

## NECESSARY AND SUFFICIENT CONDITIONS IN THE PROBLEM OF OPTIMAL INVESTMENT IN INCOMPLETE MARKETS

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Following *Ann. Appl. Probab.* **9** (1999) 904–950 we continue the study of the problem of expected utility maximization in incomplete markets. Our goal is to find *minimal* conditions on a model and a utility function for the validity of several key assertions of the theory to hold true. In the previous paper we proved that a minimal condition on the utility function *alone*, that is, a minimal *market independent* condition, is that the asymptotic elasticity of the utility function is strictly less than 1. In this paper we show that a *necessary and sufficient* condition on *both*, the utility function and the model, is that the value function of the dual problem is finite.

**1. Introduction and main results.** We study the same financial framework as in [10] and refer to this paper for more details and references. We consider a model of a security market which consists of  $d + 1$  assets, one bond and  $d$  stocks. We work in discounted terms, that is, we suppose that the price of the bond is constant, and denote by  $S = (S^i)_{1 \leq i \leq d}$  the price process of the  $d$  stocks. The process  $S$  is assumed to be a semimartingale on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ . Here  $T$  is a finite time horizon. To simplify notation we assume that  $\tilde{\mathcal{F}} = \mathcal{F}_T$ .

A (self-financing) portfolio  $\Pi$  is defined as a pair  $(x, H)$ , where the constant  $x$  is the initial value of the portfolio, and  $H = (H^i)_{1 \leq i \leq d}$  is a predictable  $S$ -integrable process, where  $H_t^i$  specifies how many units of asset  $i$  are held in the portfolio at time  $t$ . The value process  $X = (X_t)_{0 \leq t \leq T}$  of such a portfolio  $\Pi$  is given by

$$(1) \quad X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \leq t \leq T.$$

We denote by  $\mathcal{X}(x)$  the family of wealth processes with nonnegative capital at any instant, that is,  $X_t \geq 0$  for all  $t \in [0, T]$ , and with initial value equal to  $x$ . In other words

$$\mathcal{X}(x) = \{X \geq 0 : X \text{ is defined by (1) with } X_0 = x\}.$$

We shall use the shorter notation  $\mathcal{X}$  for  $\mathcal{X}(1)$ . Clearly,

$$\mathcal{X}(x) = x\mathcal{X} = \{xX : X \in \mathcal{X}\} \quad \text{for } x \geq 0.$$

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A probability measure  $\mathbb{Q} \sim \mathbb{P}$  is called an *equivalent local martingale measure* if any  $X \in \mathcal{X}$  is a local martingale under  $\mathbb{Q}$ . The family of equivalent local martingale measures will be denoted by  $\mathcal{M}$ . We assume throughout that

$$(2) \quad \mathcal{M} \neq \emptyset.$$

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [4, 5] for precise statements and references.

We also consider an economic agent in our model, whose preferences are modeled by a utility function  $U: (0, \infty) \rightarrow \mathbf{R}$  for wealth at maturity time  $T$ . Hereafter we will assume that the function  $U$  is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions

$$(3) \quad \begin{aligned} U'(0) &= \lim_{x \rightarrow 0} U'(x) = \infty, \\ U'(\infty) &= \lim_{x \rightarrow \infty} U'(x) = 0. \end{aligned}$$

For a given initial capital  $x > 0$ , the goal of the agent is *to maximize the expected value of terminal utility*. The value function of this problem is denoted by

$$(4) \quad u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Intuitively speaking, the value function  $u$  plays the role of the utility function of the investor at time 0, if she subsequently invests in an optimal way. A well-known tool in studying the optimization problem (4) is the use of duality relationships in the spaces of convex functions and semimartingales; see, for example, [1–3, 6–11, 13].

