The Annals of Applied Probability
2003, Vol. 13, No. 4, 1474–1493

ASYMPTOTICALLY EXACT ANALYSIS OF A LOSS NETWORK WITH CHANNEL CONTINUITY

BY MURAT ALANYALI

Boston University

Two channel assignment policies are considered for a Kelly type loss network with an additional channel continuity requirement. It is assumed that the channels on any given link have distinct identities, and that a connection should be assigned channels with a common identity on all links of its route. Such constraints arise in circuit switched WDM optical networks and wireless cellular networks. A functional law of large numbers, which was previously developed by Hunt and Kurtz and later refined by Zachary and Ziedins, is adapted to analyze a network with two links and three connection types. Asymptotically exact fluid-type approximations for the network process are obtained and their operating points are characterized. The results lead to asymptotic call blocking rates and point out that in cases of practical interest, random channel assignment has asymptotically the same blocking performance with more sophisticated channel assignment policies.

1. Introduction and statement of results. We consider a two-link loss network of the type considered by Kelly (1991) with an additional channel continuity requirement. Namely, it is assumed that channels on any given link have distinct identities, and that the channels assigned to a connection on the links of its route must have a common identity. This situation arises in circuit switched wavelength division multiplexed (WDM) optical networks that employ optical switches. Here channels are identified by carrier frequencies called wavelengths, and in typical applications optical switches lack the capability to switch data between different wavelengths due to technological limitations. Another instance of channel continuity arises in wireless cellular networks in which a communication channel that is used to establish a radio connection in one cell cannot be used in neighboring cells due to electromagnetic interference.

Channel assignment under the continuity constraint has been studied extensively in the context of both WDM and cellular wireless networks. Kelly (1985) obtains explicit expressions for call blocking probability under a dynamic channel assignment policy on a cellular wireless network with linear topology. Pallant and Taylor (1995) study reversible models for dynamic channel assignment on general cellular topologies. Both articles assume a rearrangeable network in which channel assignments can be altered at desired instants. When this assumption is not valid, as usually is the case in applications, the resulting model is typically too complex

Received September 2001; revised January 2003.
AMS 2000 subject classifications. Primary 60K30; secondary 90B22, 68M20, 93E20.
Key words and phrases. Loss networks, Hunt–Kurtz theory, WDM optical networks, wavelength continuity, wireless cellular networks.

for explicit analysis. A variety of heuristic assignment principles have been considered for this realistic case in the context of both wireless cellular and WDM networks. The reader is referred to Katzela and Naghshineh (1996) for a survey on channel assignment in cellular networks, and to Zang, Jue and Mukherjee (2000) for an overview of wavelength assignment in WDM networks.

The present paper provides asymptotically exact analysis of two channel assignment policies for the smallest nonrearrangeable network in which channel continuity is binding. It is shown that the techniques of Hunt and Kurtz (1994) that provide a functional law of large numbers for Kelly type loss networks can be applied here to obtain the asymptotic dynamics of a network process. The original technique of Hunt and Kurtz, however, may not uniquely identify the limiting dynamics on the boundaries of its state space. This gap was recently addressed by a refinement due to Zachary and Ziedins (2002), and the analysis given in the present paper hinges on an adaptation of this refinement.

The main contributions of the paper are given by Proposition 2.2 and Theorems 1.1 and 1.2. Proposition 2.2 provides a streamlined approach to adapt Theorem 2.1 of Zachary and Ziedins (2002) and to identify the limiting network dynamics in the present setting. Solving for the limiting trajectories leads to Theorems 1.1 and 1.2, which in turn lead to a nontrivial description of the operating regime of the network. In particular, if the network is properly dimensioned then channel continuity asymptotically entails no compromise in blocking, even if channels are assigned randomly. Connection blocking rates are identified explicitly in terms of equilibrium distributions of certain lower-dimensional ergodic Markov processes. It appears difficult to apply the technique employed here to general topologies, although the present analysis yields some insight on the general case as mentioned later in this section.

In the rest of this section we provide a formal description of the network model and state the main results. The network considered here is comprised of two links and three connection types. W, λ_1, λ_2, λ_3 are fixed positive numbers and γ is a positive scaling factor such that each link has $\lfloor \gamma W \rfloor$ channels, and connection requests of type $r = 1, 2, 3$ arrive according to independent Poisson processes with rate $\gamma\lambda_r$. It is assumed that channels on a link are indexed from 1 to $\lfloor \gamma W \rfloor$, and that each channel is identified by its index. We shall use the terms channel and index interchangeably. A type 1 (resp. type 2) connection requires one channel from link 1 (link 2). A type 3 connection requires one channel from each link, furthermore these channels should be identical. If a connection request is admitted, then the connection remains in the network for the duration of its holding time. Holding times are exponentially distributed with unit mean and are drawn independently of the past history of the system at the time of arrival. Rearrangement is not allowed, so an admitted connection maintains its original channel assignment throughout its holding time. Requests that are not admitted are lost and have no further influence on the network state.

Two channel assignment policies are considered separately. *Matching Assignment* policy assigns a type 1 (resp. type 2) connection to an available channel that is occupied on link 2 (link 1) if such a channel exists at the time of arrival. If all available channels are idle in both links, then one such channel is assigned arbitrarily. *Random Assignment* policy assigns each connection to a channel selected randomly among all available channels. Neither policy employs admission control, hence a request is granted whenever possible.

For each $t \geq 0$ let

$$X_t(r) = \text{number of type } r,\ r = 1, 2, 3, \text{ connections in the network at time } t,$$

$$X_t(4) = \text{number of channels that are idle on both links at time } t,$$

and $X_t^\gamma = \gamma^{-1}(X_t(1), X_t(2), X_t(3), X_t(4))$. Under both assignment policies considered here, the process $X^\gamma = (X_t^\gamma : t \geq 0)$ is Markovian on the state space $S = \{x \in \mathbb{R}_+^4 : x(1) + x(3) + x(4) \leq W,\ x(2) + x(3) + x(4) \leq W\}$. Here and in the rest of the paper $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of nonnegative real numbers and the set of nonnegative integers, respectively. We shall describe the evolution of X^γ via the *free channel process* $(M_t : t \geq 0)$, where $M_t = (M_t(1), M_t(2), M_t(3))$ is defined by

$$\begin{aligned} M_t(r) &= \lfloor \gamma W \rfloor - X_t(r) - X_t(3) - X_t(4), \qquad r = 1, 2, \\ M_t(3) &= X_t(4). \end{aligned}$$

Namely, $M_t(1)$ [resp. $M_t(2)$] denotes the number of channels that are idle exclusively on link 1 (link 2) at time t. Figure 1(b) provides an illustration of these variables. A type $r = 1, 2$ request that arrives at time t is granted if $M_{t^-}(r) + M_{t^-}(3) > 0$, and its assigned channel is idle on both links with probability $p_r(M_{t^-})$, where $p_r : \mathbb{Z}_+^3 \mapsto \mathbb{R}_+$ is given by

$$p_r(m) = \begin{cases} I\{m(r) + m(3) > 0\} m(3) / \big(m(r) + m(3)\big), & \\ & \text{under Random Assignment,} \\ I\{m(r) = 0, m(3) > 0\}, & \text{under Matching Assignment.} \end{cases}$$

Here $I\{\cdot\}$ denotes the indicator function. With the complementary probability $1 - p_r(M_{t^-})$ the assigned channel is idle exclusively on link r. A type 3 request is granted if $M_{t^-}(3) > 0$, in which case it is assigned a channel that is idle on both links.

