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APPROXIMATION OF SMOOTH CONVEX BODIES BY RANDOM CIRCUMSCRIBED POLYTOPES

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Choose n independent random points on the boundary of a convex body $K \subset \mathbb{R}^d$. The intersection of the supporting halfspaces at these random points is a random convex polyhedron. The expectations of its volume, its surface area and its mean width are investigated. In the case that the boundary of K is sufficiently smooth, asymptotic expansions as $n \to \infty$ are derived even in the case when the curvature is allowed to be zero. We compare our results to the analogous results for best approximating polytopes.

1. Introduction and statement of results. Let K be a compact convex set in $\mathbb{R}^d$ with nonempty interior and with boundary of differentiability class $\mathcal{C}^2$. Choose n random points $X_1, \dots, X_n$ on the boundary ∂K, independently and identically distributed with respect to a given density function d_K. Denote by $H_+(X_i)$ the supporting halfspace to ∂K at X_i, and define the random polyhedron as the intersection $\bigcap_{i=1,\dots,n} H_+(X_i)$. If n is sufficiently large, then with high probability the random polyhedron is a quite precise approximation of the convex body K. Clearly, if $d_K > 0$ the random polyhedron tends to K with probability one as n tends to infinity and, by continuity, volume, surface area and mean width of the random polyhedron tend, respectively, to the volume, surface area and mean width of K. It is of interest to determine the rate of convergence.

In this paper we investigate the expectation of the difference of volume V, surface area S and mean width W of the random polyhedron and the convex body K. It should be noted that with small but positive probability the random polyhedron is unbounded. Hence, with positive probability, the volume of the random polyhedron will be infinite and the expectation will not exist. To ensure that the random polyhedron becomes bounded we intersect it with a large cube C which contains the convex body K in its interior. Thus we define

$$P_{(n)} := \bigcap_{i=1}^{n} H_+(X_i) \cap C$$

and investigate $V(P_{(n)}) - V(K)$, $S(P_{(n)}) - S(K)$ and $W(P_{(n)}) - W(K)$. Note that our results will not depend on the actual choice of C.

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Approximation of convex bodies by polytopes has been considered frequently for nearly 100 years. Most of the investigations focused on the one hand side on best-approximating inscribed and circumscribed polytopes, and on the other hand, on inscribed random polytopes. Interestingly enough, it turns out that only a few papers deal with circumscribed random polytopes and that volume, surface area and mean width of random circumscribed polytopes have not been investigated at all so far. In this paper we fill this gap and then compare our results to those for best-approximating circumscribed polytopes.

The problem to construct a best-approximating circumscribed polytope can be formulated in the following way: Choose n points $X_1, \dots, X_n$ on the boundary of K such that the polytope $P_{(n)}^{\mathrm{best}} = \cap H_+(X_i)$ is as close as possible to K, that is, the difference of the volume of $P_{(n)}^{\mathrm{best}}$ and the volume of K should be minimal among all possible choices of points $X_i \in \partial K$. For convex bodies $K \in \mathcal{C}^2$ with positive Gaussian curvature, Gruber [8] proved

$$V(P_{(n)}^{\mathrm{best}}) - V(K) = \tfrac{1}{2}\mathrm{div}_{d-1}\Omega(K)^{(d+1)/(d-1)} n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big) \tag{1}$$

as $n \to \infty$, where div_{d-1} is a constant depending on the dimension only. Here $\Omega(K)$ denotes the affine surface area of K,

$$\Omega(K) = \int_{\partial K} H_{d-1}(x)^{1/(d+1)}\, dx,$$

where $H_{d-1}(x)$ denotes the Gaussian curvature of ∂K at x and dx denotes integration with respect to the $(d-1)$-dimensional Hausdorff measure on ∂K. Formula (1) has been generalized to $K \in \mathcal{C}^2$ (allowing the Gaussian curvature to be zero) by Böröczky [2].

It turns out that the corresponding result for random polytopes has the same order of approximation as occurs in (1). But, for random polytopes we can even prove an asymptotic expansion for $\mathbb{E}V(P_{(n)}) - V(K)$ if the convex body is sufficiently smooth. In general the existence of asymptotic expansions for best-approximating polytopes seems to be unknown, except in the case $d = 2$ (Ludwig [12] and Tabachnikov [25]), but see also Remark 2 at the end of this section.

THEOREM 1. *Let $K \in \mathcal{C}^2$ and choose n random points on ∂K independently and according to a continuous density function $d_K > 0$. Then*

$$\begin{aligned} \mathbb{E}V(P_{(n)}) - V(K) &= \frac{1}{2}\kappa_{d-1}^{-2/(d-1)}\Gamma\left(\frac{2}{d-1}+1\right) \\ &\quad \times \int_{\partial K} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\, dx\, n^{-2/(d-1)} \\ &\quad + o\big(n^{-2/(d-1)}\big) \end{aligned} \tag{2}$$

as $n \to \infty$. *Moreover, if* $K \in \mathcal{C}^k$ *with positive Gaussian curvature*, $k \geq 3$ *and* $d_K \in \mathcal{C}^{k-2}$ *with* $d_K > 0$, *then*

$$\mathbb{E}V(P_{(n)}) - V(K) = c_2^V(K)\, n^{-2/(d-1)} + c_3^V(K) n^{-3/(d-1)} + \cdots + c_{k-1}^V(K) n^{-(k-1)/(d-1)} + O\big(n^{-k/(d-1)}\big)$$

as $n \to \infty$. *The constants* $c_m^V(K)$ *satisfy* $c_{2m-1}^V(K) = 0$ *for* $m \leq d/2$ *if* d *is even, and* $c_{2m-1}^V(K) = 0$ *for all* m *if* d *is odd.*

The asymptotic formula (2) was already treated by Kaltenbach [11] if $K \in \mathcal{C}^3$ with positive Gaussian curvature. Recently, an estimate for the variance of $V(P_{(n)}) - V(K)$ led in [19] to a stronger version of (2): for any positive density function d_K a strong law of large numbers holds for $V(P_{(n)}) - V(K)$.

Using Hölder's inequality it is easy to observe that for given K the right-hand side of (2) is minimized if the density function d_K equals

$$d_K^{\min}(x) = \frac{H_{d-1}(x)^{1/(d+1)}}{\int_{\partial K} H_{d-1}(x)^{1/(d+1)}\, dx}.$$

Hence choosing random points on ∂K according to $d_K^{\min}$ by Theorem 1 gives

$$\mathbb{E}V(P_{(n)}) - V(K) = \frac{1}{2}\kappa_{d-1}^{-2/(d-1)} \Gamma\left(\frac{2}{d-1} + 1\right) \Omega(K)^{(d+1)/(d-1)} n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big)$$

as $n \to \infty$, which should be compared to formula (1) for best-approximating polytopes. In particular, (1) and (2) immediately imply for $K \in \mathcal{C}^2$

$$\lim_{n\to\infty} \frac{\mathbb{E}V(P_{(n)}) - V(K)}{V(P_{(n)}^{\text{best}}) - V(K)} = \frac{\kappa_{d-1}^{-2/(d-1)} \Gamma(\frac{2}{d-1} + 1)}{\operatorname{div}_{d-1}}, \tag{3}$$

which is independent of the convex body K. Thus it is of interest to compare the arising constants. With respect to div_{d-1}, its value is only known for $d = 2, 3$, but its asymptotic behavior as $d \to \infty$ has been determined by Zador [26], see also Conway and Sloane ([5], page 58),

$$\operatorname{div}_{d-1} = \frac{1}{2e\pi} d + o(d).$$

On the other hand, Stirling's formula yields

$$\kappa_{d-1}^{-2/(d-1)} \Gamma\left(\frac{2}{d-1} + 1\right) = \frac{1}{2e\pi} d + o(d)$$

as $d \to \infty$, which implies that the right-hand side of (3) tends to one as the dimension tends to infinity. In particular, as the dimension tends to ∞, approximation of convex bodies by random circumscribed polytopes is as good as approximation of convex bodies by best-approximating circumscribed polytopes.

In order to state the corresponding results for the surface area of random polytopes we need the following notion: for $K \in \mathcal{C}^2$ denote by $k_1(x), \dots, k_{d-1}(x)$ the principal curvatures of K at x. In particular, the mean curvature $H_1(x)$ is $(d-1)^{-1} \sum k_i(x)$ and the Gaussian curvature $H_{d-1}(x)$ is $\prod k_i(x)$.

The result for the surface area is surprisingly complicated and reads as follows:

THEOREM 2. *Let $K \in \mathcal{C}^2$ and choose n random points on ∂K independently and according to a continuous density function $d_K > 0$. Then*

$$\begin{aligned} \mathbb{E}S(P_{(n)}) - S(K) &= (d-1)\kappa_{d-1}^{-(d+1)/(d-1)} \Gamma\left(\frac{2}{d-1}+1\right) \\ &\quad \times \int_{\partial K} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)} \left\{ H_1(x) \frac{1}{2(d+1)} M(x) \right\} dx\, n^{-2/(d-1)} \\ &\quad + o\left(n^{-2/(d-1)}\right) \end{aligned} \tag{4}$$

as $n \to \infty$ where

$$M(x) := \frac{1}{(d-1)\kappa_{d-1}} \int_{S^{d-2}} \left(\sum_i k_i(x)^2 v_i^2 \right) \left(\sum_i k_i(x) v_i^2 \right)^{-(d+1)/2} dv$$

with $v = (v_1, \dots, v_{d-1})$. Moreover, if $K \in \mathcal{C}^k$ with positive Gaussian curvature, $k \geq 3$ and $d_K \in \mathcal{C}^{k-2}$ with $d_K > 0$, then

$$\begin{aligned} \mathbb{E}S(P_{(n)}) - S(K) &= c_2^S(K) n^{-2/(d-1)} + c_3^S(K) n^{-3/(d-1)} + \cdots \\ &\quad + c_{k-1}^S(K) n^{-(k-1)/(d-1)} + O\left(n^{-k/(d-1)}\right) \end{aligned}$$

as $n \to \infty$. The constants $c_m^S(K)$ satisfy $c_{2m-1}^S(K) = 0$ for $m \leq d/2$ if d is even and $c_{2m-1}^S(K) = 0$ for all m if d is odd.

Note that the right-hand side of (4) is minimized if d_K equals the following complicated density function:

$$d_K^{\min}(x) = \frac{H_{d-1}(x)^{1/(d+1)} \{H_1(x) + 1/(2(d+1)) M(x)\}^{(d-1)/(d+1)}}{\int_{\partial K} H_{d-1}(x)^{1/(d+1)} \{(H_1(x) + 1/(2(d+1)) M(x)\}^{(d-1)/(d+1)}\, dx}.$$

It is a difficult open problem to deduce an analogous formula for best-approximating polytopes.

As a third notion of "distance" between $P_{(n)}$ and K the difference of the mean width can be investigated; as for best-approximating polytopes, choose $X_i \in \partial K$ such that $P_{(n)}^{\text{best}}$ is close to K in the sense that $W(P_{(n)}^{\text{best}}) - W(K)$ is minimal.

The deduction of the asymptotic behavior of $W(P_{(n)}^{\mathrm{best}}) - W(K)$ for $K \in \mathcal{C}^2$ with positive Gaussian curvature is due to Glasauer and Gruber [7],

$$W(P_{(n)}^{\mathrm{best}}) - W(K) = \frac{1}{d\kappa_d} \mathrm{del}_{d-1} \left(\int_{\partial K} H_{d-1}(x)^{d/(d+1)}\, dx \right)^{(d+1)/(d-1)} n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big) \tag{5}$$

as $n \to \infty$ and the generalization to $K \in \mathcal{C}^2$ is due to Böröczky [2]. The constant del_{d-1} depends on the dimension only.

In the paper [7] an idea of Glasauer relates the mean width of a convex body K to a certain integral over the polar body K^* of K. This can be used to determine the asymptotic behavior of the expected mean width $\mathbb{E}W(P_{(n)})$ of the random polytope. Note that this method requires a stronger differentiability class for proving asymptotic expansions.

THEOREM 3. *Let $K \in \mathcal{C}^2$ and choose n random points on ∂K independently and according to a continuous density function $d_K > 0$. Then*

$$\mathbb{E}W(P_{(n)}) - W(K) = \frac{(d-1)\Gamma(d+1+\frac{2}{d-1})}{d(d+1)!\kappa_d \kappa_{d-1}^{2/(d-1)}} \int_{\partial K} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{d/(d-1)}\, dx\, n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big) \tag{6}$$

as $n \to \infty$. Moreover, if $K \in \mathcal{C}^{k+2}$ with positive Gaussian curvature, $k \geq 3$ and $d_K \in \mathcal{C}^k$ with $d_K > 0$, then

$$\mathbb{E}W(P_{(n)}) - W(K) = c_2^W(K)\, n^{-2/(d-1)} + c_3^W(K) n^{-3/(d-1)} + \cdots + c_{k-1}^W(K) n^{-(k-1)/(d-1)} + O\big(n^{-k/(d-1)}\big)$$

as $n \to \infty$. The constants $c_m^W(K)$ satisfy $c_{2m-1}^W(K) = 0$ for $m \leq d/2$ if d is even and $c_{2m-1}^W(K) = 0$ for all m if d is odd.