The conjugate function  $V$  to the utility function  $U$  is defined as

$$(5) \quad V(y) = \sup_{x > 0} [U(x) - xy], \quad y > 0.$$

It is well known (see, e.g., [12]) that if  $U$  satisfies the hypotheses stated above, then  $V$  is a continuously differentiable, decreasing, strictly convex function satisfying  $V'(0) = -\infty$  and  $V'(\infty) = 0$ ,  $V(0) = U(\infty)$ ,  $V(\infty) = U(0)$ , and the following relation holds true:

$$U(x) = \inf_{y > 0} [V(y) + xy], \quad x > 0.$$

In addition the derivative of  $U$  is the inverse function of the negative of the derivative of  $V$ , that is,

$$U'(x) = y \iff x = -V'(y).$$

Further, we define the family  $\mathcal{Y}$  of nonnegative semimartingales, which is dual to  $\mathcal{X}$  in the following sense:

$$\mathcal{Y} = \{Y \geq 0: Y_0 = 1 \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}\}.$$

Note that, as  $1 \in \mathcal{X}$ , any  $Y \in \mathcal{Y}$  is a supermartingale. Note also that the set  $\mathcal{Y}$  contains the density processes of all  $\mathbb{Q} \in \mathcal{M}$ . For  $y > 0$ , we define

$$\mathcal{Y}(y) = y\mathcal{Y} = \{yY : Y \in \mathcal{Y}\}$$

and consider the following optimization problem:

$$(6) \quad v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)].$$

The next result from [10] shows that the value functions  $u$  and  $v$  to the optimization problems (4) and (6) are conjugate.

THEOREM 1 ([10], Theorem 2.1). *Assume that (2) and (3) hold true and*

$$(7) \quad u(x) < \infty \quad \text{for some } x > 0.$$

Then:

1.  $u(x) < \infty$  for all  $x > 0$ , and there exists  $y_0 \geq 0$  such that  $v(y)$  is finitely valued for  $y > y_0$ . The value functions  $u$  and  $v$  are conjugate

$$(8) \quad \begin{aligned} v(y) &= \sup_{x>0} [u(x) - xy], & y > 0, \\ u(x) &= \inf_{y>0} [v(y) + xy], & x > 0. \end{aligned}$$

The function  $u$  is continuously differentiable on  $(0, \infty)$  and the function  $v$  is strictly convex on  $\{v < \infty\}$ .

The functions  $u'$  and  $v'$  satisfy

$$\begin{aligned} u'(0) &= \lim_{x \rightarrow 0} u'(x) = \infty, \\ v'(\infty) &= \lim_{y \rightarrow \infty} v'(y) = 0. \end{aligned}$$

2. The optimal solution  $\widehat{Y}(y) \in \mathcal{Y}(y)$  to (6) exists and is unique provided that  $v(y) < \infty$ .

As in [10] we are interested in the following questions related to the optimization problems (4) and (6):

1. Does the optimal solution  $\widehat{X}(x) \in \mathcal{X}(x)$  to (4) exist?
2. Does the value function  $u(x)$  satisfy the usual properties of a utility function, that is, is it increasing, strictly concave, continuously differentiable and such that  $u'(0) = \infty, u'(\infty) = 0$ ?
3. Does the dual value function  $v$  have the representation

$$(9) \quad v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E} \left[ V \left( y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right],$$

where  $\frac{d\mathbb{Q}}{d\mathbb{P}}$  denotes the Radon–Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  on  $(\Omega, \mathcal{F}) = (\Omega, \mathcal{F}_T)$ ?

In [10] (see Theorem 2.2 and the counterexamples in Section 5) we proved that a minimal assumption on the utility function  $U$ , which implies positive answers to these questions for an *arbitrary* financial model, is the condition on the asymptotic behavior of the elasticity of  $U$ ,

$$AE(U) \triangleq \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

The subsequent theorem, which is the main result of the present paper, and Note 1 below imply that a necessary and sufficient condition for all three assertions to have positive answers in the framework of a *particular* financial model is the finiteness of the dual value function.