It is established in Section 2 that the sequence $(X^\gamma : \gamma > 0)$ is tight in the Skorokhod space $D_{\mathbb{R}_+^4}[0, \infty)$ of right-continuous functions with left limits. It is also shown that the limit $(x_t : t \geq 0)$ along a convergent subsequence of $(X^\gamma : \gamma > 0)$ complies with certain differential equalities. We shall focus mainly on the steady state behavior of possible limit trajectories, which leads to the long term connection rejection rates in the large γ limit, via standard methods applied

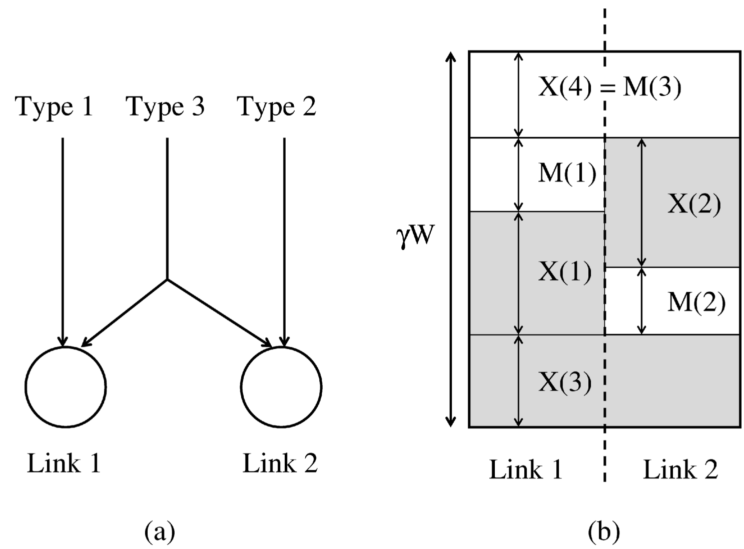

FIG. 1. (a) *The considered network*, (b) *network state illustration*.

in Bean, Gibbens and Zachary (1997) and in Alanyali and Hajek (1997). Sections 3 and 4, respectively, prove the following two theorems.

THEOREM 1.1. *Under the Random Assignment policy*, $\lim_{t\to\infty} x_t = x_*$, *where*:

(a) *if* $\lambda_1 + \lambda_3 < W$ *and* $\lambda_2 + \lambda_3 < W$, *then*

$$x_* = \big(\lambda_1, \lambda_2, \lambda_3,\ (W - \lambda_1 - \lambda_3)(W - \lambda_2 - \lambda_3)/(W - \lambda_3)\big);$$

(b) *if* $\lambda_1 + \lambda_3 > W$ *and* $\lambda_2 + \lambda_3 < W$, *then*

$$x_* = \big(\lambda_1 \pi\big(m(1) + m(3) > 0\big),\ \lambda_2,\ \lambda_3 \pi\big(m(3) > 0\big),\ 0\big),$$

where π *is the unique equilibrium distribution of the Markov process whose state space is* $\mathbb{Z}_+^2$ *and whose transition rates are given by Table* 1;

TABLE 1
Transition rates of the Markov process $(m(1), m(3))$ *that determines the limit point of the partially overloaded system under Random Assignment*

Jump of $(m(1), m(3))$	**Rate**
(1,0)	λ_2
(−1,0)	$\lambda_1 I\{m(1) + m(3) > 0\} m(1)/(m(1) + m(3))$
(0,1)	$W - \lambda_2$
(0, −1)	$\lambda_3 I\{m(3) > 0\} + \lambda_1 I\{m(1) + m(3) > 0\} m(3)/(m(1) + m(3))$

(c) *if* $\lambda_1+\lambda_3 > W$, $\lambda_2+\lambda_3 > W$ *and without loss of generality* $\lambda_1 \ge \lambda_2$, *then*

$$x_* = \begin{cases} \textit{as in part (b)}, & \textit{if } \lambda_2 < W \textit{ and } \lambda_2 < \lambda_1, \\ (\min\{W,\lambda_1\},\ \min\{W,\lambda_2\},\ \max\{W-\lambda_1,0\},\ 0), & \textit{otherwise}. \end{cases}$$

THEOREM 1.2. *Under the Matching Assignment policy*, $\lim_{t\to\infty} x_t = x^*$, *where*:

(a) *if* $\lambda_1+\lambda_3 < W$ *and* $\lambda_2+\lambda_3 < W$, *then*

$$x^* = (\lambda_1,\ \lambda_2,\ \lambda_3,\ W - \max\{\lambda_1,\lambda_2\} - \lambda_3);$$

(b) *if* $\lambda_1+\lambda_3 > W$ *and* $\lambda_2+\lambda_3 < W$, *then*

$$x^* = \big(\lambda_1\pi\big(m(1)+m(3)>0\big),\ \lambda_2,\ \lambda_3\pi\big(m(3)>0\big),\ 0\big),$$

where π *is the unique equilibrium distribution of the Markov process whose state space is* $\mathbb{Z}_+^2$ *and whose transition rates are given by Table* 2;

(c) *if* $\lambda_1+\lambda_3 > W$, $\lambda_2+\lambda_3 > W$ *and* $\lambda_1 \ge \lambda_2$, *then*

$$x^* = \begin{cases} \textit{as in part (b)}, & \textit{if } \lambda_2 < W \textit{ and } \lambda_2 < \lambda_1, \\ (\min\{W,\lambda_1\},\ \min\{W,\lambda_2\},\ \max\{W-\lambda_1,0\},\ 0), & \textit{otherwise}. \end{cases}$$

We next interpret the theorems under a natural categorization. In the following discussion, arguments are understood to hold with high probability for large γ, and $\theta(\gamma)$ refers to a quantity such that $\theta(\gamma)/\gamma$ is bounded below by a positive constant.

