)) = 1$ for every stopping time $0 \le \tau \le 1$. Let us assume that there exists a portfolio π that satisfies properties (a) and (b) in Definition 3, the definition of arbitrage. Let τ be the following stopping time:

$$\tau = \begin{cases} \inf\{0 \le t \le 1: X(t) \ne 0\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases}$$

It follows that $\pi'(t)B^{-1}(t)\sigma(t) = 0$ on $\{t \le \tau\}$. So

$$X(t) = \int_0^t \pi'(s)B^{-1}(s)\sigma(s)\left[dW(s) + \{\tau \le s\}\theta(s)\,ds\right]. \tag{3.1}$$

Since $E(Z_\tau(1)) = 1$ we can define a probability measure Q by $dQ = Z_\tau(1)\,dP$. We observe that $Z_\tau(t)X(t)$ is a P-local martingale:

$$dX(t) = \pi'(t)B^{-1}(t)\sigma(t)(dW(t) + \{\tau \le t\}\theta(t)\,dt),$$
$$dZ_\tau(t) = -Z_\tau(t)\{\tau \le t\}\theta'(t)\,dW(t).$$

So

$$\begin{aligned} d(Z_\tau(t)X(t)) &= Z_\tau(t)\,dX(t) + X(t)\,dZ_\tau(t) + dX(t)\,dZ_\tau(t) \\ &= Z_\tau(t)\left[\pi'(t)B^{-1}(t)\sigma(t) - X(t)\{\tau \le t\}\theta'(t)\right]dW(t). \end{aligned}$$

Since $Z_\tau(t)X(t)$ is a P-local martingale, it follows that $\{X(t): 0 \le t \le 1\}$ is a Q-local martingale. However, $X(t)$ is P-bounded from below by a constant, hence also Q-bounded from below by a constant, so $X(t)$ is actually a Q-supermartingale and so from $Q(X(0) = 0) = 1$ we conclude that

$$E_Q(X(1)) \le 0. \tag{3.2}$$

However, $Q(X(1) \ge 0) = 1$ since $P(X(1) \ge 0) = 1$. It follows that

$$Q(X(1) = 0) = 1. \tag{3.3}$$

This implies $Q(X(t) = 0,\ 0 \le t \le 1) = 1$, because $X(t)$ is a continuous Q-supermartingale. We conclude that

$$P(Z_\tau(t)X(t) = 0, 0 \le t \le 1) = 1. \tag{3.4}$$

However, $Z_\tau(\tau) = 1$ and $Z_\tau(t)$ is continuous because $P(\alpha = 1) = 1$, so on the event $\{\tau < 1\}$ there exists a random $\delta > 0$ such that $Z_\tau(t) > 0$, $\tau \le t \le \tau + \delta$. We conclude that $X(t) = 0$ on $\tau \le t \le \tau + \delta$, which contradicts the definition of τ. Therefore $P(\tau = 1) = 1$, so $P(X(1) = 0) = 1$ and π is not an arbitrage.

(Necessity.) It follows from Lemma 2 that $P(\alpha = 1) = 1$. This condition is now being assumed. Let us assume that there exists $r \in [0, 1)$ with $E(Z_r(1)) = c < 1$. Define the stopping time

$$\xi = \begin{cases} \inf\{0 \le t \le 1: Z_r(t) = 0\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{3.5}$$

Obviously $P(\xi > r) = 1$. Choose $\delta > 0$ such that $\delta + c < 1$. Let

$$Z^*(t) = E(Z_r(1) + \delta | F_t). \tag{3.6}$$

The process $Z^*(t)$ is a positive martingale process, so it follows from the martingale representation theorem [see Lipster and Shiryayev (1977), Theorem 5.9, page 171] that there is an adapted, R^d-valued process $\varphi(t)$ that satisfies $P(\int_0^1 \|\varphi(s)\|^2\, ds < \infty) = 1$, such that $Z^*(t)$ satisfies

$$\begin{aligned} dZ^*(t) &= Z^*(t)\varphi(t)\, dW(t), \\ Z^*(0) &= c + \delta. \end{aligned} \tag{3.7}$$