By Hölder's inequality, the right-hand side of (6) is minimized if d_K equals

$$d_K^{\min}(x) = \frac{H_{d-1}(x)^{d/(d+1)}}{\int_{\partial K} H_{d-1}(x)^{d/(d+1)}\, dx}.$$

Then by Theorem 3,

$$\mathbb{E}W(P_{(n)}) - W(K) = \frac{(d-1)\Gamma(d+1+\frac{2}{d-1})}{d(d+1)!\kappa_d \kappa_{d-1}^{2/(d-1)}} \left(\int_{\partial K} H_{d-1}(x)^{d/(d+1)}\, dx \right)^{(d+1)/(d-1)} n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big) \tag{7}$$

as $n \to \infty$. We compare this result to the one for best-approximating polytopes (5). Since in both cases the order of approximation is $n^{-2/(d-1)}$ and also the dependence on the convex body K is the same, we only have to compare the coefficients occurring in (5) and (6). The asymptotic behavior of del_{d-1} as $d \to \infty$ was determined by Mankiewicz and Schütt [13, 14],

$$\mathrm{del}_{d-1} = \frac{1}{2e\pi}\, d + o(d).$$

On the other hand, by Stirling's formula,

$$\frac{(d-1)\Gamma(d+1+\frac{2}{d-1})}{(d+1)!\kappa_{d-1}^{2/(d-1)}} = \frac{1}{2e\pi}\, d + o(d)$$

as $d \to \infty$, which implies for convex bodies $K \in \mathcal{C}^2$ with positive Gaussian curvature

$$\lim_{n\to\infty} \frac{\mathbb{E}W(P_{(n)}) - W(K)}{W(P_{(n)}^{\mathrm{best}}) - W(K)} = \frac{(d-1)\Gamma(d+1+\frac{2}{d-1})}{(d+1)!\kappa_{d-1}^{2/(d-1)}\,\mathrm{del}_{d-1}} \to 1 \qquad \text{as } d \to \infty.$$

Thus, also for the mean width, approximation of convex bodies by random circumscribed polytopes is as good as approximation of convex bodies by best-approximating circumscribed polytopes as the dimension tends to ∞.

REMARK 1. For convex bodies of class $\mathcal{C}^\infty$ and with positive Gaussian curvature, Theorems 1 to 3 yield asymptotic expansions for $\mathbb{E}V(P_{(n)}) - V(K)$, $\mathbb{E}S(P_{(n)}) - S(K)$, and $\mathbb{E}W(P_{(n)}) - W(K)$ as $n \to \infty$.

REMARK 2. As already mentioned, in case of best approximation of a planar convex domain K by circumscribed polygons of n sides, Tabachnikov [25] verified the analogue of Theorem 1; namely, the existence of a Taylor expansion in terms of n^{-2} if $K \in \mathcal{C}^\infty$ with positive Gaussian curvature. In light of Theorem 1, it might be surprising that one does not have the Taylor expansion in terms of n^{-1} in case of best approximation of the unit three-ball by circumscribed polytopes with n faces. This was proved by Böröczky and Fejes Tóth [3].

REMARK 3. Analogous results for best-approximating inscribed polytopes are due to Glasauer and Gruber [7] and Gruber [8] (investigating the asymptotic behavior of best-approximating inscribed polytopes with n vertices as $n \to \infty$).

Analogous results for inscribed random polytopes mostly deal with random polytopes chosen in the interior of the convex body K (cf., e.g., Bárány [1] and Reitzner [17]). Only recently, systematic research investigating random polytopes with vertices chosen on the boundary of K was done (after work of Buchta, Müller and Tichy [4] and Müller [15, 16]; also see Schütt and Werner [23, 24] and Reitzner [18]).

2. Tools.

2.1. The first tool is a precise description of the local behavior of the boundary of a convex body $K \in \mathcal{C}^2$. For convex bodies $K \in \mathcal{C}^2$ with positive Gaussian curvature the following lemma was proved in [18]. The generalization to convex bodies $K \in \mathcal{C}^2$ is a straightforward task and thus we omit the proof.

Fix $K \in \mathcal{C}^2$. Consider that part of the boundary of K where all principal curvatures $k_i(x)$ are bounded away from zero by a given constant, say $\varepsilon > 0$. Denote the set of such boundary points by $\partial K_+ = \partial K_+(\varepsilon)$. At every boundary point $x \in \partial K_+$ there is a paraboloid $Q_2^{(x)}$—given by a quadratic form $b_2^{(x)}$—osculating ∂K at x. $Q_2^{(x)}$ and $b_2^{(x)}$ can be defined in the following way: identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. Then there is a convex function $f^{(x)}(y) \in \mathcal{C}^2$, $y = (y^1, \dots, y^{d-1}) \in \mathbb{R}^{d-1}$ representing ∂K in a neighborhood of x, that is, $(y, f^{(x)}(y)) \in \partial K$. Denote by $f_{ij}^{(x)}(0)$ the second partial derivatives of $f^{(x)}$ at the origin. Then

$$b_2^{(x)}(y) := \frac{1}{2} \sum_{i,j} f_{ij}^{(x)}(0) y^i y^j$$

and

$$Q_2^{(x)} := \{(y, z) \mid z \geq b_2^{(x)}(y)\}.$$

The essential point in the following lemma is the fact that these paraboloids approximate the boundary of K uniformly for all $x \in \partial K_+$. Note that "flat" parts of ∂K cannot be approximated by paraboloids. Thus this approximation works only in suitable neighborhoods U^λ of $x \in \partial K_+$ which do not intersect those parts of ∂K where the Gaussian curvature vanishes.

LEMMA 1. *Let $K \in \mathcal{C}^2$ and $\varepsilon > 0$ in the definition of ∂K_+ be given. Choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$ only depending on δ, ε and K, such that for each point x of ∂K_+ the following holds: identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. The λ-neighborhood U^λ of x in ∂K defined by $\operatorname{proj}_{\mathbb{R}^{d-1}} U^\lambda = \lambda B^{d-1}$ can be represented by a convex function $f^{(x)}(y) \in \mathcal{C}^2$, $y \in \lambda B^{d-1}$. Furthermore*

$$(8) \qquad (1+\delta)^{-1} b_2^{(x)}(y) \leq f^{(x)}(y) \leq (1+\delta)\, b_2^{(x)}(y) \qquad \textit{for } y \in \lambda B^{d-1},$$

$$(9) \qquad \sqrt{1 + |\operatorname{grad} f^{(x)}(y)|^2} \leq (1+\delta) \qquad \textit{for } y \in \lambda B^{d-1},$$

$$(10) \qquad (1+\delta)^{-1} d_K(x) \leq d_K(p) \leq (1+\delta)\, d_K(x) \qquad \textit{for } p \in U^\lambda$$

and

$$(11) \qquad (1+\delta)^{-1} 2b_2^{(x)}(y) \leq (y, 0) \cdot n_K(y) \leq (1+\delta)\, 2b_2^{(x)}(y) \qquad \textit{for } y \in \lambda B^{d-1},$$

where $n_K(y)$ is the outer unit normal vector of K at the boundary point $(y, f^{(x)}(y))$.

We need the following refinement of (9):

SUPPLEMENT TO LEMMA 1. *For $\delta > 0$ and thus $\lambda > 0$ sufficiently small the following holds for each x of ∂K_+:*

$$(1+\delta)^{-1}\frac{1}{2}\sum_i\left(\sum_j f_{ij}^{(x)} y_j\right)^2 \tag{12}$$
$$\leq \sqrt{1+\sum_i f_i^{(x)}(y)^2}-1 \leq (1+\delta)\frac{1}{2}\sum_i\left(\sum_j f_{ij}^{(x)} y_j\right)^2$$

for $y \in \lambda B^{d-1}$.

This can be readily proved using the methods of the proof of Lemma 2 in [18] and using the facts that $\sqrt{1+\sum_i f_i^{(x)}(y)^2}-1$ can be replaced by $(1/2)\sum_i f_i^{(x)}(y)^2$, that

$$f_i^{(x)}(y) = f_i^{(x)}(x) + \sum_j f_{ij}^{(x)}(x+\theta y)y_j$$

for a suitable θ, and the continuity of $f_{ij}^{(x)}$.

2.2. The next tool is a description of the boundary of a convex body $K \in \mathcal{C}^k$ with positive Gaussian curvature for $k \geq 3$. It is a straightforward generalization of a result of Schneider [21] concerning convex bodies of class $\mathcal{C}^3$ to convex bodies of class $\mathcal{C}^k$.

LEMMA 2. *Let $K \in \mathcal{C}^k$ with positive Gaussian curvature, $k \geq 3$, be given. Then there are constants α, $\beta > 0$ only depending on K such that the following holds for every boundary point $x \in \partial K$: identify the support plane of K at x with $\mathbb{R}^{d-1}$ and x with the origin. Then the α-neighborhood of x in ∂K can be represented by a convex function $f(y)$ of differentiability class $\mathcal{C}^k$, $y \in \mathbb{R}^{d-1}$. Furthermore the absolute values of the partial derivatives of $f(y)$ up to order k are uniformly bounded by β.*

2.3. The third tool concerns the Taylor expansion of inverse functions. It is a refinement of well-known results on the inversion of analytic functions (cf., e.g., [10], Sections 1.7. and 1.9.) due to Gruber [9].

LEMMA 3. *Let*

$$z = z(w,t) = b_m(w)t^m + \cdots + b_k(w)t^k + O(t^{k+1})$$

for $0 \leq t \leq \alpha$, $2 \leq m \leq k$, be a strictly increasing function in t for each fixed w in a given set. Assume that $b_m(\cdot)$ is bounded between positive constants, that

$b_{m+1}(\cdot), \dots, b_k(\cdot)$ are bounded and that the constant in $O(\cdot)$ may be chosen independent of w. Then there are coefficients $c_1(\cdot), \dots, c_{k-m+1}(\cdot)$, and a constant $\gamma > 0$ independent of w, such that for each fixed w the inverse function $t(w, \cdot)$ of $z(w, \cdot)$ has the representation

$$t = t(w, z) = c_1(w) z^{1/m} + \dots + c_{k-m+1}(w) z^{(k-m+1)/m} + O\big(z^{(k-m+2)/m}\big)$$

for $0 \le z \le \gamma$. The coefficients $c_1(\cdot), \dots, c_{k-m+1}(\cdot)$ can be determined explicitly in terms of $b_m(\cdot), \dots, b_k(\cdot)$; in particular,

$$c_1(\cdot) = \frac{1}{b_m(\cdot)^{1/m}}, \qquad c_2(\cdot) = -\frac{b_{m+1}(\cdot)}{m b_m(\cdot)^{(m+2)/m}},$$

$$c_3(\cdot) = -\frac{b_{m+2}(\cdot)}{m b_m(\cdot)^{(m+3)/m}} + \frac{(m+3) b_{m+1}(\cdot)^2}{2m^2 b_m(\cdot)^{(2m+3)/m}},$$

$$c_4(\cdot) = -\frac{b_{m+3}(\cdot)}{m b_m(\cdot)^{(m+4)/m}} + \frac{(m+4) b_{m+1}(\cdot) b_{m+2}(\cdot)}{m^2 b_m(\cdot)^{(2m+4)/m}} - \frac{(m+2)(m+4) b_{m+1}(\cdot)^3}{3m^3 b_m(\cdot)^{(3m+4)/m}}.$$

The coefficients are bounded and if $b_m(\cdot), \dots, b_k(\cdot)$ are continuous, so are $c_1(\cdot), \dots, c_{k-m+1}(\cdot)$ and the constant in $O(\cdot)$ may be chosen independent of w.

REMARK. It is easy to check the following additional property of the coefficients $c_i(\cdot)$: if $b_m(w), b_{m+2}(w), b_{m+4}(w), \dots$ are even functions, and $b_{m+1}(w)$, $b_{m+3}(w), b_{m+5}(w), \dots$ are odd functions of w, then $c_1(w), c_3(w), c_5(w), \dots$ are even functions, and $c_2(w), c_4(w), c_6(w), \dots$ are odd functions of w. Further if $b_{m+1}(w), b_{m+3}(w), b_{m+5}(w), \dots$ vanish, then also $c_2(w), c_4(w), c_6(w), \dots$ vanish.