Underloaded network: $\lambda_1+\lambda_3 < W$, $\lambda_2+\lambda_3 < W$. "Since $x_*(4)$ and $x^*(4)$ are positive in this case, both policies maintain $\theta(\gamma)$ channels that are idle in both links, and typically all requests are granted." The operating point x^* indicates that Matching Assignment results in packing the utilized channels almost perfectly; in particular, Matching Assignment achieves the largest possible value for the normalized number of channels that are idle in both links. The number of such channels is inevitably smaller under Random Assignment, due to the liberal assignment rule adopted by this policy. However, since under any policy both

TABLE 2
Transition rates of the Markov process $(m(1), m(3))$ *that determines the limit point of the partially overloaded system under Matching Assignment*

Jump of $(m(1), m(3))$	**Rate**
(1,0)	λ_2
(−1,0)	$\lambda_1 I\{m(1)>0\}$
(0,1)	$W-\lambda_2$
(0, −1)	$\lambda_3 I\{m(3)>0\} + \lambda_1 I\{m(1)=0, m(3)>0\}$

$M_t(1)$ and $M_t(2)$ are $\theta(\gamma)$ in equilibrium, the dynamic behavior of Random Assignment becomes increasingly similar to that of Matching Assignment as the channels that are idle in both links are depleted. In turn the two policies have the same underload condition.

It is remarkable that the underload condition given here applies in the absence of the channel continuity constraint as well. Hence, as far as call blocking is concerned, channel continuity is asymptotically not binding for the considered topology. Numerical experimentation, which is not reported in the present paper, suggests that this property extends to a fairly large class of topologies in which the channel continuity constraint is not binding for a static channel assignment problem with mean load. One may thus conjecture that random channel assignment asymptotically entails no blocking if the mean load can be accommodated in the face of the channel continuity constraint. It is worthwhile to investigate this direction further since such topologies are desirable in applications for their capability of achieving high channel utilization.

Partially overloaded network: $\lambda_1 + \lambda_3 > W$, $\lambda_2 + \lambda_3 < W$. There is virtually no contention in the underloaded link 2, thus typically all type 2 requests are granted. In contrast type 1 and type 3 requests compete for the available capacity on link 1, and both types experience blocking. Exact blocking rates can be computed by solving the processes of Tables 1 and 2 in equilibrium.

We briefly comment on Matching Assignment. On the one hand, type 3 requests have a disadvantage against type 1 requests since they may not be assigned the channels that are occupied by type 2 connections on link 2. On the other hand, Matching Assignment assigns these channels with priority to incoming type 1 arrivals. Approximating the process of Table 2 by replacing $I\{m(1) = 0\}$ with its mean $\pi(m(1) = 0) = 1 - (\lambda_2/\lambda_1)$ leads to the approximate limit point

$$\left(\lambda_2 + (W - \lambda_2)\frac{\lambda_1 - \lambda_2}{\lambda_3 + \lambda_1 - \lambda_2},\ \lambda_2,\ (W - \lambda_2)\frac{\lambda_3}{\lambda_3 + \lambda_1 - \lambda_2},\ 0\right).$$

This point corresponds to an operating regime in which a fraction λ_2/λ_1 of all type 1 requests are assigned channels which are assigned to type 2 connections on link 2, the remaining type 1 requests compete with type 3 requests for the remaining fraction $(W - \lambda_2)/W$ of the channels on link 1, and the fraction of claimed channels is proportional to the associated arrival rate. Although this description offers an appealing intuition, it is not exact and it slightly overestimates the fraction of channels assigned to type 1 connections.

Totally overloaded case: $\lambda_1 + \lambda_3 > W$, $\lambda_2 + \lambda_3 > W$. In this case there is contention for capacity on both links, however, if $\lambda_2 < \lambda_1$, then type 3 requests face harsher competition on link 1. The vast majority of rejected type 3 requests are rejected due to congestion on link 1; in turn type 2 requests are not challenged by type 3 requests. In particular, if $\lambda_2 < W$, then type 2 requests enjoy the perception of a virtually underloaded link. If both λ_1 and λ_2 exceed W, then type 1 and type 2 requests suffice to overload the two links, and almost all type 3 requests are rejected in this case. This is an extreme manifestation of the well-known unfairness of WDM networks for long connections.

2. Weak convergence. This section establishes the tightness of the sequence $(X^\gamma : \gamma > 0)$ and identifies the dynamics of the limiting trajectories. The discussion is built upon the work of Hunt and Kurtz (1994), hence the reader is urged to read that paper in order to better understand the technique used here.

The generator of X^γ leads via Proposition 4.1.7 of Ethier and Kurtz (1986) to the representation

$$\begin{aligned} X_t^\gamma(r) = X_0^\gamma(r) &+ \int_0^t g_r(X_s^\gamma)\,ds \\ &+ \int_0^t h_r(M_s)\,ds + \eta_t^\gamma(r), \qquad r = 1, 2, 3, 4, \end{aligned} \tag{2.1}$$

where $g_r : S \mapsto \mathbb{R}$ and $h_r : \mathbb{Z}_+^3 \mapsto \mathbb{R}$ are defined by

$$\begin{aligned} g_r(x) &= -x(r) \qquad \text{for } r = 1, 2, 3, \\ g_4(x) &= 2W - x(1) - x(2) - x(3) - 2x(4), \\ h_r(m) &= \lambda_r I\{m(r) + m(3) > 0\} \qquad \text{for } r = 1, 2, 3, \\ h_4(m) &= -\lambda_1 p_1(m) - \lambda_2 p_2(m) - \lambda_3 I\{m(3) > 0\}, \end{aligned}$$

and $\eta^\gamma(r) = (\eta_t^\gamma(r) : t \geq 0)$ is a martingale such that $\eta_0^\gamma(r) = 0$. The sequence $(\eta^\gamma(r) : \gamma > 0)$ converges weakly to zero via Doob's L^2 inequality, and the boundedness of g_r and h_r imply tightness of $(\int_0^\cdot (g_r(X_s^\gamma) + h_r(M_s))\,ds : \gamma > 0)$ in $D_{\mathbb{R}_+}[0, \infty)$ via Corollary 3.7.4 of Ethier and Kurtz (1986). The sequence $(X_0^\gamma : \gamma > 0)$ is also tight due to the compactness of S, therefore $(X^\gamma : \gamma > 0)$ is tight in $D_{\mathbb{R}_+^4}[0, \infty)$.

For notational convenience let $(X^\gamma : \gamma > 0)$ represent a convergent subsequence, and let $(x_t : t \geq 0)$ denote the corresponding limiting trajectory. Appeal to Skorokhod's theorem to construct the processes $(X^\gamma : \gamma > 0)$ and $(x_t : t \geq 0)$ on some probability space $(\Omega, \mathcal{F}, P)$ so that the convergence is almost sure. Let $\mathbb{Z}_+^\Delta = \mathbb{Z}_+ \cup \{\infty\}$ be endowed with the conventional one-point compactification of the discrete topology, and define $E = (\mathbb{Z}_+^\Delta)^3$ with the product topology. Extend the definitions of p_1, p_2 and h_1, h_2, h_3, h_4 so that each mapping is defined on the set E, in compliance with the convention $\infty + k = \infty$ and $k/\infty = 0$ for each integer k. Let $\mathcal{B}(E)$ denote the Borel sets of E, and let $\mathcal{L}_0(E)$ denote the space of Borel measures μ on the product space $[0, \infty) \times E$ such that $\mu([0, t) \times E) = t$ for $t \geq 0$. Endow $\mathcal{L}_0(E)$ with the topology of weak convergence of measures restricted to $[0, t) \times E$ for each $t \geq 0$.