**THEOREM 2.** *Assume that (2) and (3) hold true and*

$$(10) \quad v(y) < \infty \quad \forall y > 0.$$

*Then in addition to the assertions of Theorem 1 we have the following:*

1. *The value functions  $u$  and  $-v$  are continuously differentiable, increasing and strictly concave on  $(0, \infty)$  and satisfy*

$$\begin{aligned} u'(\infty) &= \lim_{x \rightarrow \infty} u'(x) = 0, \\ -v'(0) &= \lim_{y \rightarrow 0} -v'(y) = \infty. \end{aligned}$$

2. *The optimal solution  $\widehat{X}(x) \in \mathcal{X}(x)$  to (4) exists, for any  $x > 0$ , and is unique. In addition, if  $y = u'(x)$  then*

$$U'(\widehat{X}_T(x)) = \widehat{Y}_T(y),$$

*where  $\widehat{Y}(y) \in \mathcal{Y}(y)$  is the optimal solution to (6). Moreover, the process  $\widehat{X}(x)\widehat{Y}(y)$  is a martingale.*

3. *The dual value function  $v$  satisfies (9).*

**PROOF.** Theorem 2 is a rather straightforward consequence of its “abstract version,” Theorem 4. Admitting Theorem 4 as well as Proposition 1, the proof of Theorem 2 proceeds as follows.

For  $x > 0$  and  $y > 0$ , let

$$(11) \quad \mathcal{C}(x) = \{g \in \mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq g \leq X_T, \text{ for some } X \in \mathcal{X}(x)\},$$

$$(12) \quad \mathcal{D}(y) = \{h \in \mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq h \leq Y_T, \text{ for some } Y \in \mathcal{Y}(y)\}.$$

In other words,  $\mathcal{C}(x)$  and  $\mathcal{D}(y)$  are the sets of random variables dominated by the final values of elements from  $\mathcal{X}(x)$  and  $\mathcal{Y}(y)$ , respectively. With this notation, the value functions  $u$  and  $v$  take the form

$$\begin{aligned} u(x) &= \sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)], \\ v(y) &= \inf_{h \in \mathcal{D}(y)} \mathbb{E}[V(h)]. \end{aligned}$$

According to Proposition 3.1 in [10] the sets  $\mathcal{C}(x)$ ,  $x > 0$ , and  $\mathcal{D}(y)$ ,  $y > 0$ , satisfy the conditions (16)–(18). Hence Theorem 4 implies assertions 1 and 2 of Theorem 2, except for the claim that the product  $\widehat{X}(x)\widehat{Y}(y)$  is a martingale. To prove the martingale property, note that  $\widehat{X}(x)\widehat{Y}(y)$  is a positive supermartingale [by the construction of the set  $\mathcal{Y}(y)$ ] and that we obtain the following equality from item 2 of Theorem 4:

$$\mathbb{E}[\widehat{X}_T(x)\widehat{Y}_T(y)] = xy = \widehat{X}_0(x)\widehat{Y}_0(y).$$

This readily implies the martingale property of  $\widehat{X}(x)\widehat{Y}(y)$ .

To prove the final assertion 3, we use Proposition 1. We denote by  $\widetilde{\mathcal{D}}$  the set of Radon–Nikodym derivatives of equivalent martingale measures

$$\widetilde{\mathcal{D}} = \left\{ h = \frac{d\mathbb{Q}}{d\mathbb{P}}, \mathbb{Q} \in \mathcal{M} \right\}.$$

The set  $\widetilde{\mathcal{D}}$  is closed under countable convex combinations. In addition,

$$g \in \mathcal{C} \Leftrightarrow g \geq 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}[g] \leq 1 \quad \forall \mathbb{Q} \in \mathcal{M}$$

by the general duality relationships between the terminal values of strategies and the densities of equivalent martingale measures (see [4] and [5]). Hence the set  $\widetilde{\mathcal{D}}$  satisfies the assumptions of Proposition 1 and the result follows.  $\square$

NOTE 1. In view of the duality relation (8), condition (10) is equivalent to

$$u'(\infty) = \lim_{x \rightarrow \infty} u'(x) = 0,$$

which may equivalently be restated as

$$\lim_{x \rightarrow \infty} \frac{u(x)}{x} = 0.$$

In particular, this shows the necessity of (10) for Theorem 2 to hold true.