3. Proof of Theorem 1. The proof is divided into two parts. In Sections 3.1–3.5 we prove the existence of an asymptotic expansion

$$\mathbb{E}V(P_{(n)}) - V(K) = c_2^V(K) n^{-2/(d-1)} + c_3^V(K) n^{-3/(d-1)} + \cdots + c_{k-1}^V(K) n^{-(k-1)/(d+1)} + O\big(n^{-k/(d+1)}\big)$$

as $n \to \infty$ for convex bodies of differentiability class $\mathcal{C}^k$ with positive Gaussian curvature, $k \ge 3$, and show the properties of the coefficients $c_i^V(K)$ stated at the end of Theorem 1, and in Sections 3.6–3.9 we prove the first part of Theorem 1 concerning convex bodies of differentiability class $\mathcal{C}^2$.

3.1. Let $K \in \mathcal{C}^2$ be given and choose n random points $X_1, \dots, X_n$ on ∂K according to the density function d_K. The intersection of the supporting halfspaces

$H_+(X_j)$ is a random polyhedron which contains K and by definition

$$P_{(n)} := \bigcap_{i=1}^{n} H_+(X_i) \cap C,$$

where C is a large cube which contains K in its interior. Thus $P_{(n)}$ has n facets F_j, $j = 1, \ldots, n$, generated by the supporting hyperplanes $H(X_j) = \partial H_+(X_j)$, and, maybe, further facets generated by the boundary of C.

It is clear that we can restrict our attention to random polytopes $P_{(n)}$ with $P_{(n)} \to K$ as $n \to \infty$. To make things more precise we assume that the Hausdorff distance $d(P_{(n)}, K)$ is less or equal ϑ, that is, $P_{(n)}$ is contained in $K + \vartheta B^d$, where $\vartheta > 0$ is chosen sufficiently small such that $K + \vartheta B^d$ is also contained in C. We add a suitable error term which takes into account those cases where $d(P_{(n)}, K) > \vartheta$.

Let $K \in \mathcal{C}^k$ with positive Gaussian curvature, $k \geq 3$. Denote by $H_j(x)$ the jth normalized elementary symmetric function of the principal curvatures of K at x; thus $H_0(x) = 1$ and

$$H_j(x) = \binom{d-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq d-1} k_{i_1}(x) \cdots k_{i_j}(x).$$

In particular, $H_{d-1}(x)$ is the Gaussian curvature and $H_1(x)$ the mean curvature of K at x.

Consider two convex bodies K and L, $K \in \mathcal{C}^2$, with $K \subset L$. Then from a local version of Steiner's formula for parallel bodies (cf. Sangwine-Yager [20]) it follows that

$$V(L) - V(K) = \frac{1}{d} \sum_{m=0}^{d-1} \binom{d}{m} \int_{\partial K} r(x)^{d-m} H_{d-1-m}(x)\, dx. \tag{13}$$

Here $r(x)$ is the distance of the point x to ∂L in direction normal to ∂K, that is, if $n_K(x)$ denotes the outer unit normal vector of ∂K at x then $x + r(x)n_K(x)$ is contained in ∂L.

We are interested in the particular case where L is the intersection of supporting halfspaces of K. Hence $r(x)$ is determined by the intersection of the ray $\{x + sn_K(x),\ s \geq 0\}$, with a hyperplane $H(y)$ tangent to the boundary of K at $y \in \partial K$. For each point $y \in \partial K$ this point of intersection is determined by $r_y(x) \geq 0$ where

$$r_y(x) = \sup\{s \mid x + sn_K(x) \in H_+(y)\},$$

and where $r_y(x) = \infty$ if the intersection of the halfline $x + sn_K(x)$, $s \geq 0$, with the halfspace $H_+(y)$ is unbounded.

Let $X_1, \ldots, X_n$ be random points chosen according to the density function d_K on the boundary of K, and let $L = P_{(n)} = \bigcap H_+(X_j) \cap C$. Fix $x \in \partial K$ and denote by $r_{\min}(x)$ the "first" point of intersection on the ray $\{x + sn_K(x), s \geq 0\}$, that is,

$$r_{\min}(x) = \min_{j=1,\ldots,n} r_{X_j}(x).$$

To determine the distribution of $r_{\min}$ we introduce the following notion: fix $x \in \partial K$ and define $G(x,s)$ as that part of ∂K which is "visible" from $x + s n_K(x)$, that is,

$$G(x,s) = \{y \in \partial K \mid r_y(x) \le s\},$$

and let $g(x,s)$ be the weighted surface area of $G(x,s)$,

$$g(x,s) = \int_{G(x,s)} d_K(y)\, dy.$$

By definition this is just the probability that $r_Y(x) \le s$ for a random point Y chosen according to the density function d_K on ∂K. Since the points $X_1, \dots, X_n$ are chosen independently,

$$\mathbb{P}\big(r_{\min}(x) > s\big) = \big(1 - g(x,s)\big)^n. \tag{14}$$

The fact that for $s \to \infty$ the function $g(x,s)$ does not tend to one corresponds to the fact that even for n large with positive probability the random polyhedron $\bigcap H_+(X_j)$ is unbounded.

Now note that $d(P_{(n)}, K) > \vartheta$ if there is a point $x \in \partial K$ such that either $x + r_{\min}(x) n_K(x)$ is a vertex of $P_{(n)}$ with $r_{\min}(x) > \vartheta$ or $r_{\min}(x) = \infty$. Since for given K the function $g(x,\vartheta)$ is bounded from below by a positive constant η for all $x \in \partial K$, it is immediate that

$$\mathbb{P}\big(d(P_{(n)}, K) > \vartheta\big) \le \binom{n}{d} (1-\eta)^{n-d} = O\big(n^d (1-\eta)^n\big). \tag{15}$$

This and (13) now implies

$$\begin{aligned} &\mathbb{E} V(P_{(n)}) - V(K) \\ &\quad = \frac{1}{d} \sum_{m=0}^{d-1} \binom{d}{m} \int_{\partial K} \mathbb{E}\big((\min\{r_{\min}(x), \vartheta\})^{d-m}\big) H_{d-1-m(x)}\, dx \\ &\qquad + O\big(n^d (1-\eta)^n\big), \end{aligned} \tag{16}$$

where we already know that

$$\mathbb{E}\big((\min\{r_{\min}(x), \vartheta\})^{d-m}\big) = \int_0^{\vartheta} s^{d-m}\, d\mathbb{P}\big(r_{\min}(x) \le s\big) + O\big(n^d (1-\eta)^n\big).$$

It is an easy observation that—for $K \in \mathcal{C}^2$ with positive Gaussian curvature and for fixed x—the function $g(x,s)$ is differentiable, increasing, and thus the inverse function $s(x,g)$ of $g(x,s)$ exists [cf. (25) and (27)]. Hence

$$\begin{aligned} &\mathbb{E}\big((\min\{r_{\min}(x), \vartheta\})^{d-m}\big) \\ &\quad = n \int_0^{\vartheta} s^{d-m} \big(1 - g(x,s)\big)^{n-1} \frac{\partial g(x,s)}{\partial s}\, ds + O\big(n^d (1-\eta)^n\big) \\ &\quad = n \int_0^{\eta} s(x,g)^{d-m} (1-g)^{n-1}\, dg + O\big(n^d (1-\eta)^n\big). \end{aligned} \tag{17}$$

Combining (16) and (17) yields for $K \in \mathcal{C}^2$ with positive Gaussian curvature

$$\mathbb{E}V(P_{(n)}) - V(K)$$

$$= \frac{1}{d}\sum_{m=0}^{d-1}\binom{d}{m}\int_{\partial K} n\int_0^{\eta} s(x,g)^{d-m}(1-g)^{n-1}\,dg\, H_{d-1-m}(x)\,dx \tag{18}$$

$$+ O\big(n^d(1-\eta)^n\big).$$

3.2. Let $K \in \mathcal{C}^k$ with positive Gaussian curvature, fix u and let x be the point on ∂K with outer unit normal vector u. In this section we give a local representation of K and the outer normal vectors in a neighborhood of x using cylinder coordinates. Thus a point in $\mathbb{R}^d$ is denoted by (rv, z) with $r \in \mathbb{R}^+$, $v \in S^{d-2}$ and $z \in \mathbb{R}$. Identify the support plane of ∂K at x with the plane $z = 0$ and x with the origin such that K is contained in the halfspace $z \geq 0$. Since $K \in \mathcal{C}^k$, by Lemma 2 there is a neighborhood of x in ∂K such that ∂K can be represented by a convex function $f(rv)$ which in polar coordinates reads as

$$z = f(rv) = b_2(v)r^2 + b_3(v)r^3 + \cdots + b_{k-1}(v)r^{k-1} + O(r^k). \tag{19}$$

The coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. By choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ the coefficient $b_2(v)$ can be written as

$$b_2(v) = \tfrac{1}{2}(k_1 v_1^2 + \cdots + k_{d-1}v_{d-1}^2),$$

where $v = (v_1, \ldots, v_{d-1})$ and since for all boundary points of K the principal curvatures k_i are bounded from below and above by positive constants, the same holds for $b_2(v)$. Since (19) is the Taylor expansion, the coefficients $b_2(v), b_4(v), b_6(v), \ldots$ are even functions and $b_3(v), b_5(v), b_7(v)$ are odd functions of $v \in S^{d-2}$.

On the other hand, the Taylor expansion of $f(y)$, $y \in \mathbb{R}^{d-1}$, implies the Taylor expansion of $f(y)_i$, $i = 1, \ldots, d-1$, where $f(y)_i$ is the ith partial derivative of $f(y)$. In cylinder coordinates this Taylor expansion reads as

$$f(rv)_i = c_{i,1}(v)r + c_{i,2}(v)r^2 + \cdots + c_{i,k-2}(v)r^{k-2} + O(r^{k-1}). \tag{20}$$

The coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $c_{i,1}(v), c_{i,3}(v), \ldots$ are odd functions and $c_{i,2}(v), c_{i,4}(v), \ldots$ are even functions of $v \in S^{d-2}$.

For computing $g(x, s)$ we need to determine those points $(rv, z) \in \partial K$ with

$$s \geq rv \cdot \operatorname{grad} f(rv) - f(rv)$$

and thus by (19) and (20) we need the solution of the equation

$$s = d_2(v)r^2 + d_3(v)r^3 + \cdots + d_{k-1}(v)r^{k-1} + O(r^k).$$

The coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. By choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ it is easy to see that the coefficient $d_2(v)$ equals $b_2(v)$ and thus is bounded from below and above by positive constants. The coefficients $d_3(v), d_5(v), \dots$ are odd functions and $d_2(v), d_4(v), \dots$ are even functions of $v \in S^{d-2}$.

Inverting this series using Lemma 3 gives

$$(21)\qquad r = r(v,s) = e_1(v)s^{1/2} + e_2(v)s + \cdots + e_{k-2}(v)(v)s^{(k-2)/2} + O\big(s^{(k-1)/2}\big).$$

The coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $e_2(v), e_4(v), \dots$ are odd functions and $e_1(v), e_3(v), \dots$ are even functions of $v \in S^{d-2}$. Note that $r = r(v,s)$ is the radial function of the projection of the set $G(x,s)$ of "visible" points onto $\mathbb{R}^{d-1}$.

3.3. Now we prove that the function $g(x,s)$ has a Taylor expansion in $s^{1/2}$. By definition

$$g(x,s) = \int_{G(x,s)} d_K(y)\,dy.$$

We rewrite this integral using cylinder coordinates

$$(22)\qquad g(x,s) = \int_{S^{d-2}} \int_{r \le r(v,s)} d_K(rv)\sqrt{1 + |\operatorname{grad} f(rv)|^2}\, r^{d-2}\,dr\,dv,$$

where $r(v,s)$ is defined in (21).

First note that $d_K \in \mathcal{C}^{k-2}$ and thus we obtain a Taylor expansion for d_K in terms of r where the first term equals $d_K(x)$, which implies the existence of functions $d_{K,m}(v)$ with

$$(23)\qquad d_K(rv) = d_K(x) + d_{K,1}(v)r + \cdots + d_{K,k-3}(v)r^{k-3} + O(r^{k-2}).$$

All coefficients are bounded by a constant independent of x and v. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $d_{K,2}(v), d_{K,4}(v), \dots$ are even functions and the coefficients $d_{K,1}(v), d_{K,3}(v), \dots$ are odd functions of $v \in S^{d-1}$.