Recall that $\{Z_r(t)\}$ satisfied the following SDE:

$$\begin{aligned} dZ_r(t) &= -Z_r(t)\{r \le t\}\theta'(t)\, dW(t), \\ Z_r(0) &= 1. \end{aligned} \tag{3.8}$$

The quadratic variation of $\{Z_r(t)\}$ satisfies

$$d[Z_r, Z_r](t) = Z_r^2(t)\theta'(t)\theta(t)\{r \le t\}\, dt. \tag{3.9}$$

The quadratic covariation of $\{Z_r(t)\}$ and $\{Z^*(t)\}$ satisfies

$$d[Z_r, Z^*](t) = -Z_r(t)Z^*(t)\{r \le t\}\varphi'(t)\theta(t)\, dt. \tag{3.10}$$

It follows from Itô's formula that on $\{r \le t < \xi\}$ we have

$$\begin{aligned} d\left(\frac{Z^*(t)}{Z_r(t)}\right) &= \frac{1}{Z_r(t)}\, dZ^*(t) - \frac{Z^*(t)}{Z_r^2(t)}\, dZ_r(t) \\ &\quad + \frac{Z^*(t)}{Z_r^3(t)}\, d[Z_r, Z_r](t) - \frac{1}{Z_r^2(t)}\, d[Z_r, Z^*](t) \\ &= \frac{Z^*(t)}{Z_r(t)}\varphi'(t)\, dW(t) + \frac{Z^*(t)}{Z_r(t)}\theta'(t)\, dW(t) \\ &\quad + \frac{Z^*(t)}{Z_r(t)}\theta'(t)\theta(t)\, dt \\ &\quad + \frac{Z^*(t)}{Z_r(t)}\varphi'(t)\theta(t)\, dt \\ &= \frac{Z^*(t)}{Z_r(t)}(\varphi'(t) + \theta'(t))(dW(t) + \theta(t)\, dt). \end{aligned} \tag{3.11}$$

Define the stopping time

$$\beta = \begin{cases} \inf\{0 \le t \le 1 \colon Z_r(t) = Z^*(t)\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{3.12}$$

Since $Z_r(0) = 1 > c + \delta = Z^*(0)$ and $Z_r(1) < Z_r(1) + \delta = Z^*(1)$, we have $P(\beta < 1) = 1$. Since $Z_r(\beta) = Z^*(\beta) \ge \delta$, we actually have $P(\beta < \xi) = 1$.

Next we define the following portfolio:

$$\pi(t) = \begin{cases} B(t)(\sigma^{-1}(t))' \dfrac{Z^*(t)}{Z_r(t)}(\varphi(t) + \theta(t)), & r \le t \le \beta, \\ 0, & \text{otherwise}. \end{cases} \tag{3.13}$$

The capital gain process $X(t)$ for the portfolio π satisfies the following:

$$\begin{aligned} &\text{if } \beta \le r, \text{ then} \quad X(t) = 0, \quad 0 \le t \le 1; \\ &\text{if } \beta > r, \text{ then} \quad X(t) = 0, \quad 0 \le t \le r; \end{aligned} \tag{3.14}$$

$$\begin{aligned} X(t) &= \frac{Z^*(t)}{Z_r(t)} - \frac{Z^*(r)}{Z_r(r)} \\ &\ge -Z^*(r) \ge -1, \qquad r \le t \le \beta; \end{aligned} \tag{3.15}$$

$$\begin{aligned} X(t) &= \frac{Z^*(\beta)}{Z_r(\beta)} - \frac{Z^*(r)}{Z_r(r)} \\ &= 1 - Z^*(r) > 0, \qquad \beta \le t \le 1. \end{aligned} \tag{3.16}$$

We observe that $P(\beta \le r) < 1$ because $E(Z^*(\beta)) = E(Z^*(1)) = c + \delta < 1$ and $Z^*(\beta) = 1$ on the event $\{\beta \le r\}$. It follows that π is an arbitrage and we have a contradiction. This completes the proof. □

COROLLARY 1. *Assume* (1.10). *Then there is no arbitrage if and only if* $E(Z_0(1)) = 1$.