NOTE 2. In [10], Theorem 2.2, we proved that the assertions of Theorem 2 follow from the assumptions of Theorem 1 and the condition  $AE(U) < 1$  on the asymptotic elasticity of  $U$ . Let us now deduce this result as an easy consequence of Theorem 2.

We need to show that  $AE(U) < 1$  implies that  $v(y) < \infty$  for all  $y > 0$ . By Theorem 1 there is  $y_0 > 0$  such that

$$(13) \quad v(y) < \infty, \quad y > y_0.$$

Further, the condition  $AE(U) < 1$  is equivalent to the following property of  $V$  (see Lemma 6.3 in [10]): there are positive constants  $c_1$  and  $c_2$  such that

$$(14) \quad V\left(\frac{y}{2}\right) \leq c_1 V(y) + c_2, \quad y > 0.$$

The finiteness of  $v$  now follows from (13) and (14).

NOTE 3. Condition (10) may also be stated in the following equivalent form:

$$(15) \quad \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E} \left[ V \left( y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] < \infty \quad \forall y > 0.$$

Indeed, the implication (15)  $\Rightarrow$  (10) is trivial, as the density processes of martingale measures belong to  $\mathcal{Y}$ . The more difficult reverse implication follows from Theorem 2.

**2. The abstract version of the theorem.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be nonempty sets of positive random variables such that

1. The set  $\mathcal{C}$  is bounded in  $L^0(\Omega, \mathcal{F}, \mathbb{P})$  and contains the constant function  $g = 1$ ,

$$(16) \quad \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{C}} \mathbb{P}[|g| \geq n] = 0,$$

$$(17) \quad 1 \in \mathcal{C}.$$

2. The sets  $\mathcal{C}$  and  $\mathcal{D}$  satisfy the bipolar relations

$$(18) \quad \begin{aligned} g \in \mathcal{C} &\Leftrightarrow g \geq 0 \quad \text{and} \quad \mathbb{E}[gh] \leq 1 && \forall h \in \mathcal{D}, \\ h \in \mathcal{D} &\Leftrightarrow h \geq 0 \quad \text{and} \quad \mathbb{E}[gh] \leq 1 && \forall g \in \mathcal{C}. \end{aligned}$$

For  $x > 0$  and  $y > 0$ , we define the sets

$$\begin{aligned} \mathcal{C}(x) &= x\mathcal{C} = \{xg : g \in \mathcal{C}\}, \\ \mathcal{D}(y) &= y\mathcal{D} = \{yh : h \in \mathcal{D}\}, \end{aligned}$$

and the optimization problems

$$(19) \quad u(x) = \sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)],$$

$$(20) \quad v(y) = \inf_{h \in \mathcal{D}(y)} \mathbb{E}[V(h)].$$

Here  $U = U(x)$  and  $V = V(y)$  are the functions defined in Section 1. If  $\mathcal{C}(x)$  and  $\mathcal{D}(y)$  are defined by (11) and (12), these value functions coincide with the value functions defined in (4) and (6).

Let us recall the following result from [10], which is the abstract version of Theorem 1.

**THEOREM 3** (Theorem 3.1 in [10]). *Assume that the sets  $\mathcal{C}$  and  $\mathcal{D}$  satisfy (16)–(18). Assume also that the utility function  $U$  satisfies (3) and that*

$$(21) \quad u(x) < \infty \quad \text{for some } x > 0.$$

Then:

1.  $u(x) < \infty$  for all  $x > 0$ , and there exists  $y_0 \geq 0$  such that  $v(y)$  is finitely valued for  $y > y_0$ . The value functions  $u$  and  $v$  are conjugate:

$$(22) \quad \begin{aligned} v(y) &= \sup_{x>0} [u(x) - xy], & y > 0, \\ u(x) &= \inf_{y>0} [v(y) + xy], & x > 0. \end{aligned}$$

The function  $u$  is continuously differentiable on  $(0, \infty)$ , and the function  $v$  is strictly convex on  $\{v < \infty\}$ .