Second, by (20),

$$|\operatorname{grad} f(rv)|^2 = \bar{c}_2(v)r^2 + \bar{c}_3(v)r^3 + \cdots + \bar{c}_{k-1}(v)r^{k-1} + O(r^k),$$

where the coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $\bar{c}_2(v), \bar{c}_4(v), \dots$ are even functions and $\bar{c}_3(v), \bar{c}_5(v), \dots$ are odd functions of $v \in S^{d-2}$. Therefore the element of surface area $\sqrt{1+|\operatorname{grad} f|^2}$ has the Taylor expansion up to order $O(r^k)$:

$$\begin{aligned}(24)\qquad &\sqrt{1+|\operatorname{grad} f(rv)|^2}\\ &\quad = 1 + f_2(v)r^2 + f_3(v)r^3 + \cdots + f_{k-1}(v)r^{k-1} + O(r^k).\end{aligned}$$

All coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $f_2(v), f_4(v), \ldots$ are even functions and $f_3(v), f_5(v), \ldots$ are odd functions of $v \in S^{d-2}$.

Thus the integrand (22) has the Taylor expansion

$$\begin{aligned}&d_K(rv)\sqrt{1+|\operatorname{grad} f(rv)|^2}\, r^{d-2}\\ &\quad = g_0(v)r^{d-2} + g_1(v)r^{d-1} + \cdots + g_{k-3}(v)r^{d+k-5} + O(r^{d+k-4}),\end{aligned}$$

where all coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $g_2(v), g_4(v), \ldots$ are even functions and $g_1(v), g_3(v), \ldots$ are odd functions of $v \in S^{d-2}$ and $g_0(v) = d_K(x)$ is independent of v. Thus the integrations in (22) and the definition of $r(v,s)$ in (21) imply the existence of coefficients g_i with

$$\begin{aligned}(25)\qquad g = g(x,s) &= g_1 s^{(d-1)/2} + g_2 s^{d/2} + \cdots + g_{k-2} s^{(d+k-4)/2}\\ &\quad + O\big(s^{(d+k-3)/2}\big),\end{aligned}$$

where the coefficients $g_2, g_4, \ldots$ vanish, g_1 is bounded away from zero, all coefficients are bounded by a constant independent of x, and the constant in $O(\cdot)$ can be chosen independent of x.

3.4. In the last step we investigate the moments $\mathbb{E}((\min\{r_{\min}(x), \vartheta\})^{d-m})$ and thus by (18)

$$n\int_0^\eta s(x,g)^{d-m}(1-g)^{n-1}\,dg.$$

It is easy to see that this can be written as

$$(26)\qquad n\int_0^\eta (1-g)^{n-1} g^{2(d-m)/(d-1)}\Big(P^{\lfloor (k-3)/2\rfloor}\big(g^{2/(d-1)}\big) + O\big(g^{(k-2)/(d-1)}\big)\Big)\,dg,$$

where $P^{\lfloor (k-3)/2\rfloor}(z)$ is a polynomial in the variable z of degree $\lfloor (k-3)/2\rfloor$. Inverting the Taylor expansion (25) using Lemma 3 gives

$$\begin{aligned}(27)\qquad s = s(x,g) &= s_2 g^{2/(d-1)} + s_3 g^{3/(d-1)} + \cdots + s_{k-1} g^{(k-1)/(d-1)}\\ &\quad + O\big(g^{k/(d-1)}\big),\end{aligned}$$

where $s_3, s_5, \ldots$ are vanishing. Defining $P^{\lfloor (k-3)/2 \rfloor}(\cdot)$ by

$$s^{d-m} = g^{2(d-m)/(d-1)}\Big(P^{\lfloor (k-3)/2 \rfloor}\big(g^{2/(d-1)}\big) + O\big(g^{(k-2)/(d-1)}\big)\Big)$$

proves (26). The coefficients of $\bar{P}^{\lfloor (k-3)/2 \rfloor}(\cdot)$ and the constant in $O(\cdot)$ are bounded independent of x.

Finally, the integral

$$\int_0^{\eta} (1-g)^{n-1} g^{2(d-m)/(d-1)}\Big(P^{\lfloor (k-3)/2 \rfloor}\big(g^{2/(d-1)}\big) + O\big(g^{(k-2)/(d-1)}\big)\Big)\, dg$$

can be evaluated by the substitution $e^{-t} := 1-g$. Consider the integral of a single term $g^{l/(d-1)}$. Up to an error term which decreases exponentially in n this is the following Laplace transform:

$$\begin{aligned}
&\int_0^{\eta} (1-g)^{n-1} g^{l/(d-1)}\, dg \\
&\quad = \int_0^{\infty} e^{-tn}(1-e^{-t})^{l/(d-1)}\, dt + O\big(n^d(1-\eta)^n\big) \\
&\quad = \mathcal{L}\{(1-e^{-t})^{l/(d-1)}\}(n) + O\big(n^d(1-\eta)^n\big) \\
&\quad = \mathcal{L}\left\{t^{l/(d-1)}\left(1 - \frac{l}{2(d-1)}t + \cdots\right)\right\}(n) + O\big(n^d(1-\eta)^n\big).
\end{aligned}$$

Using an Abelian theorem (cf., e.g., Doetsch [6], Chapter 3, Section 1) we obtain

$$\begin{aligned}
&= \Gamma\left(\frac{l}{d-1}+1\right)n^{-l/(d-1)-1} - \frac{l}{2(d-1)}\Gamma\left(\frac{l}{d-1}+2\right)n^{-l/(d-1)-2} \\
&\quad + \cdots + O\big(n^d(1-\eta)^n\big).
\end{aligned} \tag{28}$$

In particular,

$$\mathcal{L}\{O\big(t^{l/(d-1)+j+1}\big)\}(n) = O\big(n^{-l/(d-1)-j-2}\big) \qquad \text{as } n \to \infty.$$

Therefore, terminating the Taylor expansion of $(1-e^{-t})^{l/(d-1)}$ after the term of order $t^{l/(d-1)+j}$ and taking into account the error term $O(t^{l/(d-1)+j+1})$ of the same order as the first term omitted, results in an expansion of the Laplace transform up to order $n^{-l/(d-1)-j-1}$ with an error term of order $O(n^{-l/(d-1)-j-2})$. Choose j as the smallest integer such that $l/(d-1)+j+1 > 2(d-m)/(d-1) + (k-2)/(d-1)$. Then

$$\begin{aligned}
&n\int_0^{\eta} (1-g)^{n-1} g^{2(d-m)/(d-1)}\big(P^{\lfloor (k-3)/2 \rfloor}\big(g^{2/(d-1)}\big) + O\big(g^{(k-2)/(d-1)}\big)\big)\, dg \\
&\quad = h_0 n^{-2(d-m)/(d-1)} + h_1 n^{-(2(d-m)+1)/(d-1)} \\
&\qquad + \cdots + h_{k-3} n^{-(2(d-m)+k-3)/(d-1)} + O\big(n^{-(2(d-m)+k-2)/(d-1)}\big),
\end{aligned}$$

where the coefficients $h_m = h_m(x)$ and the constant in $O(\cdot)$ are bounded independent of x. Combined with (18) this implies

$$\begin{aligned}\mathbb{E}V(P_{(n)}) - V(K) \\ &= c_2^V(K)n^{-2/(d-1)} + c_3^V(K)n^{-3/(d-1)} + \cdots + c_{k-1}^V(K)n^{-(k-1)/(d+1)} \\ &\quad + O\big(n^{-k/(d+1)}\big).\end{aligned}$$

3.5. The following facts concerning the coefficients $c_m^V(K)$ are easily checked:

if $d-1$ is even then the expansion of the Laplace transform in (28) is a series in powers $2/(d-1)$ of n^{-1}—observe that l is even—which yields that for odd d,

$$c_3^V(K) = c_5^V(K) = \cdots = 0.$$

Let $d-1$ be odd. Then (28) proves that for d even,

$$c_3^V(K) = \cdots = c_{d-1}^V(K) = 0.$$

3.6. Now we come to the proof of the first part of Theorem 1 concerning convex bodies of differentiability class $\mathcal{C}^2$.

In a first step we prove that in general $g(x,s)$ is at least of order $s^{2/(d-1)}$ for $K \in \mathcal{C}^2$ [cf. (25)]. This follows from the fact that all principal curvatures of ∂K are bounded by a constant and thus each boundary point x of K is contained in a ball of radius $\rho > 0$ which is itself contained in K. Fix $x \in \partial K$. Identify the support plane of ∂K at x with the hyperplane $\mathbb{R}^{d-1}$ and represent the boundary of K locally at x by a function $f(y)$, $y \in \mathbb{R}^{d-1}$. By definition

$$\begin{aligned}g(x,s) &= \int_{G(x,s)} d_K(y)\sqrt{1 + |\operatorname{grad} f(y)|^2}\,dy \\ &\geq \min_{x\in\partial K} d_K(x) V_{d-1}\big(\{y : (y, f(y)) \in G(x,s)\}\big).\end{aligned}$$

It is clear that the set $\{y : (y, f(y)) \in G(x,s)\}$—which is the projection of $G(x,s)$ onto $\mathbb{R}^{d-1}$—contains the intersection of $\mathbb{R}^{d-1}$ with the convex hull of K and $(x - se_d)$, and thus also the intersection of $\mathbb{R}^{d-1}$ with the convex hull of $K \cap H(t)$ and $(x - se_d)$ for any hyperplane $H(t)$ parallel to $\mathbb{R}^{d-1}$ with distance t to the origin: $H(t) = \mathbb{R}^{d-1} + te_d$. Choose in particular $t = s$: then

$$g(x,s) \geq \min_{x\in\partial K} d_K(x)\big(\tfrac{1}{2}\big)^{d-1} V_{d-1}\big(K \cap H(s)\big). \tag{29}$$

To estimate $V_{d-1}(K \cap H(s))$ note that this intersection contains the intersection of $H(s)$ with the ball of radius ρ. This immediately implies

$$g(x,s) \geq cs^{(d-1)/2} \tag{30}$$

for $s \leq \rho$, where c depends on K and d_K.

Now an easy argument proves that it is enough to take into account the term $m = d - 1$ in (16),

$$\mathbb{E}\big((\min\{r_{\min}(x), \vartheta\})^k\big) \le t^k + \vartheta^k \mathbb{P}(r_{\min} > t)$$

for $t \le \vartheta$. Choose in particular $t = n^{-3/(2(d-1))}$. Then by (14),

$$\mathbb{E}\big((\min\{r_{\min}(x), \vartheta\})^k\big) \le n^{-3k/(2(d-1))} + \vartheta^k (1 - cn^{-3/4})^n = O\big(n^{-3/(d-1)}\big)$$

for $k \ge 2$, and thus

$$\mathbb{E}V(P_{(n)}) - V(K) = \int_{\partial K} \mathbb{E}\big((\min\{r_{\min}(x), \vartheta\})\big)\, dx + O\big(n^{-3/(d-1)}\big). \tag{31}$$

Since the Gaussian curvature H_{d-1} and thus the principal curvatures k_i, $i = 1, \dots, d-1$, now are allowed to become zero, we again split the boundary of K and thus the integration in (18) into the parts ∂K_+ and ∂K_0, where (cf. Section 2.1)

$$\partial K_+ = \partial K_+(\varepsilon) := \big\{x \in \partial K : k_i(x) \ge \varepsilon,\ i = 1, \dots, d-1\big\}$$

and

$$\partial K_0 := \partial K \setminus \partial K_+$$

for given $\varepsilon > 0$. Note that by definition $\partial K_+(\varepsilon)$ is contained in $\partial K_+(\varepsilon/2)$. Now we choose the maximal distance ϑ between $P_{(n)}$ and K [used in (15)] such that the following two conditions are satisfied:

(i) the set $G(x, \vartheta)$ for $x \in \partial K_+(\varepsilon)$ is contained in $\partial K_+(\varepsilon/2)$, and

(ii) for any boundary point $x \in \partial K_0$ with principal curvatures $k_i(x)$ there is a paraboloid Q with principal curvatures $k_i(x) + \varepsilon$ which touches ∂K at x from "inside" and is contained in K up to height ϑ.