PROOF. (Necessity.) It follows from Theorem 1 that the no-arbitrage assumption implies $E(Z_0(1)) = 1$.

(Sufficiency.) Assume $E(Z_0(1)) = 1$. Since (1.10) clearly implies $P(\alpha = 1) = 1$, all that we need to show in order to establish absence of arbitrage is $E(Z_r(1)) = 1$ for every $0 < r < 1$. Fix $0 < r < 1$. Since $Z_0(t)$ is a martingale, then

$$E(Z_0(1)|F_r) = Z_0(r). \tag{3.17}$$

On the other hand,

$$E(Z_0(1)|F_r) = Z_0(r)E(Z_r(1)|F_r). \tag{3.18}$$

Putting (3.17) and (3.18) together gives

$$Z_0(r) = Z_0(r)E(Z_r(1)|F_r). \tag{3.19}$$

Since (1.10) implies $P(Z_0(r) > 0) = 1$, it follows from (3.19) that $E(Z_r(1)|F_r) = 1$, which implies $E(Z_r(1)) = 1$. □

COROLLARY 2. *Assume* (1.11). *Then there is no arbitrage if and only if there exists a probability measure* Q *equivalent to* P *such that* $W^*(t)$ *is a* Q*-Brownian motion.*

PROOF. This follows immediately from Corollary 1 and the Girsanov theorem [see Lipster and Shiryayev (1977), Theorem 6.3, page 232]. □

COROLLARY 3. *Assume* (1.11). *There is an arbitrage π whose capital gain process X satisfies $P(X(t) > -1, 0 \le t \le 1) = 1$, if and only if there exists φ, an adapted, R^d-valued process that satisfies $P(\int_0^1 \|\varphi(s)\|^2\, ds < \infty) = 1$, so that*

$$\pi(t) = \frac{Y(t)}{Z_0(t)} B(t)(\sigma^{-1}(t))'(\varphi(t) + \theta(t)),$$

where the process $Y(t)$, defined by

$$Y(t) = \exp\left\{\int_0^t \varphi'(s)\, dW(s) - \int_0^t \frac{\|\varphi(s)\|^2}{2}\, ds\right\},$$

satisfies $P(Y(1) \ge Z_0(1)) = 1$ and $P(Y(1) > Z_0(1)) > 0$. The capital gain process of π is $X(t) = Y(t)/Z_0(t) - 1$.

PROOF. (Sufficiency.) This is proved by performing the same calculation as (3.11) with $Y(t)$ and $Z_0(t)$ taking the role of $Z^*(t)$ and $Z_r(t)$, respectively:

$$d\left(\frac{Y(t)}{Z_0(t)}\right) = \frac{Y(t)}{Z_0(t)}(\varphi'(t) + \theta'(t))(dW(t) + \theta(t)\, dt).$$

By choosing π as prescribed in the corollary, we see that its capital gain is $X(t) = Y(t)/Z_0(t) - 1$, so π is an arbitrage.

(Necessity.) Let us assume that there is an arbitrage π whose capital gain process X satisfies $P(X(t) + 1 > 0, 0 \le t \le 1) = 1$. Let f be the following R^d-valued process:

$$f'(t) = \frac{\pi'(t) B^{-1}(t) \sigma(t)}{1 + X(t)}.$$

From assumptions (1.4) and (1.5) and from the fact that $\min\{X(t) + 1: 0 \le t \le 1\} > 0$, a.s., it follows that $P(\int_0^1 \|f(s)\|^2\, ds < \infty) = 1$.