TABLE 3
Transition rates of the process $(m^x(1), m^x(2), m^x(3))$ *for* $x \in S$ *in the discussion of Random Assignment. It is understood that* $\infty + k = \infty$ *and* $k/\infty = 0$ *for each integer* k

Jump	Rate
$(1, 0, 0)$	$x(1) + x(2) + x(3) + x(4) - W$
$(0, 1, 0)$	$x(1) + x(2) + x(3) + x(4) - W$
$(0, 0, 1)$	$2W - x(1) - x(2) - x(3) - 2x(4)$
$(-1, 0, 0)$	$\lambda_1 I\{m^x(1) + m^x(3) > 0\} m^x(1)/(m^x(1) + m^x(3))$
$(0, 1, -1)$	$\lambda_1 I\{m^x(1) + m^x(3) > 0\} m^x(3)/(m^x(1) + m^x(3))$
$(0, -1, 0)$	$\lambda_2 I\{m^x(2) + m^x(3) > 0\} m^x(2)/(m^x(2) + m^x(3))$
$(1, 0, -1)$	$\lambda_2 I\{m^x(2) + m^x(3) > 0\} m^x(3)/(m^x(2) + m^x(3))$
$(0, 0, -1)$	$\lambda_3 I\{m^x(3) > 0\}$

Let the random measure $\nu^\gamma \in \mathcal{L}_0(E)$ be defined by

$$\nu^\gamma\big([0, t) \times B\big) = \int_0^t I\{M_s \in B\}\, ds, \qquad t \geq 0,\ B \in \mathcal{B}(E).$$

The compactness of E implies compactness of $\mathcal{L}_0(E)$ via Prohorov's theorem, in turn $(\nu^\gamma : \gamma > 0)$ is tight in $\mathcal{L}_0(E)$. Theorem 3 of Hunt and Kurtz (1994) can be adapted to establish that the weak limit ν along a convergent subsequence of $(\nu^\gamma : \gamma > 0)$ has the form

$$\nu\big([0, t) \times B\big) = \int_0^t \pi_{x_s}(B)\, ds, \qquad t \geq 0,\ B \in \mathcal{B}(E),$$

where π_{x_s} is an equilibrium distribution of a Markov process m^{x_s} that takes values in E and is identified by the transition rates given in Table 3 for Random Assignment and in Table 4 for Matching Assignment. Informally speaking, the process m^{x_s} mimics the (unnormalized) free channel process in slow motion, in a short period of time during which the normalized process X^γ remains close to x_s.

TABLE 4
Transition rates of the process $(m^x(1), m^x(2), m^x(3))$ *for* $x \in S$ *in the discussion of Matching Assignment*

Jump	Rate
$(1, 0, 0)$	$x(1) + x(2) + x(3) + x(4) - W$
$(0, 1, 0)$	$x(1) + x(2) + x(3) + x(4) - W$
$(0, 0, 1)$	$2W - x(1) - x(2) - x(3) - 2x(4)$
$(-1, 0, 0)$	$\lambda_1 I\{m^x(1) > 0\}$
$(0, 1, -1)$	$\lambda_1 I\{m^x(1) = 0, m^x(3) > 0\}$
$(0, -1, 0)$	$\lambda_2 I\{m^x(2) > 0\}$
$(1, 0, -1)$	$\lambda_2 I\{m^x(2) = 0, m^x(3) > 0\}$
$(0, 0, -1)$	$\lambda_3 I\{m^x(3) > 0\}$

The limit trajectory $(x_t : t \geq 0)$ is characterized in terms of the distributions $(\pi_x, x \in S)$ as stated by the following proposition.

PROPOSITION 2.1. *The process* $(x_t : t \geq 0)$ *satisfies*

$$\dot{x}_t(r) = g_r(x_t) + \pi_{x_t}(h_r), \qquad r = 1, 2, 3, 4,$$

for almost all t, *where* $\pi_{x_t}(h_r)$ *denotes the expectation of* h_r *with respect to* π_{x_t}.

PROOF. The claim is proved by identifying the limiting forms of the second and third terms in the representation (2.1). By Theorem 3.10.2 and Lemma 3.10.1 of Ethier and Kurtz (1986) $(x_t : t \geq 0)$ is continuous and $(X^\gamma : \gamma > 0)$ converges uniformly on compact time sets; in turn the uniform continuity of $g_r : S \mapsto \mathbb{R}$ implies that $\int_0^t g_r(X_s^\gamma)\, ds \to \int_0^t g_r(x_s)\, ds$ as $\gamma \to \infty$. A standard integration argument is employed next to establish the limit of the third term in (2.1). Fix $K > 0$, and set $\bar{h}_r = \sup\{|h_r(m)| : m \in E\} < \infty$. For each integer k let $F_{r,K,k} = \{m \in E : (k/K) \leq h_r(m) < (k+1)/K\}$, and let $h_r^K : E \mapsto \mathbb{R} : h_r^K(m) = \sum_{k=-\lceil \bar{h}_r K \rceil}^{\lceil \bar{h}_r K \rceil} (k/K) I\{m \in F_{r,K,k}\}$. The arbitrariness of K leads to

$$\begin{aligned}
&\lim_{\gamma \to \infty} \left| \int_0^t h_r(M_s)\, ds - \int_0^t \pi_{x_s}(h_r)\, ds \right| \\
&\quad \leq \lim_{K \to \infty} \left| \int_0^t h_r(M_s)\, ds - \int_0^t h_r^K(M_s)\, ds \right| \\
&\qquad + \lim_{\gamma \to \infty} \left| \int_0^t h_r^K(M_s)\, ds - \int_0^t \sum_{k=-\lceil \bar{h}_r K \rceil}^{\lceil \bar{h}_r K \rceil} (k/K) \pi_{x_s}(F_{r,K,k})\, ds \right| \\
&\qquad + \lim_{K \to \infty} \left| \int_0^t \sum_{k=-\lceil \bar{h}_r K \rceil}^{\lceil \bar{h}_r K \rceil} (k/K) \pi_{x_s}(F_{r,K,k})\, ds - \int_0^t \pi_{x_s}(h_r)\, ds \right|.
\end{aligned}$$

Since $|\int_0^t h_r(M_s)\, ds - \int_0^t h_r^K(M_s)\, ds| \leq t/K$ the first term on the right-hand side vanishes. The function $I\{m \in F_{r,K,k}\} : E \mapsto R$ is continuous via the definition of h_r under either policy; in turn the second term also vanishes by the continuous mapping theorem. Finally, the last term is zero due to the monotone convergence theorem, and so the proposition follows. $\square$