For abbreviation we fix ε and write from now on ∂K_+ and ∂K_0 instead of $\partial K_+(\varepsilon)$ and $\partial K_0(\varepsilon)$. This definition of ϑ now guarantees that the method developed in Sections 3.1–3.4 also works for $x \in \partial K_+$. In particular, for $x \in \partial K_+$ and $s \le \vartheta$ the function $g(x, s)$ is differentiable, increasing, thus the inverse function $s(x, g)$ of $g(x, s)$ exists and (17) holds. For $x \in \partial K_0$ we have to modify formula (17) slightly by using partial integration. Combining this with (31) yields

$$\begin{aligned} &\mathbb{E}V(P_{(n)}) - V(K) \\ &\quad = \int_{\partial K_+} n \int_0^{\eta} s(x, g)(1 - g)^{n-1}\, dg\, dx + \int_{\partial K_0} \int_0^{\vartheta} \big(1 - g(x, s)\big)^n\, ds\, dx \\ &\qquad + O\big(n^{-3/(d-1)}\big). \end{aligned} \tag{32}$$

Denote the first expression concerning the difference of the volume for $x \in \partial K_+$ by $\mathbb{E}\Delta V_+$ and the second expression concerning $x \in \partial K_0$ by $\mathbb{E}\Delta V_0$. We prove estimates for both expressions: the estimate (36) for $\mathbb{E}\Delta V_+$ gives the right asymptotic behavior and the estimate (37) proves that $\mathbb{E}\Delta V_0$ is of smaller order. Combined this implies Theorem 1.

3.7. First fix $x \in \partial K_+$. For abbreviation write $b_2(\cdot)$ and $f(\cdot)$ instead of $b_2^{(x)}(\cdot)$ and $f^{(x)}(\cdot)$. Identify the support plane of ∂K at x with the plane $z = 0$ and x with the origin such that K is contained in the halfspace $z \geq 0$. Since $x \in \partial K_+$ there is a neighborhood of x in ∂K such that ∂K can be represented by a convex function $f(rv)$ which satisfies Lemma 1. [We apply Lemma 1 with arbitrary, but sufficiently small $\delta > 0$ such that U^λ is contained in $\partial K_+(\varepsilon/2)$.]

Equations (8)–(11) imply that for given s the solution of the equation

$$s = rv \cdot \operatorname{grad} f(rv) - f(rv) = (y, 0) \cdot n_K(y)\sqrt{1 + |\operatorname{grad} f^{(x)}(y)|^2} - f(y)$$

with $y = rv$ satisfies

$$(1+\delta)^{-3}\, b_2(v)^{-1/2} s^{1/2} \leq r \leq (1+\delta)^2\, 2^{-1/2} b_2(v)^{-1/2} s^{1/2} \tag{33}$$

for $r \leq \lambda$ and δ sufficiently small. Recall that $r = r(v, s)$ is the radial function of the projection of $G(x, s)$ onto $\mathbb{R}^{d-1}$.

As in (22) we have

$$g(x, s) = \int_{S^{d-2}} \int_{r \leq r(v,s)} d_K(rv) \sqrt{1 + |\operatorname{grad} f(rv)|^2}\, r^{d-2}\, dr\, dv$$

and hence for $r \leq \lambda$ by (10) and (9),

$$\begin{aligned}(1+\delta)^{-1} d_K(x) &\int_{S^{d-2}} \int_{r \leq r(v,s)} r^{d-2}\, dr\, dv \\ &\leq g(x, s) \leq (1+\delta)^2 d_K(x) \int_{S^{d-2}} \int_{r \leq r(v,s)} r^{d-2}\, dr\, dv,\end{aligned}$$

where $r(v, s)$ satisfies (33). The integral in the last expression equals the $(d-1)$-dimensional volume of the convex body with radial function $r(v, s)$ which by (33)—up to a factor $(1+\delta)^{\pm 1} s^{1/2}$—is the indicatrix of K at x. Therefore

$$\begin{aligned}&(1+\delta)^{-3d+2} 2^{(d-1)/2} \kappa_{d-1} d_K(x) H_{d-1}(x)^{-1/2} s^{(d-1)/2} \\ &\quad \leq g = g(x, s) \leq (1+\delta)^{2d} 2^{(d-1)/2} \kappa_{d-1} d_K(x) H_{d-1}(x)^{-1/2} s^{(d-1)/2}\end{aligned} \tag{34}$$

for s sufficiently small, where κ_{d-1} denotes the $(d-1)$-dimensional volume of B^{d-1}.

Now fix $x \in \partial K_0$. To estimate $V_{d-1}(K \cap H(s))$ in (29) note that for $s \leq \vartheta$ this intersection contains the intersection of $H(s)$ with a paraboloid with principal curvatures $k_i(x) + \varepsilon$. Since the principal curvatures are bounded from above, and at least one principal curvature is at most ε, this implies

$$g(x, s) \geq c\varepsilon^{-1/2} s^{(d-1)/2} \tag{35}$$

for $s \leq \vartheta$, where c depends on K and d_K.

3.8. Inequality (34) is equivalent to

$$(1+\delta)^{-4d/(d-1)}\tfrac{1}{2}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)} g^{2/(d-1)}$$
$$\le s \le (1+\delta)^{2(3d-2)/(d-1)}\tfrac{1}{2}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)} g^{2/(d-1)},$$

which by (28) implies, for

$$n\int_0^\eta s(g)\,(1-g)^{n-1}\,dg,$$

the following upper and lower bound:

$$\begin{aligned}
&(1+\delta)^{-4d/(d-1)}\frac{1}{2}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\\
&\quad\times\Gamma\Big(\frac{2}{d-1}+1\Big) n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big)\\
&\quad\le n\int_0^\eta s(g)(1-g)^{n-1}\,dg\\
&\quad\le (1+\delta)^{2(3d-2)/(d-1)}\frac{1}{2}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\\
&\qquad\times\Gamma\Big(\frac{2}{d-1}+1\Big) n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big),
\end{aligned}$$

where the constant in $O(\cdot)$ is bounded independent of $x\in\partial K_+$. This proves

$$\begin{aligned}
&(1+\delta)^{-4d/(d-1)}\frac{1}{2}\kappa_{d-1}^{-2/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\
&\quad\times\int_{\partial K_+} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\,dx\, n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big)\\
&\quad\le \mathbb{E}\Delta V_+ \le (1+\delta)^{2(3d-2)/(d-1)}\frac{1}{2}\kappa_{d-1}^{-2/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\
&\qquad\times\int_{\partial K_+} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\,dx\, n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big)
\end{aligned}\tag{36}$$

for arbitrary $\delta\ge 0$.

For $x\in\partial K_0$ inequality (35) and formula (28) imply

$$\begin{aligned}
\int_0^\vartheta (1-g(x,s))^n\,ds &\le \int_0^1 \big(1-c\,\varepsilon^{-1/2}s^{(d-1)/2}\big)^n\,ds\\
&\le \Gamma\Big(\frac{2}{d-1}+1\Big)c^{-2/(d-1)}\varepsilon^{1/(d-1)}n^{-2/(d-1)}.
\end{aligned}$$

This proves

$$\mathbb{E}\Delta V_0 \le \Gamma\Big(\frac{2}{d-1}+1\Big)c^{-2/(d-1)}\int_{\partial K_0}\varepsilon^{1/(d-1)}\,dx\,n^{-2/(d-1)},\tag{37}$$

where c depends on K and d_K.

We combine our results. By (36) and (37) we have

$$\begin{aligned}
&(1+\delta)^{-4d/(d-1)}\frac{1}{2}\kappa_{d-1}^{-2/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\
&\quad\times\int_{\partial K} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\,dx\, n^{-2/(d-1)}\\
&\quad+O\big(n^{-4/(d-1)}\big)+O\big(\varepsilon^{1/(d-1)}\big)\\
&\quad\le \mathbb{E}\Delta V_+ + \mathbb{E}\Delta V_0\\
&\quad\le (1+\delta)^{2(3d-2)/(d-1)}\frac{1}{2}\kappa_{d-1}^{-2/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\
&\qquad\times\int_{\partial K} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\,dx\, n^{-2/(d-1)}\\
&\qquad+O\big(n^{-4/(d-1)}\big)+O\big(\varepsilon^{1/(d-1)}\big),
\end{aligned}$$

which by (32) proves Theorem 1 since δ and ε can be chosen arbitrarily small. □

4. Proof of Theorem 2. We present the proof of Theorem 2 in an order similar to that of the proof of Theorem 1. We only work out in detail those parts of the proof which differ from the proof of Theorem 1.

4.1. In a first step we develop a formula analogous to (13) for the difference of the surface area of two convex bodies K and L with $K \in \mathcal{C}^2$ and $K \subset L$. By (13) we have for the volume of $L + tB^d$

$$V(L+tB^d) = V(K) + \frac{1}{d}\sum_{m=0}^{d-1}\binom{d}{m}\int_{\partial K} r(t,x)^{d-m} H_{d-1-m}(x)\,dx,$$

where $r(t,x)$ denotes the distance of the point $x \in \partial K$ to $\partial(L+tB^d)$ in direction orthogonal to ∂K. [Note that $r(0,x) = r(x)$.] Let x_L be the unique point in ∂L such that $x_L = x + r(0,x)n_K(x)$. Then

$$r(t,x) = r(0,x) + \frac{1}{n_K(x)\cdot n_L(x_L)}t + o(t)$$

as $t \to 0$ and thus

$$\frac{\partial r(t,x)}{\partial t}\bigg|_{t=0} = \frac{1}{n_K(x)\cdot n_L(x_L)}.$$

This and the fact that $r(t,x) \le r(0,x) + (n_K(x)\cdot n_L(x_L))^{-1}t$ immediately imply

$$\begin{aligned}
S(L) &= \frac{\partial V(L+tB^d)}{\partial t}\bigg|_{t=0}\\
&= \frac{1}{d}\sum_{m=0}^{d-1}\binom{d}{m}\int_{\partial K}(d-m)r(0,x)^{d-m-1}\frac{1}{n_K(x)\cdot n_L(x_L)}H_{d-1-m}(x)\,dx
\end{aligned}$$

which in turn proves the following lemma:

LEMMA 4. *Let K and L be two convex bodies with $K \in \mathcal{C}^2$ and $K \subset L$. Then*

$$
\begin{aligned}
&S(L) - S(K)\\
&\quad = \int_{\partial K} \left(\frac{1}{n_K(x) \cdot n_L(x_L)} - 1 \right) dx \\
&\qquad + \sum_{m=0}^{d-2} \binom{d-1}{m} \int_{\partial K} r(x)^{d-1-m} \frac{1}{n_K(x) \cdot n_L(x_L)} H_{d-1-m}(x)\, dx,
\end{aligned} \tag{38}
$$

where $r(x)$ denotes the distance of the point x to ∂L in direction orthogonal to ∂K and $x_L = x + r(x) n_K(x) \in \partial L$.

Let $K \in \mathcal{C}^2$ be given and let L be $P_{(n)}$. Then for given $x \in \partial K$ we are interested in the expectations

$$
\mathbb{E}\left(\frac{1}{n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}})} - 1 \right) \quad \text{and} \quad \mathbb{E}\left(\min(r_{\min}(x), \vartheta)^j \frac{1}{n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}})} \right).
$$

Let again $P_{(n)}$ satisfy $d(P_{(n)}, K) \le \vartheta$ whence by (15) we have to add an error term $O(n^d(1-\eta)^n)$. Further we assume without loss of generality that the point X_1 satisfies $r_{X_1}(x) = r_{\min}(x)$ which means that $r_{X_i}(x) \ge r_{X_1}(x)$ for all $i = 2, \dots, n$. By definition this happens with probability $(1 - g(x, r_{X_1}(x)))^{n-1}$. It is also clear that in this case $x_{P_{(n)}}$ lies in the supporting hyperplane $H(X_1)$ and thus the outer unit normal vector to $P_{(n)}$ at $x_{P_{(n)}}$ equals $n_K(X_1)$.

$$
\begin{aligned}
&\mathbb{E}\left(\min(r_{\min}(x), \vartheta)^j \frac{1}{n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}})} \right) \\
&\quad = n \int_{\partial K} \min(r_y(x), \vartheta)^j \frac{1}{n_K(x) \cdot n_K(y)} (1 - g(x, r_y(x)))^{n-1} d_K(y)\, dy \\
&\qquad + O(n^d(1-\eta)^n).
\end{aligned}
$$

We define

$$
h(x, s) := \int_{r_y(x) \le s} \frac{1}{n_K(x) \cdot n_K(y)} d_K(y)\, dy
$$

[which is close to $g(x, s)$ for s sufficiently small]. Observe that for $K \in \mathcal{C}^2$ with positive Gaussian curvature the function $h(x, s)$ is differentiable for s sufficiently small. Thus in this case we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\min(r_{\min}(x), \vartheta)^j \frac{1}{n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}})} \right) \\
&\quad = n \int_0^{\vartheta} s^j (1 - g(x, s))^{n-1} \frac{\partial h(x, s)}{\partial s}\, ds + O(n^d(1-\eta)^n)
\end{aligned} \tag{39}
$$

for s sufficiently small. Analogously

$$\mathbb{E}\Big(\frac{1}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})}-1\Big) \tag{40}$$
$$= n\int_0^{\vartheta}\big(1-g(x,s)\big)^{n-1}\frac{\partial(h(x,s)-g(x,s))}{\partial s}\,ds + O\big(n^d(1-\eta)^n\big).$$

Combining (38), (39) and (40) we obtain for $K\in\mathcal{C}^2$ with positive Gaussian curvature

$$\begin{aligned}\mathbb{E}S(P_{(n)}) - S(K) &= \int_{\partial K} n\int_0^{\vartheta}\big(1-g(x,s)\big)^{n-1}\frac{\partial(h(x,s)-g(x,s))}{\partial s}\,ds\,dx\\ &\quad+\sum_{m=0}^{d-2}\binom{d-1}{m}\int_{\partial K} n\int_0^{\vartheta} s^{d-m-1}\big(1-g(x,s)\big)^{n-1}\\ &\qquad\times\frac{\partial h(x,s)}{\partial s}\,ds\,H_{d-1-m}(x)\,dx\\ &\quad+O\big(n^d(1-\eta)^n\big).\end{aligned} \tag{41}$$

4.2. As in Section 3.2 let $K\in\mathcal{C}^k$ with positive Gaussian curvature, fix u, let x be the point on ∂K with outer unit normal vector u, and denote by (rv,z) a point in $\mathbb{R}^d$, $v\in S^{d-2}$, $r\in\mathbb{R}^+$, $z\in\mathbb{R}$. Identify the support plane of ∂K at x with the plane $z=0$ and x with the origin. In Section 3.2 we proved

$$r = r(v,s) = e_1(v)s^{1/2} + e_2(v)s + \cdots + e_{k-2}(v)(v)s^{(k-2)/2} + O\big(s^{(k-1)/2}\big).$$

The coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $e_2(v), e_4(v),\ldots$ are odd functions and $e_1(v), e_3(v),\ldots$ are even functions of $v\in S^{d-2}$.