From (1.8) we get

$$\begin{aligned} 1 + X(t) &= 1 + \int_0^t (1 + X(s)) f'(s)[dW(s) + \theta(s)\, ds] \\ &= \exp\left\{\int_0^t f'(s)\, dW(s) - \int_0^t \left(\frac{\|f(s)\|^2}{2} - f'(s)\theta(s)\right) ds\right\}. \end{aligned}$$

Now set $\varphi(t) = f(t) - \theta(t)$. Obviously $P(\int_0^t \|\varphi(s)\|^2\, ds \le \infty) = 1$. A simple calculation shows that

$$\frac{Y(t)}{Z_0(t)} = \exp\left\{\int_0^t f'(s)\, dW(s) - \int_0^t \left(\frac{\|f(s)\|^2}{2} - f'(s)\theta(s)\right) ds\right\}.$$

We conclude that $1 + X(t) = Y(t)/Z_0(t)$. Since π is an arbitrage it follows that $P(Y(1)/Z_0(1) \ge 1) = 1$ and $P(Y(1)/Z_0(1) > 1) > 0$. □

THEOREM 2. *There is no approximate arbitrage if and only if there exists a probability measure Q equivalent to P such that $W^*(t)$ is a Q-Brownian motion.*

PROOF. (Sufficiency.) Let $\{\pi_n\}$ be a sequence of portfolios that satisfies (a) and (b) in Definition 4. Let $\{X_n\}$ be its respective sequence of discounted capital gains; $X_n(t)$ is a Q-local martingale which is bounded below by a constant, and hence it is a Q-supermartingale. So, for all n,

$$E_Q(X_n(1)) \le 0. \tag{3.20}$$

Now, it follows from Definition 4(b) that $P(X_n(1) < 0) \to 0$, so we have $Q(X_n(1) < 0) \to 0$. Since $Q(X_n(1) > C) = 1$ we must have $E_Q(X_n(1) \wedge 0) \to 0$. However, $E_Q(X_n(1) \wedge 0) \to 0$ and (3.20) together imply

$$E_Q(|X_n(1)|) \to 0. \tag{3.21}$$

So $X_n(1) \to 0$ in Q-probability, hence in P-probability. This means that $\{\pi_n\}$ does not satisfy Definition 4(c), and it is not an approximate arbitrage.

(Necessity.) If there is no approximate arbitrage, then, of course, there is no arbitrage. So we learn from Theorem 1 that $P(\alpha = 1) = 1$ and $E(Z_0(1)) = 1$. All we need to prove is that $P(Z_0(1) > 0) = 1$, because then an application of the Girsanov theorem will finish the proof. We will assume that $P(Z_0(1) = 0) > 0$ and construct an approximate arbitrage.

For all n let the following hold:

$$\tau_n = \begin{cases} \inf\{0 \le t \le 1: Z_0(t) = 1/n\}, \\ 1, \quad \text{if no such } t \text{ exists;} \end{cases}$$

$$\beta_n = \begin{cases} \inf\left\{\tau_n \le t \le 1: \int_{\tau(n)}^t \theta'(s)\,dW(s) + \int_{\tau(n)}^t \frac{\|\theta(s)\|^2}{2}\,ds = 1\right\}, \\ 1, \quad \text{if no such } t \text{ exists;} \end{cases} \tag{3.22}$$

$$Y_n(t) = \exp\left\{\int_0^t \{\tau_n \le s \le \beta_n\}\left(\theta'(s)\,dW(s) + \frac{\|\theta(s)\|^2}{w}\,ds\right)\right\};$$

$$\pi_n(t) = \{\tau_n \le t \le \beta_n\} Y_n(t) B(t)\left(\sigma^{-1}(t)\right)'\theta(t).$$