Proposition 2.1 does not provide a complete description of the dynamics of $(x_t : t \geq 0)$, since for each t the process m^{x_t} is reducible due to the isolated states at infinity and so it admits multiple equilibrium distributions. A partial description of the right distribution π_{x_t} can be obtained via the principles laid out in the original work of Hunt and Kurtz (1994). In the present case these principles lead to the following three observations:

$$\text{if } x_t(1) + x_t(3) + x_t(4) < W \qquad \text{then } \pi_{x_t}\big(m(1) = \infty\big) = 1, \tag{2.2}$$

$$\text{if } x_t(2) + x_t(3) + x_t(4) < W \qquad \text{then } \pi_{x_t}\big(m(2) = \infty\big) = 1, \tag{2.3}$$

$$\text{if } x_t(4) > 0 \qquad \text{then } \pi_{x_t}\big(m(3) = \infty\big) = 1, \tag{2.4}$$

which can be proved as done for a similar setting in Lemma 2.1 of Alanyali (1999). In particular, (2.2)–(2.4) identify π_{x_t} completely when x_t is in the interior of its state space S. Note that these observations are consistent with the informal interpretation of the process m^{x_t}, in that positive normalized values of the free channel process indicate large unnormalized values.

Identifying the distribution that describes the limiting dynamics on the boundaries of the state space has been a common challenge in applications of the Hunt–Kurtz technique. Related previous work by Hunt and Kurtz (1994), Hunt (1995), Hunt and Laws (1997), Bean, Gibbens and Zachary (1995, 1997) and Alanyali (1999) characterize this distribution in sufficient detail by exploiting the fact that the limit trajectory is confined to the relevant state space. Nevertheless this approach may fail in the general case, as argued by Zachary and Ziedins (2002). In the present case also, a close examination of the discussion given in Sections 3 and 4 yields that this approach may be adopted to prove Theorem 1.2, however one needs a sharper characterization of π_{x_t} in order to prove Theorem 1.1. The following proposition, which is based on the theme of Theorem 2.1 of Zachary and Ziedins (2002), extracts a further property of π_{x_t} that leads to the proofs presented next. We start with some definitions.

Let J denote the common set of jumps listed in Tables 3 and 4. Given $j \in J$, $x \in S$ and $m \in E$, let $\mu_j^x(m)$ denote the instantaneous rate of jump j of the process m^x at state m, as given by Tables 3 and 4. Given $f : E \times S \mapsto \mathbb{R}$ and $y \in S$, define the function $D_{x,y} f : E \mapsto \mathbb{R}$ by

$$D_{x,y} f(m) = \sum_{j \in J} \big(f(m + j, x) - f(m, x)\big)\mu_j^y(m).$$

In particular, $D_{x,y} f(m)$ is the mean drift of the process $f(m^y, x)$ given m^y is at state m. Designate the following boundaries of S:

$$B_1 = \{x \in S : x(1) + x(3) + x(4) = W\},$$

$$B_2 = \{x \in S : x(2) + x(3) + x(4) = W\},$$

$$B_3 = \{x \in S : x(4) = 0\},$$

and for $L \subseteq \{1,2,3\}$, let $L^c = \{1,2,3\} - L$ and

$$S_L = \{x \in S : x \in B_r \text{ for } r \in L,\ x \notin B_r \text{ for } r \in L^c\},$$

$$E_L = \{m \in E : m(r) \in \mathbb{Z}_+ \text{ for } r \in L,\ m(r) = \infty \text{ for } r \in L^c\},$$

$$\hat{E}_L = \{m \in E : m(r) \in \mathbb{Z}_+ \text{ for } r \in L\}.$$

PROPOSITION 2.2. *Given $L \subseteq \{1,2,3\}$ and $S_o \subseteq S_L$ if there exists $f : \hat{E}_L \times S_o \mapsto \mathbb{R}$ such that for each $x \in S_o$:*

(i) *$f(m,x)$ does not depend on $m(r) : r \in L^c$,*
(ii) *$\sup\{|f(m+j,x) - f(m,x)| : m \in \hat{E}_L,\ j \in J\}$ is finite,*
(iii) *there exist real numbers $\varepsilon_x > 0$, $\delta_x > 0$, A_x such that $D_{x,y} f(m) < -\varepsilon_x$ whenever $m \in \hat{E}_L$, $f(m,x) > A_x$ and $y \in S : |x-y| < \delta_x$,*
(iv) *for each real number a, the set $\{m \in E_L : f(m,x) \le a\}$ is finite,*

then $\pi_{x_t}(E_L) = 1$ for almost all $t \ge 0$ such that $x_t \in S_o$.

PROOF. Fix $x \in S_o$, and let $[t_1, t_2]$ be an interval such that $x_t \in S_o$ and $|x - x_t| < \delta_x/2$ for $t \in [t_1, t_2]$. Denote by τ_n the time of the nth jump of $(M_t : t \ge t_1)$. For $y \in S$ define the function $\hat{D}_{x,y} f : \hat{E}_L \mapsto \mathbb{R}$ by

$$\hat{D}_{x,y} f(m) = \sum_{j \in J} \big(f(m+j,x) - f(m,x)\big) P\Big(M_{\tau_{n+1}^-} - M_{\tau_n^-} = j \Big| X^\gamma_{\tau_n^-} = y\Big),$$

provided that $\{X^\gamma_{\tau_n^-} = y\}$ has positive probability. It can be verified via Tables 3 and 4 that

$$\hat{D}_{x,X^\gamma_{\tau_n^-}} f\big(M_{\tau_n^-}\big) = D_{x,X^\gamma_{\tau_n^-}} f\big(M_{\tau_n^-}\big) \Big/ \sum_{j \in J} \mu_j^{X^\gamma_{\tau_n^-}}\big(M_{\tau_n^-}\big). \tag{2.5}$$

Fix arbitrary $\sigma_1, \sigma_2, \sigma_3 > 0$ and let $\beta = \inf\{W - x_t(r) - x_t(3) - x_t(4) : t \in [t_1,t_2], r \in L^c\} > 0$. Since $(X^\gamma : \gamma > 0)$ converges uniformly on compact time sets, there exists a real number γ_o such that for each $\gamma > \gamma_o$ the following conditions hold on a set $\Omega_\gamma \subseteq \Omega$ of probability at least $1 - \sigma_1$:

$$\sup\big\{|X^\gamma_t - x_t| : t \in [t_1,t_2]\big\} < \delta_x/2, \tag{2.6}$$

$$\sup\{M_t(r) : t \in [t_1,t_2], r \in L\} < \sigma_2 \gamma, \tag{2.7}$$

$$\inf\{M_t(r) : t \in [t_1,t_2], r \in L^c\} > \beta\gamma/2. \tag{2.8}$$

Appeal to hypotheses (i) and (ii) to choose σ_2 smaller if necessary so that

$$f(M_{t_1}, x) < \sigma_3 \gamma$$

on Ω_γ. Inequality (2.6) implies via hypothesis (iii) that

$$P\big(D_{x,X^\gamma_t} f(m) < -\varepsilon_x \text{ for all } m \in \hat{E}_L$$
$$\text{such that } f(m,x) > A_x, \text{ and all } t \in [t_1,t_2] \big| \Omega_\gamma\big) = 1.$$

In turn, inequalities (2.7) and (2.8) imply via equality (2.5) that for sufficiently small σ_2,

$$P\Big(\hat{D}_{x,X^{\gamma}_{\tau_n^-}} f(M_{\tau_n^-}) < -\varepsilon_x/2\mu^* \text{ for all } n \text{ such that } f(M_{\tau_n^-},x) > A_x,\ \tau_n \in [t_1,t_2] \Big| \Omega_\gamma\Big) = 1, \tag{2.9}$$

where $\mu^* = \sup\{\sum_{j\in J} \mu_j^x(m) : x \in S,\ m \in E\} < \infty$.