4.3. Analogously to the expansion of $g(x,s)$ in Section 3.3 we now expand the functions $h(x,s)$ and $(h(x,s)-g(x,s))$:

$$\begin{aligned}h(x,s) &= \int_{S^{d-2}}\int_{r\le r(v,s)}\frac{1}{n_K(x)\cdot n_K(rv)}\,d_K(rv)\\ &\qquad\times\sqrt{1+|\operatorname{grad} f(rv)|^2}\,r^{d-2}\,dr\,dv.\end{aligned} \tag{42}$$

Since $(n_K(x)\cdot n_K(rv))^{-1} = \sqrt{1+|\operatorname{grad} f(rv)|^2}$, formulae (23) and (24) show that

$$\begin{aligned}&d_K(rv)\big(1+|\operatorname{grad} f(rv)|^2\big)r^{d-2}\\ &\quad= h_0(v)r^{d-2} + h_1(v)r^{d-1} + \cdots + h_{k-3}(v)r^{d+k-5} + O(r^{d+k-4}),\end{aligned}$$

where all coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $h_0(v), h_2(v), \dots$ are even functions and $h_1(v), h_3(v), \dots$ are odd functions of $v \in S^{d-2}$. Thus the integrations in (42) and the definition of $r(v, s)$ in (21) imply the existence of coefficients h_i with

$$h = h(x,s) = h_1 s^{(d-1)/2} + h_2 s^{d/2} + \cdots + h_{k-2} s^{(d+k-4)/2} + O\big(s^{(d+k-3)/2}\big), \tag{43}$$

where the coefficients $h_2, h_4, \dots$ vanish, h_1 is bounded away from zero, all coefficients are bounded by a constant independent of x, and the constant in $O(\cdot)$ can be chosen independent of x.

In the same way (23) and (24) imply the existence of coefficients h_i^- with

$$h(x,s) - g(x,s) = h_1^- s^{(d+1)/2} + h_2^- s^{(d+2)/2} + \cdots + h_{k-2}^- s^{(d+k-2)/2} + O\big(s^{(d+k-1)/2}\big), \tag{44}$$

where the coefficients $h_2^-, h_4^-, \dots$ vanish, h_1^- is bounded away from zero, all coefficients are bounded by a constant independent of x, and the constant in $O(\cdot)$ can be chosen independent of x.

4.4. The arguments used in Section 3.4 show that there are asymptotic expansions for

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{n_K(x)\cdot n_{P_{(n)}}(x)_{P_{(n)}}} - 1\right)\\ &\quad = n\int_0^{\vartheta} \big(1 - g(x,s)\big)^{n-1} \frac{\partial(h(x,s) - g(x,s))}{\partial s}\, ds + O\big(n^d (1-\eta)^n\big)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(\min\big(r_{\min}(x), \vartheta\big)^j \frac{1}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})}\right)\\ &\quad = n\int_0^{\vartheta} s^j \big(1 - g(x,s)\big)^{n-1} \frac{\partial h(x,s)}{\partial s}\, ds + O\big(n^d (1-\eta)^n\big)\end{aligned}$$

which yield

$$\begin{aligned}&\mathbb{E}S(P_{(n)}) - S(K)\\ &\quad = c_2^S(K) n^{-2/(d-1)} + c_3^S(K) n^{-3/(d-1)} + \cdots + c_{k-1}^S(K) n^{-(k-1)/(d+1)}\\ &\qquad + O\big(n^{-k/(d+1)}\big).\end{aligned}$$

4.5. If $d-1$ is even, then $c_3^S(K) = c_5^S(K) = \cdots = 0$ and for $d-1$ odd we have $c_3^S(K) = \cdots = c_{d-1}^S(K) = 0$.

4.6. Now we come to the proof of the first part of Theorem 2. By (30) and by the argument presented in Section 3.6, we get

$$\begin{aligned}\mathbb{E}S(P_{(n)}) - S(K) &= \int_{\partial K} \mathbb{E}\Big(\frac{1}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})} - 1\Big)\,dx\\ &\quad + (d-1)\int_{\partial K} \mathbb{E}\Big(\min\big(r_{\min}(x),\vartheta\big)\frac{1}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})}\Big)\,dx\\ &\quad + O\big(n^{-3/(d-1)}\big).\end{aligned}$$

Again we split the boundary of K into the parts $\partial K_+ = \partial K_+(\varepsilon)$ and $\partial K_0 = \partial K_0(\varepsilon)$ for given $\varepsilon > 0$. Choose the maximal distance ϑ between $P_{(n)}$ and K [used in (15)] such that the following conditions are satisfied:

(i) the set $G(x,\vartheta)$ for $x \in \partial K_+(\varepsilon)$ is contained in $\partial K_+(\varepsilon/2)$,

(ii) for any boundary point $x \in \partial K_0$ with principal curvatures $k_i(x)$ there is a paraboloid Q with principal curvatures $k_i(x)+\varepsilon$ which touches ∂K at x from "inside" and is contained in K up to height ϑ and

(iii) $\vartheta \le \rho$, where ρ is the radius of the ball touching ∂K from inside at every boundary point (cf. Section 3.6).

This guarantees that the method developed in 4.1–4.4 also works for $x \in \partial K_+$, for $x \in \partial K_+$ and $s \le \vartheta$ the functions $h(x,s)$ and $h(x,s) - g(x,s)$ are differentiable, increasing, and thus the inverse functions $s(x,h)$ [resp. $s(x,h-g)$] exist. Thus

$$\begin{aligned}&\mathbb{E}S(P_{(n)}) - S(K)\\ &\quad= \int_{\partial K_+} n\int_0^{\eta(h-g)} \big(1 - g(x, s(x,h-g))\big)^{n-1}\,d(h-g)\,dx\\ &\quad\quad + (d-1)\int_{\partial K_+} n\int_0^{\eta(h)} s(x,h)\big(1-g(x,s(x,h))\big)^{n-1}\,dh\,dx\\ &\quad\quad + \int_{\partial K_0}\mathbb{E}\Big(\frac{1}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})} - 1 + (d-1)\frac{\min(r_{\min}(x),\vartheta)}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})}\Big)\,dx\\ &\quad\quad + O\big(n^{-3/(d-1)}\big),\end{aligned}\tag{45}$$

where $\eta_{(h-g)}$ and $\eta_{(h)}$ are suitable positive constants. Denote the first expressions concerning the difference of the surface area for $x \in \partial K_+$ by $\mathbb{E}\Delta S_+$ and the expression concerning $x \in \partial K_0$ by $\mathbb{E}\Delta S_0$. We prove estimates for both expressions which imply Theorem 2.

4.7. Fix $x \in \partial K_+$ and recall the notations introduced in Section 3.7. Here we need analogous results for $h(x,s)$ and $h(x,s) - g(x,s)$. As in (42) we have

$$h(x,s) = \int_{S^{d-2}}\int_{r\le r(v,s)} d_K(rv)\big(1 + |\operatorname{grad} f(rv)|^2\big)r^{d-2}\,dr\,dv$$

and hence for $r \leq \lambda$ by (10) and (9),

$$(1+\delta)^{-1} d_K(x) \int_{S^{d-2}} \int_{r \leq r(v,s)} r^{d-2}\, dr\, dv \\ \leq h(x,s) \leq (1+\delta)^3 d_K(x) \int_{S^{d-2}} \int_{r \leq r(v,s)} r^{d-2}\, dr\, dv,$$

where $r(v,s)$ satisfies (33). This leads to

$$\begin{aligned} &(1+\delta)^{-3d+2} \kappa_{d-1} d_K(x) H_{d-1}(x)^{-1/2} s^{(d-1)/2} \\ &\quad \leq h = h(x,s) \leq (1+\delta)^{2d+1} \kappa_{d-1} d_K(x) H_{d-1}(x)^{-1/2} s^{(d-1)/2}, \end{aligned} \tag{46}$$

where κ_{d-1} denotes the $(d-1)$-dimensional volume of B^{d-1}.

By (42) and (22) we have

$$h(x,s) - g(x,s) = \int_{S^{d-2}} \int_{r \leq r(v,s)} \left(\sqrt{1 + |\operatorname{grad} f(rv)|^2 - 1}\right) d_K(rv) \\ \times \sqrt{1 + |\operatorname{grad} f(rv)|^2}\, r^{d-2}\, dr\, dv.$$

Hence for $x \in \partial K_+$ and $r \leq \lambda$ by (10), (9), (12) and (33),

$$\begin{aligned} &(1+\delta)^{-3d-5} \frac{d-1}{2(d+1)} \kappa_{d-1} d_K(x) M(x) s^{(d+1)/2} \\ &\quad \leq h - g = h(x,s) - g(x,s) \\ &\quad \leq (1+\delta)^{2d+5} \frac{d-1}{2(d+1)} \kappa_{d-1} d_K(x) M(x) s^{(d+1)/2}, \end{aligned} \tag{47}$$

where κ_{d-1} denotes the $(d-1)$-dimensional volume of B^{d-1} and

$$M(x) := \frac{1}{(d-1)\kappa_{d-1}} \int_{S^{d-2}} \sum_i \left(\sum_j f_{ij}^{(x)} v_j \right)^2 b_2(v)^{-(d+1)/2}\, dv.$$

Choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ this integral takes the form

$$M(x) := \frac{1}{(d-1)\kappa_{d-1}} \int_{S^{d-2}} \left(\sum_i k_i(x)^2 v_i^2 \right) \left(\sum_i k_i(x) v_i^2 \right)^{-(d+1)/2} dv,$$

where $k_1(x), \ldots, k_{d-1}(x)$ denote the principal curvatures of ∂K at $x \in \partial K_+$.

Fix $x \in \partial K_0$. Because $P_{(n)}$ is close to K the boundary of $P_{(n)}$ consists of supporting hyperplanes and because ∂K is touched from inside by a ball B of radius ρ it is immediate that

$$n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}}) = n_K(x) \cdot n_K(X_k) \geq n_K(x) \cdot n_B(\tilde{X}_k)$$

with a suitable random point $X_k \in \partial K$ and where $\tilde{X}_k \in \partial B$ is chosen such that the tangent hyperplane to ∂B at $\tilde{X}_k$ contains the point $x + r_{X_k}(x) n_K(x)$, that is, it intersects the halfline $\{x + s n_K(x), s \geq 0\}$ in the same point as the hyperplane tangent to ∂K at X_k. Since

$$n_K(x) \cdot n_B(\tilde{X}_k) = \frac{\rho}{\rho + r_y(x)}$$

for $r_{X_k}(x) \leq \rho$, we obtain

$$\frac{1}{n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}})} - 1 \leq c \min\bigl(r_{\min}(x), \vartheta\bigr) \tag{48}$$

and

$$\min\bigl(r_{\min}(x), \vartheta\bigr) \frac{1}{n_K(x) \cdot n_{P_{(n)}}(x_{P_{(n)}})} \leq c \min\bigl(r_{\min}(x), \vartheta\bigr) \tag{49}$$

with a suitable constant c depending on K.