The functions  $u'$  and  $-v'$  satisfy

$$\begin{aligned} u'(0) &= \lim_{x \rightarrow 0} u'(x) = \infty, \\ v'(\infty) &= \lim_{y \rightarrow \infty} v'(y) = 0. \end{aligned}$$

2. If  $v(y) < \infty$ , then the optimal solution  $\hat{h}(y) \in \mathcal{D}(y)$  to (19) exists and is unique.

We now state the abstract version of Theorem 2. This theorem refines Theorem 3.2 in [10] in the sense that the condition  $AE(U) < 1$  is replaced by the weaker condition (23) requiring the finiteness of the function  $v(y)$ , for all  $y > 0$ .

**THEOREM 4.** Assume that the utility function  $U$  satisfies (3), the sets  $\mathcal{C}$  and  $\mathcal{D}$  satisfy (16)–(18), and that the value function  $v$  defined in (20) is finite

$$(23) \quad v(y) < \infty \quad \forall y > 0.$$

Then, in addition to the assertions of Theorem 3, we have the following:

1. The value functions  $u$  and  $-v$  are continuously differentiable, increasing and strictly concave on  $(0, \infty)$  and satisfy

$$\begin{aligned} u'(\infty) &= \lim_{x \rightarrow \infty} u'(x) = 0, \\ -v'(0) &= \lim_{y \rightarrow 0} -v'(y) = \infty. \end{aligned}$$

2. The optimal solution  $\hat{g}(x) \in \mathcal{C}(x)$  to (19) exists, for all  $x > 0$ , and is unique. In addition, if  $y = u'(x)$ , then

$$\begin{aligned} U'(\hat{g}(x)) &= \hat{h}(y), \\ \text{and } \mathbb{E}[\hat{g}(x)\hat{h}(y)] &= xy, \end{aligned}$$

where  $\hat{h}(y) \in \mathcal{D}(y)$  is the optimal solution to (20).

The proof of Theorem 4 is based on the following lemma.

LEMMA 1. Assume that the set  $\mathcal{C}$  satisfies (16)–(18) and the value function  $u(x)$  defined in (19) is finite (for some or, equivalently, for all  $x > 0$ ) and satisfies

$$(24) \quad \lim_{x \rightarrow \infty} \frac{u(x)}{x} = 0.$$

Then the optimal solution  $\widehat{g}(x) \in \mathcal{C}(x)$  exists for all  $x > 0$ .

PROOF. The assertion that  $u(x) < \infty$ , for some  $x > 0$ , iff  $u(x) < \infty$ , for all  $x > 0$ , is a straightforward consequence of the concavity and monotonicity of  $u$  and the fact that  $u \geq U$ . Also observe that, as remarked in Note 1, assertion (24) is equivalent to (23).

Fix  $x > 0$ . Let  $(f^n)_{n \geq 1}$  be a sequence in  $\mathcal{C}(x)$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[U(f^n)] = u(x).$$

We can find a sequence of convex combinations  $g^n \in \text{conv}(f^n, f^{n+1}, \dots)$  which converges almost surely to a random variable  $\widehat{g}$  with values in  $[0, \infty]$ ; see, for example, [4], Lemma A1.1. Since the set  $\mathcal{C}(x)$  is bounded in  $\mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ , we deduce that  $\widehat{g}$  is almost surely finitely valued. By (18) and Fatou’s lemma,  $\widehat{g}$  belongs to  $\mathcal{C}(x)$ . We claim that  $\widehat{g}$  is the optimal solution to (19), that is,

$$\mathbb{E}[U(\widehat{g})] = u(x).$$

Let us denote by  $U^+$  and  $U^-$  the positive and negative parts of the function  $U$ . From the concavity of  $U$  we deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}[U(g^n)] = u(x)$$

and from Fatou’s lemma that

$$\liminf_{n \rightarrow \infty} \mathbb{E}[U^-(g^n)] \geq \mathbb{E}[U^-(\widehat{g})].$$

The optimality of  $\widehat{g}$  will follow if we show that

$$(25) \quad \lim_{n \rightarrow \infty} \mathbb{E}[U^+(g^n)] = \mathbb{E}[U^+(\widehat{g})].$$

If  $U(\infty) \leq 0$ , then there is nothing to prove. So we assume that  $U(\infty) > 0$ .