Let $(\mathcal{F}_n : n \geq 1)$ be the filtration generated by $(X^{\gamma}_{\tau_n^-} : n \geq 1)$, and denote by $N(\gamma)$ the number of jumps of $(M_t : t \in [t_1,t_2])$. Refer to inequality (2.9) and hypothesis (ii) to verify that for $\gamma > \gamma_o$ the sequence $((f(M_{\tau_n^-},x),\mathcal{F}_n) : 1 \leq n \leq N(\gamma))$ satisfies conditions D1 and D2 of Hajek (1982) on the set Ω_γ. Given $a > A_x$, define $n_a(\gamma) = \min\{n \geq 1 : f(M_{\tau_n^-},x) \leq a\}$. Given $\alpha > 0$, one can choose σ_3 smaller if necessary so that $n_a(\gamma) < \alpha\gamma$ on Ω_γ via Theorem 2.3 of Hajek (1982). Theorem 3.1 of Hajek (1982) now implies that for each $\varepsilon > 0$ there exist positive constants $\rho < 1$, K, L, ζ, none of which depends on γ, such that for each positive integer N,

$$P\Bigg(\frac{1}{N(\gamma)} \sum_{n=n_a(\gamma)}^{N(\gamma)} I\{f(M_{\tau_n^-},x) > a\} \geq 1 - (1 - Le^{-\zeta a})(1-\varepsilon) \Bigg| \Omega_\gamma \cap \{N(\gamma) - n_a(\gamma) = N\}\Bigg) \leq K\rho^N.$$

Choose α so that $2\alpha < (t_2 - t_1)\min\{W,\lambda_1\}$ and in turn $\liminf_{\gamma\to\infty} N(\gamma)/\gamma > 2\alpha > 0$ almost surely (a.s.). One can then choose γ_o larger if necessary so that $N(\gamma)/\gamma > 2\alpha$ on Ω_γ and so

$$P\Bigg(\frac{1}{N(\gamma)} \sum_{n=n_a(\gamma)}^{N(\gamma)} I\{f(M_{\tau_n^-},x) > a\} \geq 1 - (1 - Le^{-\zeta a})(1-\varepsilon) \Bigg| \Omega_\gamma\Bigg) \leq K\rho^{\alpha\gamma}$$

for all $\gamma > \gamma_o$. By the Borel–Cantelli lemma,

$$\limsup_{\gamma\to\infty} \frac{1}{N(\gamma)} \sum_{n=n_a(\gamma)}^{N(\gamma)} I\{f(M_{\tau_n^-},x) > a\} < 1 - (1 - Le^{-\zeta a})(1-\varepsilon)$$

a.s.; in turn the arbitrariness of α, ε leads to

$$\limsup_{\gamma\to\infty} \frac{1}{N(\gamma)} \sum_{n=1}^{N(\gamma)} I\{f(M_{\tau_n^-},x) > a\} \leq Le^{-\zeta a} \qquad \text{a.s.} \tag{2.10}$$

Let $(U_n : n \geq 1)$ be an i.i.d. sequence of exponentially distributed random variables, drawn independently of $(f(M_{\tau_n^-},x) : n \geq 1)$, so that U_1 has mean $1/(\gamma\min\{W,\lambda_1\})$. The holding time of the nth state visited by $(f(M_t,x) : t > t_1)$

is stochastically dominated by U_n; in turn, $(U_n : n \geq 1)$ can be generated on the probability space $(\Omega, \mathcal{F}, P)$ so that

$$\begin{aligned}
&\lim_{a\to\infty} \limsup_{\gamma\to\infty} \int_{t_1}^{t_2} I\{f(M_t, x) > a\}\,dt \\
&\qquad \leq \lim_{a\to\infty} \limsup_{\gamma\to\infty} \sum_{n=1}^{N(\gamma)} I\{f\bigl(M_{\tau_n^-}, x\bigr) > a\}U_n \\
&\qquad = \lim_{a\to\infty} \limsup_{\gamma\to\infty} \frac{1}{N(\gamma)} \sum_{n=1}^{N(\gamma)} I\{f\bigl(M_{\tau_n^-}, x\bigr) > a\}N(\gamma)U_n \\
&\qquad = 0 \qquad \text{a.s.},
\end{aligned} \tag{2.11}$$

where (2.11) follows via (2.10) since $\sup_\gamma E[(N(\gamma)U_1)^k] < \infty$ for $k = 1, 2$. [See also the proof of Theorem 2.1 of Zachary and Ziedins (2002) for a discussion that leads to a conclusion analogous to (2.11).]

Given $T \geq 0$, let $T(x)$ denote the interior of the set $\{t \in [0, T] : x_t \in S_o,\ |x - x_t| < \delta_x/2\}$. By Proposition 2.8 of Royden $T(x)$ is the union of a countable set of disjoint open intervals. Let the kth interval be denoted by $(t_1(k), t_2(k))$ so that

$$\begin{aligned}
\int_{T(x)} \pi_{x_t}(\hat{E}_L)\,dt &= \sum_k \int_{t_1(k)}^{t_2(k)} \pi_{x_t}(\hat{E}_L)\,dt \\
&\geq \sum_k \int_{t_1(k)}^{t_2(k)} \lim_{a\to\infty} \pi_{x_t}\bigl(m \in \hat{E}_L : f(m, x) \leq a\bigr)\,dt \\
&= \sum_k \lim_{a\to\infty} \int_{t_1(k)}^{t_2(k)} \pi_{x_t}\bigl(m \in \hat{E}_L : f(m, x) \leq a\bigr)\,dt && (2.12)\\
&\geq \sum_k \lim_{a\to\infty} \limsup_{\gamma\to\infty} \int_{t_1(k)}^{t_2(k)} I\{f(M_t, x) \leq a\}\,dt && (2.13)\\
&= \sum_k \bigl(t_2(k) - t_1(k)\bigr), && (2.14)
\end{aligned}$$

where equality (2.12) is due to the Monotone convergence theorem, inequality (2.13) is due to Theorem 3.3.1.d of Ethier and Kurtz (1986) and hypothesis (iv), which implies that the set $\{m \in \hat{E}_L : f(m, x) \leq a\}$ is closed, and the last equality follows via (2.11).