4.8. Inequality (46) is equivalent to

$$\begin{aligned}(1+\delta)^{-2(2d+1)/(d-1)} & \kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)} h^{2/(d-1)} \\ & \leq s \leq (1+\delta)^{2(3d-2)/(d-1)} \kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)} h^{2/(d-1)}\end{aligned}$$

and inequality (47) to

$$\begin{aligned}(1+\delta)^{-2(2d+5)/(d+1)} & \left(\frac{d-1}{2(d+1)}\right)^{-2/(d+1)} \kappa_{d-1}^{-2/(d+1)} d_K(x)^{-2/(d+1)} \\ & \times M(x)^{-2/(d+1)} (h-g)^{2/(d+1)} \\ & \quad \leq s \leq (1+\delta)^{2(3d+5)/(d+1)} \left(\frac{d-1}{2(d+1)}\right)^{-2/(d+1)} \kappa_{d-1}^{-2/(d+1)} d_K(x)^{-2/(d+1)} \\ & \qquad \times M(x)^{-2/(d+1)} (h-g)^{2/(d+1)}.\end{aligned}$$

Formula (34) combined with (46) gives

$$(1+\delta)^{-5d+1} h \leq g \leq (1+\delta)^{5d-2} h$$

and (34) combined with (47),

$$\begin{aligned}(1+\delta)^{-5d+1-6/(d+1)} & \left(\frac{d-1}{2(d+1)}\right)^{-(d-1)/(d+1)} \kappa_{d-1}^{2/(d+1)} d_K(x)^{2/(d+1)} \\ & \times H_{d-1}^{-1/2} M(x)^{-(d-1)/(d+1)} (h-g)^{(d-1)/(d+1)} \\ & \quad \leq g \leq (1+d)^{5d-1-4/(d+1)} \left(\frac{d-1}{2(d+1)}\right)^{-(d-1)/(d+1)} \kappa_{d-1}^{2/(d+1)} d_K(x)^{2/(d+1)} \\ & \qquad \times H_{d-1}^{-1/2} M(x)^{-(d-1)/(d+1)} (h-g)^{(d-1)/(d+1)}.\end{aligned}$$

Now (28) implies the following upper and lower bound:

$$
\begin{aligned}
&(1+\delta)^{-5d-12-12/(d-1)}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)}\\
&\quad\times H_{d-1}(x)^{1/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big)\\
&\quad\le n\int_0^{\eta(h)} s(x,h)\big(1-g(x,s(x,h))\big)^{n-1}\,dh\\
&\quad\le (1+\delta)^{5d+15+10/(d-1)}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)}\\
&\qquad\times H_{d-1}(x)^{1/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big),
\end{aligned}
$$

where the constant in $O(\cdot)$ is bounded independent of x. And analogously

$$
\begin{aligned}
&(1+\delta)^{-5d-9-4/(d-1)}\frac{d-1}{2(d+1)}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)}\\
&\quad\times H_{d-1}(x)^{1/(d-1)} M(x)\Gamma\Big(\frac{2}{d-1}+1\Big)n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big)\\
&\quad\le n\int_0^{\eta(h-g)} \big(1-g(x,s(x,h-g))\big)^{n-1}\,d(h-g)\\
&\quad\le (1+\delta)^{5d+9+2/(d-1)}\frac{d-1}{2(d+1)}\kappa_{d-1}^{-2/(d-1)} d_K(x)^{-2/(d-1)}\\
&\qquad\times H_{d-1}(x)^{1/(d-1)} M(x)\Gamma\Big(\frac{2}{d-1}+1\Big)n^{-2/(d-1)} + O\big(n^{-2/(d-1)-1}\big),
\end{aligned}
$$

where the constant in $O(\cdot)$ is bounded independent of x. This proves

$$
\begin{aligned}
&(1+\delta)^{-5d-12-12/(d-1)}(d-1)\kappa_{d-1}^{-(d+1)/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\
&\quad\times\int_{\partial K_+} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\\
&\qquad\quad\times\Big\{H_1(x)+\frac{1}{2(d+1)}M(x)\Big\}\,dx\, n^{-2/(d-1)} + O\big(n^{-4/(d-1)}\big)\\
&\quad\le \mathbb{E}\Delta S_+ \qquad\qquad (50)\\
&\quad\le (1+\delta)^{5d+15+10/(d-1)}(d-1)\kappa_{d-1}^{-(d+1)/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\
&\qquad\times\int_{\partial K_+} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)}\\
&\qquad\quad\times\Big\{H_1(x)+\frac{1}{2(d+1)}M(x)\Big\}\,dx\, n^{-2/(d-1)} + O\big(n^{-4/(d-1)}\big)
\end{aligned}
$$

for arbitrary $\delta\ge 0$.

For $\mathbb{E}\Delta S_0$ formulae (48), (49) and (31) imply

$$\begin{aligned}\mathbb{E}\Delta S_0 &= \int_{\partial K_0} \mathbb{E}\Big(\frac{1}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})} - 1 + (d-1)\frac{\min(r_{\min}(x),\vartheta)}{n_K(x)\cdot n_{P_{(n)}}(x_{P_{(n)}})}\Big)\,dx\\ &\le dc\int_{\partial K_0}\mathbb{E}\big(\min(r_{\min}(x),\vartheta)\big)\,dx\\ &\le dc\mathbb{E}\Delta V_0.\end{aligned}$$

The asymptotic behavior of $\mathbb{E}\Delta V_0$ was already determined in (37). Together with (50) this implies

$$\begin{aligned}&(1+\delta)^{-5d-12-12/(d-1)}(d-1)\kappa_{d-1}^{-(d+1)/(d-1)}\Gamma\Big(\frac{2}{d-1}+1\Big)\\ &\quad\times\int_{\partial K} d_K(x)^{-2/(d-1)}H_{d-1}(x)^{1/(d-1)}\Big\{H_1(x)+\frac{1}{2(d+1)}M(x)\Big\}\,dx\,n^{-2/(d-1)}\\ &\quad + O\big(n^{-4/(d-1)}\big)+O\big(\varepsilon^{1/(d-1)}\big)\\ &\qquad\le \mathbb{E}\Delta S_+ + \mathbb{E}\Delta S_0\\ &\qquad\le (1+\delta)^{5d+15+10/(d-1)}(d-1)\kappa_{d-1}^{-(d+1)/(d-1)}\\ &\qquad\quad\times\Gamma\Big(\frac{2}{d-1}+1\Big)\int_{\partial K} d_K(x)^{-2/(d-1)}H_{d-1}(x)^{1/(d-1)}\\ &\qquad\qquad\qquad\times\Big\{H_1(x)+\frac{1}{2(d+1)}M(x)\Big\}\,dx\,n^{-2/(d-1)}\\ &\qquad\quad + O\big(n^{-4/(d-1)}\big)+O\big(\varepsilon^{1/(d-1)}\big)\end{aligned}$$

which by (45) proves Theorem 2 since δ and ε can be chosen arbitrarily small.

5. Proof of Theorem 3. To obtain Theorem 3 we follow an idea of Glasauer described in [7]. We may assume that $0\in\operatorname{int}K$ and thus also $0\in\operatorname{int}P_{(n)}$. Then

$$\begin{aligned}W(P_{(n)})-W(K) &= \frac{2}{d\kappa_d}\int_{S^{d-1}}\big(h_{P_{(n)}}(u)-h_K(u)\big)\,du\\ &= \frac{2}{d\kappa_d}\int_{S^{d-1}}\Big(\frac{1}{\rho_{P^*_{(n)}}(u)}-\frac{1}{\rho_{K^*}(u)}\Big)\,du\\ &= \frac{2}{d\kappa_d}\int_{K^*\setminus P^*_{(n)}}\|x\|^{-(d+1)}\,dx\\ &= W^*(P^*_{(n)})-W^*(K^*),\end{aligned}$$

where K^* is the polar body of K and ρ_K is the radial function of K. Here we define for a convex body L

$$W^*(L) := \frac{2}{d\kappa_d} \int_{L^c} \|x\|^{-(d+1)}\, dx, \tag{51}$$

where $L^c = \{x \in \mathbb{R}^d \mid x \notin L \text{ and } [0,x] \cap L \neq \varnothing\}$. In particular, if the convex body L contains the origin in its interior then $L^c = \mathbb{R}^d \setminus L$.

If $K \in \mathcal{C}^2$ with positive Gaussian curvature, then choosing n random points on the boundary of K with respect to the density function d_K corresponds to the choice of n random points on the boundary of K^* with respect to the density function

$$d^*_{K^*}(x^*) := d_K(x) H_{d-1}(x)^{-1} \|x^*\|^{-d} \big(x^* \cdot n_{K^*}(x^*)\big), \tag{52}$$

where $x^* \in \partial K^*$ is the unique point such that $x \cdot x^* = 1$. This choice of $d^*_{K^*}$ ensures that $d^*_{K^*}$ is the corresponding density function on ∂K^*. Using polar coordinates with respect to the convex body K^* and then parametrizing the convex body K by its outer unit normal vector—observe that $n_K(x) = \frac{x^*}{\|x^*\|}$—we have

$$\begin{aligned}
\int_{\partial K^*} d^*_{K^*}(x^*)\, dx^* &= \int_{\partial K^*} d_K(x) H_{d-1}(x)^{-1} \|x^*\|^{-d} \big(x^* \cdot n_{K^*}(x^*)\big)\, dx^* \\
&= \int_{S^{d-1}} d_K(x) H_{d-1}(x)^{-1}\, du \\
&= \int_{\partial K} d_K(x)\, dx = 1.
\end{aligned}$$

But if $K \in \mathcal{C}^2$ and the Gaussian curvature vanishes at certain boundary points, $d^*_{K^*}$ will not be defined properly by (52). For this case let D_K be the distribution function with density d_K and define for ∂K^* the distribution function $D^*_{K^*}$ by

$$D^*_{K^*}\big(\text{cone}\{0, \nu(A)\} \cap \partial K^*\big) = D_K(A) \tag{53}$$

for a Borel set $A \subset \partial K$, where cone$\{0, B\}$ denotes the cone $\{x : x = ty,\ y \in B,\ t \geq 0\}$ and ν denotes the spherical image map. Note that $D^*_{K^*}$ needs neither to be differentiable nor to be continuous, in particular, if ∂K contains flat parts.

Thus investigating $W(P_{(n)}) - W(K)$ of a random polyhedron $P_{(n)}$, generated by choosing n random points on the boundary of K with respect to D_K and intersecting their supporting halfspaces, is the same as generating a random polytope P^*_n in K^* by taking the convex hull of n random points on the boundary of K^* chosen with respect to $D^*_{K^*}$ and investigating $W^*(P_n) - W^*(K^*)$.

Observe that the functional $W^*(P_n) - W^*(K^*)$ is closely related to the volume difference $V(K^*) - V(P_n)$. This means that the method used by Reitzner [18] for investigating the limit behavior of $V(K) - \mathbb{E}V(P_n)$ as n tends to infinity can also be used for the functional W^*. In [18] the limit $V(K) - \mathbb{E}V(P_n)$ was investigated for $K \in \mathcal{C}^2$ with positive Gaussian curvature, but it should be noted that for convex bodies fulfilling weaker differentiability assumptions this limit was determined before by Schütt and Werner [24] in a long and intricate proof.