Let $X_n(t)$ be the discounted capital gain of portfolio π_n at time t. We will prove that $\{\pi_n\}$ is an approximate arbitrage. We get

$$\begin{aligned} 1 + X_n(t) &= 1 + \int_0^t \pi_n'(s) B^{-1}(s)\sigma(s)[dW(s) + \theta(s)\,ds] \\ &= 1 + \int_0^t Y_n(s)\{\tau_n \le s \le \beta_n\}\left(\theta'(s)\,dW(s) + \|\theta(s)\|^2\,ds\right). \end{aligned} \tag{3.23}$$

It follows from the definition of Y_n that

$$Y_n(t) = 1 + \int_0^t Y_n(s)\{\tau_n \le s \le \beta_n\}\left(\theta'(s)\,dW(s) + \|\theta(s)\|^2\,ds\right).$$

So we have

$$1 + X_n(t) = Y_n(t). \tag{3.24}$$

Since $Y_n(t)$ is positive, we get that $\{X_n\}$ satisfies Definition 4(a) with $C = -1$. Next we observe that

$$\{Z_0(1) = 0\} \subseteq \{\tau_n < 1\}, \tag{3.25}$$

because $Z_0(t)$ is continuous. We also have

$$\{Z_0(1) = 0\} \subseteq \left\{\int_{\tau(n)}^t \theta'(s)\,dW(s) + \int_{\tau(n)}^t \frac{\|\theta(s)\|^2}{2}\,ds \to \infty, t \to 1\right\}.$$

So we get

$$\{Z_0(1) = 0\} \subseteq \{\tau_n < \beta_n < 1\}. \tag{3.26}$$

From (3.22) and (3.24) we conclude that

$$\{\tau_n < \beta_n < 1\} \subseteq \{X_n(1) = \exp(1) - 1 > 0\}. \tag{3.27}$$

It follows that $\{X_n(1) < 0\} \subseteq \{\tau_n < 1\} \cap \{Z_0(1) > 0\}$, but continuity of the process Z_0 implies that $P(\{\tau_n < 1\} \cap \{Z_0(1) > 0\}) \to 0$, and this proves that $\{X_n\}$ satisfies Definition 4(b). Definition 4(c) follows from (3.26) and (3.27):

$$P(X_n(1) \geq \exp(1) - 1) \geq P(Z_0(1) = 0) > 0 \quad \text{for all } n.$$

So $\{\pi_n\}$ is an approximate arbitrage. □

4. Some examples. Here are some simple examples that we have encountered while doing this work. In all of them we assume $d = 1$, $\sigma = 1$ and $r = 0$.

EXAMPLE 1. This is an example where (1.11) holds and $E(Z_0(1)) < 1$. It looks somehow simpler than a related example in Lipster and Shiryaev [(1977), page 222]. It follows from Corollary 2 that there is an arbitrage. Let

$$\theta(t) = \{t \leq \tau\}(1 - t)^{-1}, \tag{4.1}$$

where τ is a stopping time defined by

$$\tau = \begin{cases} \inf\left\{t\colon -\int_0^t \frac{1}{1-s}\,dW(s) - \int_0^t \frac{1}{2(1-s)^2}\,ds = -1\right\}, \\ 1, \quad \text{if no such } t \text{ exists}. \end{cases} \tag{4.2}$$

Since we have

$$-\int_0^t \frac{1}{1-s}\,dW(s) - \int_0^t \frac{1}{2(1-s)^2}\,ds \to -\infty, \qquad t \to 1 \text{ a.s.},$$

it follows that $P(\tau < 1) = 1$, so (1.11) holds and $E(Z_0(1)) = \exp(-1) < 1$.

EXAMPLE 2. In this example $P(\alpha = 1) = 1$, $E(Z_0(1)) = 1$ and $P(Z_0(1) = 0) > 0$. It follows from Theorem 2 that there is an approximate arbitrage. We will show that there is no arbitrage.