Appeal to the compactness of the closure of S_o to identify a finite set $\{x_1, x_2, \ldots, x_K\} \subset S_o$ such that $S_o \subset \bigcup_{k=1}^K \{x' \in S_o : |x' - x_k| < \delta_{x_k}/2\}$. Equality (2.14) implies that

$$\int_{\{t\in[0,T] : x_t \in S_o\}} \bigl(1 - \pi_{x_t}(\hat{E}_L)\bigr)\,dt \leq \sum_{k=1}^K \int_{T(x_k)} \bigl(1 - \pi_{x_t}(\hat{E}_L)\bigr)\,dt = 0.$$

Therefore, $\pi_{x_t}(\hat{E}_L) = 1$ for almost all $t \in [0, T]$ such that $x_t \in S_o$. The proposition now follows from observations (2.2)–(2.4) and the arbitrariness of T. $\square$

REMARK 2.1. $f(\cdot, x)$ is essentially a Lyapounov function that applies to establish ergodicity of m^y on E_L uniformly for all y in some neighborhood of x. It seems plausible to conjecture that for $x \in S_L$ ergodicity of m^x on E_L is sufficient for $\pi_x(E_L) = 1$. Although the proposition stops short of this assertion, its applications later in the paper, namely the proofs of Lemmas 3.3 and 4.1, do not provide any counter examples.

We next establish Theorems 1.1 and 1.2.

3. Random Assignment.

Under Random Assignment the limiting trajectory $(x_t : t \geq 0)$ complies with

$$\dot{x}_t(1) = \lambda_1 \pi_{x_t}\big(m(1) + m(3) > 0\big) - x_t(1), \tag{3.1}$$

$$\dot{x}_t(2) = \lambda_2 \pi_{x_t}\big(m(2) + m(3) > 0\big) - x_t(2), \tag{3.2}$$

$$\dot{x}_t(3) = \lambda_3 \pi_{x_t}\big(m(3) > 0\big) - x_t(3), \tag{3.3}$$

$$\begin{aligned} \dot{x}_t(4) = {} & 2W - x_t(1) - x_t(2) - x_t(3) - 2x_t(4) \\ & - \lambda_3 \pi_{x_t}\big(m(3) > 0\big) - \lambda_1 \pi_{x_t}(p_1) - \lambda_2 \pi_{x_t}(p_2), \end{aligned} \tag{3.4}$$

where π_{x_t} is an equilibrium distribution for the Markov process m^{x_t} that takes values in E and has transitions as given in Table 3. The following lemma provides a further property of π_{x_t} in the spirit of (2.2)–(2.4).

LEMMA 3.1. *Under Random Assignment, for $r = 1, 2$,*

$$\pi_{x_t}(p_r) = \frac{x_t(4)}{W - x_t(r) - x_t(3)} \qquad \textit{whenever } x_t(r) + x_t(3) < W.$$

PROOF. Given $T, \varepsilon > 0$ set $S^\varepsilon = \{x \in S : x(r) + x(3) \leq W - \varepsilon\}$ and $T^\varepsilon = \{t \in [0, T] : x_t \in S^\varepsilon\}$. Since $\hat{p}_r : S^\varepsilon \mapsto \mathbb{R}_+ : \hat{p}_r(x) = x(4)/(W - x(r) - x(3))$ is uniformly continuous, it follows by Lemma 3.10.1 of Ethier and Kurtz (1986) that $\lim_{\gamma \to \infty} \hat{p}_r(X_t^\gamma) = \hat{p}_r(x_t)$ uniformly for all $t \in T^\varepsilon$. In turn $\lim_{\gamma \to \infty} \int_{T^\varepsilon} p_r(M_t)\, dt = \lim_{\gamma \to \infty} \int_{T^\varepsilon} \hat{p}_r(X_t^\gamma)\, dt = \int_{T^\varepsilon} \hat{p}_r(x_t)\, dt$. The lemma follows by the arbitrariness of T and ε. $\square$

3.1. *Underloaded network* ($\lambda_1 + \lambda_3 < W$, $\lambda_2 + \lambda_3 < W$).

PROOF OF THEOREM 1.1(a). Assume without loss of generality that $\lambda_1 \geq \lambda_2$. Equalities (3.1)–(3.3) imply that $\limsup_{t \to \infty} x_t(r) \leq \lambda_r$ for $r = 1, 2, 3$, therefore one can choose τ such that $x_t(1) < (W + \lambda_1 - \lambda_3)/2$, $x_t(2) < (W + \lambda_2 - \lambda_3)/2$

and $x_t(3) < (W - \lambda_1 + \lambda_3)/2$ for $t > \tau$. If $t > \tau$ is such that $x_t(4) = 0$, then $x_t(1) + x_t(3) < W$ and $x_t(2) + x_t(3) < W$; in turn, by equality (3.4) and conditions (2.2)–(2.3),

$$\dot{x}_t(4) = \bigl(W - x_t(1) - x_t(3)\bigr) + \bigl(W - x_t(2) - \lambda_3 \pi_{x_t}\bigl(m(3) > 0\bigr)\bigr) > 0.$$

Thus $x_t(4) > 0$ for almost all $t > \tau$. Equalities (3.1)–(3.4) together with condition (2.4) and Lemma 3.1 imply that for $t > \tau$,

$$\begin{aligned}
\dot{x}_t(r) &= \lambda_r - x_t(r), \qquad r = 1, 2, 3,\\
\dot{x}_t(4) &= 2W - x_t(1) - x_t(2) - x_t(3) - 2x_t(4)\\
&\qquad - \lambda_3 - \lambda_1 \frac{x_t(4)}{W - x_t(1) - x_t(3)} - \lambda_2 \frac{x_t(4)}{W - x_t(2) - x_t(3)}.
\end{aligned}$$

The claimed limit can now be verified. □

3.2. *Partially overloaded network* ($\lambda_1 + \lambda_3 > W$, $\lambda_2 + \lambda_3 < W$).