The validity of (25) is equivalent to the uniform integrability of the sequence  $(U^+(g^n))_{n \geq 1}$ . If this sequence is not uniformly integrable then, passing if necessary to a subsequence still denoted by  $(g^n)_{n \geq 1}$ , we can find a constant  $\alpha > 0$  and a disjoint sequence  $(A^n)_{n \geq 1}$  of  $(\Omega, \mathcal{F})$ , that is,

$$A^n \in \mathcal{F}, \quad A^i \cap A^j = \emptyset \quad \text{if } i \neq j,$$

such that

$$\mathbb{E}[U^+(g^n)I(A^n)] \geq \alpha \quad \text{for } n \geq 1.$$

We define the sequence of random variables  $(h^n)_{n \geq 1}$

$$h^n = x_0 + \sum_{k=1}^n g^k I(A^k),$$

where

$$x_0 = \inf\{x > 0 : U(x) \geq 0\}.$$

For any  $f \in \mathcal{D}$ ,

$$\mathbb{E}[h^n f] \leq x_0 + \sum_{k=1}^n \mathbb{E}[g^k f] \leq x_0 + nx.$$

Hence  $h^n \in \mathcal{C}(x_0 + nx)$ . On the other hand,

$$\mathbb{E}[U(h^n)] \geq \sum_{k=1}^n \mathbb{E}[U^+(g^k) I(A^k)] \geq \alpha n,$$

and therefore

$$\limsup_{x \rightarrow \infty} \frac{u(x)}{x} \geq \limsup_{n \rightarrow \infty} \frac{\mathbb{E}[U(h^n)]}{x_0 + nx} \geq \limsup_{n \rightarrow \infty} \frac{\alpha n}{x_0 + nx} = \alpha > 0.$$

This contradicts (24). Therefore (25) holds true.  $\square$

PROOF OF THEOREM 4. Since, for  $x > 0$  and  $y > 0$ ,

$$U(x) \leq V(y) + xy,$$

and, for  $g \in \mathcal{C}(x)$  and  $h \in \mathcal{D}(y)$ ,

$$\mathbb{E}[gh] \leq xy,$$

we have

$$u(x) \leq v(y) + xy.$$

In particular, the finiteness of  $v(y)$ , for some  $y > 0$ , implies the finiteness of  $u(x)$ , for all  $x > 0$ . It follows that the conditions of Theorem 3 hold true.

From the assumption that  $v(y) < \infty$ ,  $y > 0$ , and the duality relations (22) between  $u$  and  $v$ , we deduce that

$$(26) \quad \lim_{x \rightarrow \infty} \frac{u(x)}{x} = \lim_{x \rightarrow \infty} u'(x) = 0.$$

Lemma 1 now implies that the optimal solution  $\widehat{g}(x)$  to (19) exists, for any  $x > 0$ . The strict concavity of  $U$  implies the uniqueness of  $\widehat{g}(x)$ , as well as the fact that the function  $u$  is strictly concave too. The remaining assertions of item 1 related to the function  $v$  follow from the established properties of  $u$ , because of the duality relations (22) (see, e.g., [12]).