Let

$$\theta(t) = \{t \le \beta\}(1 - t)^{-1}, \tag{4.3}$$

where β is a stopping time defined by

$$\beta = \begin{cases} \inf\left\{t: -\int_0^t \frac{1}{1-s}\,dW(s) - \int_0^t \frac{1}{2(1-s)^2}\,ds = 1\right\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{4.4}$$

We get that $Z_0(t)$ is bounded by exp(1). Since $Z_0(t)$ is a local martingale, it is actually a martingale and $E(Z_0(1)) = 1$. So we conclude from Corollary 1 that there does not exist an arbitrage. We observe next that $Z_0(1) \in \{0, \exp(1)\}$ a.s. It follows that $P(Z_0(1) = 0) = 1 - \exp(-1) > 0$.

EXAMPLE 3. This example is a minor modification of Example 2. As in Example 2 we will have $P(\alpha = 1) = 1$, $E(Z_0(1)) = 1$ and $P(Z_0(1) = 0) > 0$, but there exists an arbitrage.

Let

$$\theta(t) = \begin{cases} \{t \le \beta\}(0.5 - t)^{-1}, & \text{if } 0 \le t \le 0.5, \\ 0, & \text{if } 0.5 \le t \le 1 \text{ and } Z_0(0.5) > 0, \\ (1-t)^{-1}, & \text{if } 0.5 \le t \le 1 \text{ and } Z_0(0.5) = 0, \end{cases} \tag{4.5}$$

where β is a stopping time defined by

$$\beta = \begin{cases} \inf\left\{t \le 0.5: -\int_0^t \frac{1}{1-s}\,dW(s) - \int_0^t \frac{1}{2(1-s)^2}\,ds = 1\right\}, \\ 0.5, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{4.6}$$

By arguing as in Example 2, we get $E(Z_0(0.5)) = 1$ and $P(Z_0(0.5) = 0) > 0$. However, $Z_0(1) = Z_0(0.5)$ a.s., so indeed $E(Z_0(1)) = 1$ and $P(Z_0(1) = 0) > 0$. The existence of an arbitrage follows from Theorem 1 since $E(Z_{0.5}(1)) = \exp(-1) < 1$.

EXAMPLE 4. In this example $P(\alpha < 1) > 0$ and $E(Z_r(1)) = 1$ for every $0 \le r \le 1$. Let φ: $R \to (0.25, 1)$ be measurable and strictly increasing. Define

$$\beta = \varphi(W(0.25)), \tag{4.7}$$

where β is a stopping time and $P(\beta = r) = 0$, $0 \le r \le 1$. Define the stopping time

$$\tau = \begin{cases} \inf\left\{0.25 < t \le \beta: -\int_{0.25}^t \frac{1}{\beta - s}\,dW(s) - \int_{0.25}^t \frac{1}{2(\beta - s)^2}\,ds = 1\right\}, \\ \beta, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{4.8}$$

Now we define $\theta(t)$:

$$\theta(t) = \begin{cases} 0, & \text{if } 0 \le t \le 0.25, \\ \{t \le \tau\}(\beta - t)^{-1}, & \text{if } 0.25 < t \le \beta, \\ 0, & \text{if } \beta < t \le 1 \text{ and } \tau < \beta, \\ (t - \beta)^{-1}, & \text{if } \beta < t \le 1 \text{ and } \tau = \beta. \end{cases} \tag{4.9}$$

We have

$$P(\alpha < 1) = P(\beta = \tau) > 0. \tag{4.10}$$

Since $Z_0(t)$ is bounded and a local martingale, it is a martingale and $E(Z_0(1)) = 1$. Our goal is to show that $E(Z_r(1)) = 1$, $0 < r \le 1$. Fix r. We have

$$\begin{aligned} Z_0(r) &= E(Z_0(1)|F_r) \\ &= Z_0(r)E(Z_r(1)|F_r). \end{aligned}$$