Now most of the proof of Theorem 3 is contained in Lemma 5 and Lemma 6. In order to state Lemma 5 and to prepare for the remaining argument in the case $K \in \mathcal{C}^2$ we split the integral defining the mean width. Set $S_+^{d-1} = \nu(\partial K_+(\varepsilon))$ and $S_0^{d-1} = S^{d-1} \setminus S_+^{d-1}$. (Recall that ν denotes the spherical image map.) Further ∂K_+ and S_+^{d-1} correspond to a part $\partial K_+^* = \{x^* \in \partial K^* \mid x(x^*) \in \partial K_+\}$ of the boundary of K^* where the principal curvatures are bounded from above by a constant, and thus ∂K_+^* is locally of differentiability class $\mathcal{C}^2$ with positive Gaussian curvature (cf. Schneider [22], page 111). Observe that for $x \in \partial K_+$ the Gaussian curvature is positive and thus $D_{K^*}^*$ restricted to ∂K_+^* has a density function $d_{K^*}^*$ defined by (52). Analogous to (32) and (45) we define

$$\begin{aligned}
&W(P_{(n)}) - W(K)\\
&\quad = \mathbb{E}\Delta W_+ + \mathbb{E}\Delta W_0\\
&\quad = \frac{2}{d\kappa_d}\int_{S_+^{d-1}} \big(h_{P_{(n)}}(u) - h_K(u)\big)\,du + \frac{2}{d\kappa_d}\int_{S_0^{d-1}} \big(h_{P_{(n)}}(u) - h_K(u)\big)\,du\\
&\quad = \frac{2}{d\kappa_d}\int_{S_+^{d-1}} \left(\frac{1}{\rho_{P_{(n)}^*}(u)} - \frac{1}{\rho_{K^*}(u)}\right) du + \frac{2}{d\kappa_d}\int_{S_0^{d-1}} \left(\frac{1}{\rho_{P_{(n)}^*}(u)} - \frac{1}{\rho_{K^*}(u)}\right) du\\
&\quad = \mathbb{E}\Delta W_+^* + \mathbb{E}\Delta W_0^*.
\end{aligned}$$

Using this notation we have the following lemma:

LEMMA 5. *Let $K \in \mathcal{C}^2$ and choose random points $X_1, \dots, X_n$ on ∂K^* independently and according to the distribution function $D_{K^*}^*$ defined by a continuous density function $d_K > 0$. Denote by P_n the convex hull of the random points $X_1, \dots, X_n$. Then*

$$\begin{aligned}
\mathbb{E}\Delta W_+^* = c_2^W \int_{\partial K_+^*} d_{K^*}(x^*)^{-2/(d-1)}\, \|x^*\|^{-(d+1)}\, H_{d-1}^*(x^*)^{1/(d-1)}\, dx^*\, n^{-2/(d-1)}\\
+ o\big(n^{-2/(d-1)}\big)
\end{aligned}$$

as $n \to \infty$, where

$$c_2^W = \frac{(d-1)\Gamma(d+1+\frac{2}{d-1})}{d(d+1)!\kappa_d \kappa_{d-1}^{2/(d-1)}}.$$

The analogous result for $K \in \mathcal{C}^k$ with positive Gaussian curvature can be stated more directly since in that case the convex body $L = K^*$ is also of differentiability class $\mathcal{C}^k$ with positive Gaussian curvature and $d_L = d_{K^*}^*$ is well defined by (52).

LEMMA 6. *Let $L \in \mathcal{C}^{k+2}$ with positive Gaussian curvature, $k \geq 3$, and choose random points $X_1, \dots, X_n$ on ∂L independently and according to a density function $d_L \in \mathcal{C}^k$ with $d_L > 0$. Denote by P_n the convex hull of the random points $X_1, \dots, X_n$. Then*

$$\begin{aligned}
&\mathbb{E}W^*(P_n) - W^*(L) \\
&\quad = c_2^{W^*}(L) n^{-2/(d-1)} + c_3^{W^*}(L) n^{-3/(d-1)} + \cdots + c_{k-1}^{W^*}(L) n^{-(k-1)/(d-1)} \\
&\qquad + O\bigl(n^{-k/(d-1)}\bigr)
\end{aligned}$$

as $n \to \infty$. The constants $c_m^{W^*}(L)$ satisfy $c_{2m-1}^{W^*}(L) = 0$ for $m \leq d/2$ if d is even and $c_{2m-1}^{W^*}(L) = 0$ for all m if d is odd.

Since the proof of Lemma 5 (resp. Lemma 6), is similar to the proof of Theorem 1 (resp. Theorem 2) in [18], we will omit the proof. We only want to remark on the following: since (51) also makes sense for $(d-1)$-dimensional polytopes we have

$$W^*(P^*_{(n)}) = \sum_{F_n \text{ facet of} P^*_{(n)}} W^*(F_n).$$

Now Lemma 5 follows from the fact that $\|x\|^{-(d+1)}$ is a continuous function if x is contained in a suitable neighborhood of ∂K. The essential step in the proof of Lemma 6 is to observe that for the convex hull $\operatorname{conv}[X_1, \dots, X_d]$ of points $X_1, \dots, X_d$ in K,

$$W^*(\operatorname{conv}[X_1, \dots, X_d]) S(\operatorname{conv}[X_1, \dots, X_d])$$

is an analytic function with respect to the coordinates of the point $(X_1, \dots, X_d) \in \mathbb{R}^{d^2}$. [Recall that $S(\cdot)$ denotes the surface area.] Note that the proof of this Lemma (as in [18]) requires the stronger differentiability class $\mathcal{C}^{k+2}$ for K and $\mathcal{C}^k$ for d_L.

Using Lemma 5 we determine the asymptotic behavior of $\mathbb{E}\Delta W_+$,

$$\begin{aligned}
\mathbb{E}\Delta W_+(K) &= \mathbb{E}\Delta W_+^*(K^*) \\
&= c_2^W \int_{\partial K_+^*} d_{K^*}(x^*)^{-2/(d-1)} \|x^*\|^{-(d+1)} H_{d-1}^*(x^*)^{1/(d-1)}\, dx^*\, n^{-2/(d-1)} \\
&\quad + o\bigl(n^{-2/(d-1)}\bigr) \\
&= c_2^W \int_{\partial K_+^*} d_K(x(x^*))^{-2/(d-1)} H_{d-1}(x(x^*))^{2/(d-1)} H_{d-1}^*(x^*)^{1/(d-1)} \\
&\qquad\qquad \times \bigl(n_K(x(x^*)) \cdot n_{K^*}(x^*)\bigr)^{-(d+1)/(d-1)} \\
&\qquad\qquad \times \|x^*\|^{-d} \bigl(x^* \cdot n_{K^*}(x^*)\bigr)\, dx^*\, n^{-2/(d-1)} + o\bigl(n^{-2/(d-1)}\bigr),
\end{aligned}$$

since $n_K(x) = x^*/\|x^*\|$, where $H^*_{d-1}(x^*)$ denotes the Gaussian curvature of K^* at x^*. Using polar coordinates with respect to K^* the integral equals

$$\int_{S_+^{d-1}} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{2/(d-1)} H^*_{d-1}(x^*)^{1/(d-1)} \times \big(n_K(x)\cdot n_{K^*}(x^*)\big)^{-(d+1)/(d-1)}\, du\, n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big)$$

and parametrizing ∂K by its outer unit normal vector gives

$$\int_{\partial K_+} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{1/(d-1)} H^*_{d-1}(x^*)^{1/(d-1)} \times \big(n_K(x)\cdot n_{K^*}(x^*)\big)^{-(d+1)/(d-1)} H_{d-1}(x)^{d/(d-1)}\, dx\, n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big).$$

Since

$$H_{d-1}(x)\, H^*_{d-1}(x^*)\big(n_K(x)\cdot n_{K^*}(x^*)\big)^{-(d+1)} = 1$$

(cf. Kaltenbach [11]), we obtain

$$\mathbb{E}\Delta W_+ = c_2^W \int_{\partial K_+} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{d/(d-1)}\, dx\, n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big).$$

To deal with $\mathbb{E}\Delta W_0$ we choose the maximal distance ϑ between $P_{(n)}$ and K such that the following conditions are satisfied:

(i) the set $G(x,\vartheta)$ for $x \in \partial K_+(\varepsilon)$ is contained in $\partial K_+(\varepsilon/2)$,
(ii) the set $G(x,\vartheta)$ for $x \in \partial K_0(\varepsilon)$ is contained in $\partial K_+(2\varepsilon)$,
(iii) for any boundary point $x \in \partial K_0$ with principal curvatures $k_i(x)$ there is a paraboloid Q with principal curvatures $k_i(x)+\varepsilon$ which touches ∂K at x from "inside" and is contained in K up to height ϑ and
(iv) $\vartheta \le \rho$, where ρ is the radius of the ball touching ∂K from inside at every boundary point.

Now fix $P_{(n)}$. As in the case of the surface area we show the existence of a constant c depending on K and d_K such that

$$\mathbb{E}\Delta W_0 = \mathbb{E}\Delta W_0(\varepsilon) \le c\mathbb{E}\Delta V_0(2\varepsilon). \tag{54}$$

Here $\mathbb{E}\Delta V_0(2\varepsilon)$ corresponds to $\mathbb{E}\Delta V_0$ in (32) with $\partial K_0(\varepsilon)$ replaced by $\partial K_0(2\varepsilon)$.

To prove (54) we denote by v_j, $j = 1,\dots,N$, the vertices of $P_{(n)}$, denote by x_j the point on ∂K nearest to v_j and by s_j the distance from v_j to K, that is, $v_j = x_j + s_j n_K(x_j)$. Now without loss of generality we assume that the vertices are numbered such that the first J vertices form a "maximal sequence of disjoint sets $G(x_j,s_j)$ meeting ∂K_0." That is, for $j \in \{1,\dots,J\}$ the set $G(x_j,s_j)$ satisfies the following: $G(x_j,s_j)$ meets ∂K_0, $G(x_j,s_j)$ is disjoint from

$\bigcup_{i=1,\dots j-1} G(x_i, s_i)$, and for all $k > j$ either $s_k < s_j$ or the set $G(x_k, s_k)$ meets $\bigcup_{i=1,\dots j-1} G(x_i, s_i)$ or is contained in ∂K_+. Further for all $j \geq J$ either $G(x_j, s_j)$ intersects $\bigcup_{i=1,\dots,J} G(x_i, s_i)$ or $G(x_j, s_j)$ is contained in ∂K_+.

For abbreviation, put $U_j = \nu(G(x_j, s_j))$ and

$$\overline{U}_j = \nu\Big(\bigcup G(x_i, s_i) : G(x_i, s_i) \cap G(x_j, s_j) \neq \varnothing,\ i > j\Big).$$

Then

$$\begin{aligned}
\mathbb{E}\Delta W_0 &= \frac{2}{d\kappa_d} \int_{S_0^{d-1}} \big(h_{P_{(n)}}(u) - h_K(u)\big)\, du \\
&\leq \frac{2}{d\kappa_d} \sum_{j=1}^{J} \int_{\overline{U}_j} \big(h_{P_{(n)}}(u) - h_K(u)\big)\, du \\
&\leq \frac{2}{d\kappa_d} \sum_{j=1}^{J} s_j V_{d-1}(\overline{U}_j).
\end{aligned}$$

We show that $V_{d-1}(\overline{U}_j)$ is of order $s_j^{(d-1)/2}$. Indeed, if $y \in G(x, s)$ then there is a point y_B on the boundary of the ball B of radius ρ touching ∂K at x from inside, such that $n_K(y) = n_B(y_B)$. Clearly the hyperplane to ∂B at y_B intersects the segment $[x, x + s n_K(x)]$ and thus y_B is "visible" with respect to B from $x + s n_K(x)$. Now an elementary calculation shows that

$$\cos \angle\{n_B(y_B), n_B(x)\} = \frac{\rho}{r_{y_B}(x) + \rho}$$

which immediately implies

$$\angle\{n_K(y), n_K(x)\} \leq c_1 s^{1/2}$$

for $y \in G(x, s)$. Hence for $y \in G(x_j, s_j)$, this proves that the angle between $n_K(y)$ and $n_K(x_j)$ is bounded from above by $c_1 s_j^{1/2}$ and for $y \in G(x_i, s_i)$ where $G(x_i, s_i)$ intersects $G(x_j, s_j)$ this proves that the angle between $n_K(y)$ and $n_K(x_j)$ is bounded from above by $3c_1 s_j^{1/2}$. From this $V_{d-1}(\overline{U}_j) \leq c_2 s_j^{(d-1)/2}$ and thus

$$\mathbb{E}\Delta W_0 \leq c_3 \sum_{j=1}^{J} s_j^{(d+1)/2}$$

follows immediately with a constant c_3 depending on K. To prove (54) it remains to show that

$$\mathbb{E}\Delta V_0(2\varepsilon) \geq c_4 \sum_{j=1}^{J} s_j^{(d+1)/2}, \tag{55}$$

with a constant c_4 depending on K. To see this observe that the convex hull of v_j and $G(x_j, s_j)$ contains a cone whose height is s_j and base is a $(d-1)$-dimensional ball of radius $\sqrt{\rho s}/2$ [cf. the deduction of formula (29)]. Since $\mathbb{E}\Delta V_0(2\varepsilon)$ is bounded from below by the sum of the volumes of the convex hulls of v_j and $G(x_j, s_j)$, formula (55) follows.

Combining Lemma 5, (54) and (37) (with ε replaced by 2ε) gives

$$\mathbb{E}\Delta W_+ + \mathbb{E}\Delta W_0 = c_2^W \int_{\partial K} d_K(x)^{-2/(d-1)} H_{d-1}(x)^{d/(d-1)}\, dx\, n^{-2/(d-1)} + o\big(n^{-2/(d-1)}\big) + O\big(\varepsilon^{1/(d-1)}\big)$$

which proves the first part of Theorem 3.

The second part of Theorem 3 follows immediately from Lemma 6.

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