Let  $x > 0$ ,  $y = u'(x)$ ,  $\widehat{g}(x)$  and  $\widehat{h}(y)$  be the optimal solutions to (19) and (20), respectively. We have

$$\begin{aligned} & \mathbb{E}[|V(\widehat{h}(y)) + \widehat{g}(x)\widehat{h}(y) - U(\widehat{g}(x))|] \\ &= \mathbb{E}[V(\widehat{h}(y)) + \widehat{g}(x)\widehat{h}(y) - U(\widehat{g}(x))] \\ &\leq v(y) + xy - u(x) = 0, \end{aligned}$$

where, in the last step, we have used the relation  $y = u'(x)$ . It follows that

$$U(\widehat{g}(x)) = V(\widehat{h}(y)) + \widehat{g}(x)\widehat{h}(y).$$

This readily implies that

$$U'(\widehat{g}(x)) = \widehat{h}(y) \quad \text{a.s.}$$

and

$$\mathbb{E}[\widehat{g}(x)\widehat{h}(y)] = \mathbb{E}[U(\widehat{g}(x))] - \mathbb{E}[V(\widehat{h}(y))] = u(x) - v(y) = xy. \quad \square$$

We complete the section with Proposition 1, which was used in the proof of item 3 of Theorem 2. This proposition was proved in [10] under the additional assumption  $AE(U) < 1$ .

Let  $\widetilde{\mathcal{D}}$  be a convex subset of  $\mathcal{D}$  such that:

1. For any  $g \in \mathcal{C}$ ,

$$(27) \quad \sup_{h \in \widetilde{\mathcal{D}}} \mathbb{E}[gh] = \sup_{h \in \mathcal{D}} \mathbb{E}[gh].$$

2. The set  $\widetilde{\mathcal{D}}$  is closed under countable convex combinations, that is, for any sequence  $(h^n)_{n \geq 1}$  of elements of  $\widetilde{\mathcal{D}}$  and any sequence of positive numbers  $(a^n)_{n \geq 1}$  such that  $\sum_{n=1}^{\infty} a^n = 1$  the random variable  $\sum_{n=1}^{\infty} a^n h^n$  belongs to  $\widetilde{\mathcal{D}}$ .

**PROPOSITION 1.** *Assume that the conditions of Theorem 4 hold true and that  $\widetilde{\mathcal{D}}$  satisfies the above assertions. The value function  $v(y)$  defined in (20) then satisfies*

$$(28) \quad v(y) = \inf_{h \in \widetilde{\mathcal{D}}} \mathbb{E}[V(yh)].$$

The proof of the proposition will use the following two lemmas.

The first is an easy result, whose proof is analogous to the proof of Proposition 3.1 in [10] and is therefore skipped.

**LEMMA 2.** *Under the assumptions of Proposition 1, let  $\widehat{h}(y)$  be the optimal solution to (20). Then there exists a sequence  $(h^n)_{n \geq 1}$  in  $\widetilde{\mathcal{D}}$ , that converges almost surely to  $\widehat{h}(y)/y$ .*

LEMMA 3. *Under the assumptions of Proposition 1, we have, for each  $y > 0$ ,*

$$\inf_{h \in \tilde{\mathcal{D}}} \mathbb{E}[V(yh)] < \infty.$$

PROOF. To simplify the notation we shall prove the assertion of the lemma for the case  $y = 1$ .

Let  $(\lambda_n)_{n \geq 1}$  be a sequence of strictly positive numbers such that  $\sum_{n=1}^\infty \lambda_n = 1$ . We denote by  $\hat{h}(\lambda_n)$  the optimal solution to (20) corresponding to the case  $y = \lambda_n$ . Let  $(\delta_n)_{n \geq 2}$  be a sequence of strictly positive numbers, decreasing to 0, such that

$$(29) \quad \sum_{n=1}^\infty \mathbb{E}[V(\hat{h}(\lambda_n))I(A_n)] < \infty \quad \text{if } A_n \in \mathcal{F}, \mathbb{P}[A_n] \leq \delta_n, n \geq 2.$$

From Lemma 2 we deduce the existence of a sequence  $(h_n)_{n \geq 1}$  in  $\tilde{\mathcal{D}}$  such that

$$\mathbb{P}[V(\lambda_n h_n) > V(\hat{h}(\lambda_n)) + 1] \leq \delta_{n+1}, \quad n \geq 1.$$