It follows that

$$E(Z_r(1)\{Z_0(r) > 0\}) = P(Z_0(r) > 0). \tag{4.11}$$

Next observe that on $\{\beta < r, \tau = \beta\}$ we have

$$Z_r(1) = \exp\left\{ -\int_r^1 \frac{1}{s - \beta}\, dW(s) - \int_r^1 \frac{1}{2(s - \beta)^2}\, ds \right\}.$$

Let $P_\omega(\cdot)$ be the regular conditional probability of P given F_r. With respect to P_ω, $\omega \in \{\beta < r, \tau = \beta\}$, we have $\beta = \beta(\omega)$ is a constant, $\int_r^1 (1/(s - \beta)^2)\, ds < \infty$ and $\{W(r + h) - W(r) \colon 0 \le h\}$ is a standard Brownian motion. So, for every $\omega \in \{\beta < r, \tau = \beta\}$, $Z_r(t)$ is a P_ω-martingale and $E_\omega(Z_r(1)) = 1$. We conclude that

$$E(Z_r(1)\{\beta < r, \tau = \beta\}) = P(\beta < r, \tau = \beta). \tag{4.12}$$

Since $\{Z_0(r) = 0\} = \{\beta < r, \ \tau = \beta\}$, we get from (4.11) and (4.12) that $E(Z_r(1)) = 1$.

Example 5. Here we assume (1.11), $E(Z_0(1)) < 1$ and that there is a constant $D > 0$ such that $P(Z_0(1) < D) = 1$. Let

$$\tau = \begin{cases} \inf\{0 \le t \colon Z_0(t) = D\}, \\ 1, \quad \text{if no such } t \text{ exists} \end{cases}$$

Observe that $P(\tau < 1) > 0$; otherwise $Z_0(t)$, as a uniformly bounded local martingale, will be a martingale and then we would have $E(Z_0(1)) = 1$. Using Corollary 3, we can construct an arbitrage by taking

$$\pi(t) = \begin{cases} 0, & \text{if } 0 \le t \le \tau, \\ \dfrac{\theta(t)D}{Z_0(t)}, & \text{if } \tau < t \le 1. \end{cases}$$

The capital gain of this π satisfies

$$X(1) = \begin{cases} 0, & \text{if } \tau = 1. \\ \dfrac{D}{Z_0(1)} - 1, & \text{if } \tau < 1. \end{cases}$$

This is a different arbitrage than the one which follows from the proof of Theorem 1.

REFERENCES

DELBAEN, F. (1992). Representing martingale measures when asset prices are continuous and bounded. *Math. Finance* **2** 107–130.

DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520.

DELBAEN, F. and SCHACHERMAYER, W. (1995). The existence of absolutely continuous local martingale measures. *Ann. Appl. Probab.* **5** 926–945.

DUFFIE, D. (1992). *Dynamic Asset Pricing Theory*. Princeton Univ. Press.

DYBVIG, P. and HUANG, C. F. (1988). Nonnegative wealth, absence of arbitrage, and feasible consumption plans. *Review of Financial Studies* **1** 377–401.

FRITTELLI, M. and LAKNER, P. (1994). Arbitrage and free lunch in a general financial market model; the fundamental theorem of assest pricing. In *Mathematical Finance* (M. H. A. Davis, D. Duffie, W. H. Fleming and S. E. Shreve, eds.) Springer, New York.

HARRISON, M. and KREPS, D. (1979). Martingales and arbitrage in multiperiod security markets. *J. Econom. Theory* **20** 381–408.

HARRISON, M. and PLISKA, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260.

KARATZAS, I. (1993). *Tutorial on Mathematical Finance*. Inst. Math. Appl., Univ. Minnesota.

KARATZAS, I. and SHREVE, S. (1987). *Brownian Motion and Stochastic Calculus*. Springer, New York.

LIPSTER, R. S. and SHIRYAYEV, A. N. (1977). *Statistics of Random Processes* **1**. *General Theory*. Springer, New York.

STRICKER, C. (1990). Arbitrage et lois de Martingale. *Ann. Inst. H. Poincaré B* **26** 451–460.

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