We define the sequence of measurable sets  $(A_n)_{n \geq 1}$  as follows:

$$\begin{aligned} A_1 &= \{V(\lambda_1 h_1) \leq V(\hat{h}(\lambda_1)) + 1\} \\ &\vdots \\ A_n &= \{V(\lambda_n h_n) \leq V(\hat{h}(\lambda_n)) + 1\} \setminus \bigcup_{k=1}^{n-1} A_k. \end{aligned}$$

This sequence has the following properties:

$$\begin{aligned} A_i \cap A_j &= \emptyset \quad \text{if } i \neq j, \\ \mathbb{P}\left[\bigcup_{n=1}^\infty A_n\right] &= 1, \\ \mathbb{P}[A_n] &\leq \delta_n, \quad n \geq 2. \end{aligned}$$

We define

$$h = \sum_{n=1}^\infty \lambda_n h_n.$$

We have  $h \in \tilde{\mathcal{D}}$  because the set  $\tilde{\mathcal{D}}$  is closed under countable convex combinations. The proof now follows from the inequalities

$$\begin{aligned} \mathbb{E}[V(h)] &= \sum_{n=1}^\infty \mathbb{E}[V(h)I(A_n)] \\ &\stackrel{(i)}{\leq} \sum_{n=1}^\infty \mathbb{E}[V(\lambda_n h_n)I(A_n)] \end{aligned}$$

$$\stackrel{\text{(ii)}}{\leq} \sum_{n=1}^{\infty} \mathbb{E}[V(\widehat{h}(\lambda_n))I(A_n)] + 1$$

$$\stackrel{\text{(iii)}}{<} \infty,$$

where (i) holds true because  $V$  is a decreasing function, (ii) follows from the construction of the sequence  $(A_n)_{n \geq 1}$ , and (iii) is a consequence of (29).  $\square$

PROOF OF PROPOSITION 1. Fix  $\varepsilon > 0$  and  $y > 0$ . We have to show that there is  $h \in \widetilde{\mathcal{D}}$  such that

$$\mathbb{E}[V((y + \varepsilon)h)] \leq v(y) + \varepsilon.$$

Let  $\widehat{h} = \widehat{h}(y)$  be the optimal solution to the optimization problem (20) and  $f$  be an element of  $\widetilde{\mathcal{D}}$  such that

$$\mathbb{E}[V(\varepsilon f)] < \infty.$$

The existence of such a function  $f$  follows from Lemma 3. Let  $\delta > 0$  be a sufficiently small number such that

$$(30) \quad \mathbb{E}[ (|V(\widehat{h})| + |V(\varepsilon f)|) I(A) ] \leq \frac{\varepsilon}{2} \quad \text{if } A \in \mathcal{F}, \mathbb{P}[A] \leq \delta.$$

From Lemma 2 we deduce the existence of  $g \in \widetilde{\mathcal{D}}$  such that

$$(31) \quad \mathbb{P}\left[ V(yg) > V(\widehat{h}) + \frac{\varepsilon}{2} \right] \leq \delta.$$

Denote

$$A = \left\{ V(yg) > V(\widehat{h}) + \frac{\varepsilon}{2} \right\}$$

and define

$$h = \frac{yg + \varepsilon f}{y + \varepsilon}.$$

Since the set  $\widetilde{\mathcal{D}}$  is convex,  $h \in \widetilde{\mathcal{D}}$ . The proof now follows from the inequalities.

$$\begin{aligned} \mathbb{E}[V((y + \varepsilon)h)] &= \mathbb{E}[V(yg + \varepsilon f)] \\ &\stackrel{\text{(i)}}{\leq} \mathbb{E}[V(yg)I(A^c)] + \mathbb{E}[V(\varepsilon f)I(A)] \\ &\stackrel{\text{(ii)}}{\leq} v(y) + \varepsilon, \end{aligned}$$

where (i) holds true, because  $V$  is a decreasing function, and (ii) follows from (30) and (31).  $\square$

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