LEMMA 3.2. *$x_t(1) > x_t(2)$ for large enough t.*

PROOF. Appeal to equalities (3.2) and (3.3) to choose $\tau > 0$ such that $x_t(2) + x_t(3) < W$ for all $t > \tau$. If $t > \tau$ is such that $x_t(1) - x_t(2) \le 0$, then $x_t(1) + x_t(3) < W$ also and conditions (2.2)–(2.4) imply that $\dot{x}_t(1) - \dot{x}_t(2) = \lambda_1 - \lambda_2 - (x_t(1) - x_t(2)) \ge \lambda_1 - \lambda_2 > 0$. In turn, $x_t(1) > x_t(2)$ for $t > \tau + W/(\lambda_1 - \lambda_2)$, and the lemma follows. □

LEMMA 3.3. *If $\lambda_1 + \lambda_3 > W$, then $\pi_{x_t}(m(1) \in \mathbb{Z}_+, m(2) = \infty, m(3) \in \mathbb{Z}_+) = 1$ for almost all t such that $x_t(1) + x_t(3) = W$, $x_t(2) + x_t(3) < W$, $x_t(4) = 0$ and $0 < x_t(2) < \lambda_1$.*

PROOF. The lemma is proved via Proposition 2.2. Let $S_o = \{x \in S : x(1) + x(3) = W,\ x(2) + x(3) < W,\ x(4) = 0,\ 0 < x(2) < \lambda_1\}$. Given $x \in S_o$, let $\alpha_x = (\lambda_1 - x(2))/x(2) > 0$, and define $f : \hat{E}_{\{1,3\}} \times S_o \mapsto \mathbb{R}_+ : f(m, x) = ((m(3) - \alpha_x m(1))^2 + \alpha_x (m(1) + m(3))^2)^{1/2}$. The function $f(\cdot, x) : \hat{E}_{\{1,3\}} \mapsto \mathbb{R}_+$ satisfies conditions (i) and (iv) of Proposition 2.2. The proof of Lemma 3.3.3 of Fayolle, Malyshev and Menshikov (1995) applies to the present setting to establish that for each $j = (j(1), j(2), j(3)) \in J$,

$$\begin{aligned}
&f(m + j, x) - f(m, x)\\
&\quad = \frac{(m(3) - \alpha_x m(1))(j(3) - \alpha_x j(1)) + \alpha_x (m(1) + m(3))(j(1) + j(3))}{f(m, x)}\\
&\qquad + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $m(1)^2 + m(3)^2 \to \infty$. In particular, f satisfies condition (ii) of Proposition 2.2. One can verify via Table 3 that under the hypothesis $\lambda_1 + \lambda_3 > W$,

$$\begin{aligned} D_{x,x} f(m) = {} & \frac{1}{f(m,x)} \\ & \times \Big(m(3)(1+\alpha_x)\big(W - \lambda_1 I\{m(1)+m(3) > 0\} - \lambda_3 I\{m(3) > 0\}\big) \\ & \qquad - \lambda_1 I\{m(1)+m(3) > 0\}\big(m(3) - \alpha_x m(1)\big)^2 / \big(m(1)+m(3)\big)\Big) \\ & + o(1), \end{aligned}$$

so that there exist positive numbers ε', A such that $D_{x,x} f(m) < -\varepsilon'$ whenever $m \in E_{\{1,3\}}$ and $f(m,x) > A$. Since $(\mu_j^x(m) : m \in E, j \in J)$ is uniformly continuous in x, there exist positive numbers δ and $\varepsilon \le \varepsilon'$ such that for all $y \in S$ with $|x - y| < \delta$, $D_{x,y} f(m) < -\varepsilon$ whenever $m \in E_{\{1,3\}}$ and $f(m,x) > A$. Hence condition (iii) of Proposition 2.2 is also satisfied, and the lemma follows. $\square$

LEMMA 3.4. *For any number b and absolutely continuous* $(g_t : t \ge 0)$, $\dot{g}_t = 0$ *for almost all* $t \ge 0$ *such that* $g_t = b$.

PROOF. See, for example, Lemma 8.6 of Alanyali and Hajek (1997). $\square$

PROOF OF THEOREM 1.1(b). Let $B = \liminf_{t\to\infty}(x_t(1) - x_t(2))$. Since $B > 0$ due to Lemma 3.2, there exists a finite τ such that $x_t(1) < \lambda_1 + B/4$ and $x_t(2) < x_t(1) - B/2$ for $t > \tau$. For almost all $t > \tau$ such that $x_t(1) + x_t(3) < W$, either $x_t(4) > 0$ or $x_t(4) = 0$ and $\dot{x}_t(4) = 0$ via Lemma 3.4. In the first case, $\dot{x}_t(1) + \dot{x}_t(3) = \lambda_1 + \lambda_3 - x_t(1) - x_t(3) > \lambda_1 + \lambda_3 - W > 0$ via equalities (3.1) and (3.3) and condition (2.4). In the second case, equality (3.4) and Lemma 3.1 imply that $\dot{x}_t(4) = 2W - x_t(1) - x_t(2) - x_t(3) - \lambda_3 \pi_{x_t}(m(3) > 0)$. The choice of τ yields $x_t(2) + x_t(3) < W - B/2$, therefore it follows that $x_t(3) - \lambda_3 \pi_{x_t}(m(3) > 0) < -B/2$, and in turn, $\dot{x}_t(1) + \dot{x}_t(3) = \lambda_1 - x_t(1) + \lambda_3 \pi_{x_t}(m(3) > 0) - x_t(3) > B/4$. In particular, there exists finite $\tau' > \tau$ such that $x_t(1) + x_t(3) = W$ and thus $x_t(4) = 0$ for $t > \tau'$. Since $x_t(2) + x_t(3) + x_t(4) < W$ for $t > \tau'$, the limit point of $x(2)$ follows by equality (3.2) and condition (2.3).

We complete the proof by establishing the limit values of $x(1)$ and $x(3)$. Note that $\pi_{x_t}(E_{\{1,3\}}) = 1$ for almost all $t > \tau'$ via Lemma 3.3, and the sequence $(\pi_{x_t} : t > \tau')$ is tight on the set $E_{\{1,3\}}$ as can be verified via the Lyapounov function $f : E_{\{1,3\}} \mapsto \mathbb{R} : f(m) = ((m(3) - \alpha m(1))^2 + \alpha(m(1) + m(3))^2)^{1/2}$, where $\alpha = (\lambda_1 - \lambda_2)/\lambda_2$. Theorem 4.9.10 of Ethier and Kurtz (1986), together with the convergence of $x(2)$, yields that $(\pi_{x_t} : t > \tau')$ converges weakly to a distribution $\tilde{\pi}$ such that $\tilde{\pi}(E_{\{1,3\}}) = 1$ and $\pi = \tilde{\pi} \circ \tilde{f}$, where $\tilde{f} : E_{\{1,3\}} \mapsto \mathbb{Z}_+^2 : f(m) = (m(1), m(3))$. In particular, given $\varepsilon > 0$, there exists $\tau'' > \tau'$ such that

$$\max_{r=1,3} \left|\dot{x}_t(r) - \lambda_r \pi\big(m(r) + m(3) > 0\big) + x_t(r)\right| \le \varepsilon$$

for $t > \tau''$. Fix $t_o > \tau''$, define $y_{t_o}(r) = x_{t_o}(r)$ for $r = 1, 3$ and let $((y_t(1), y_t(3)) : t \geq t_o)$ satisfy the above inequality with the right-hand side equal to 0. Since $\max_{r=1,3} x_{t_o}(r) \leq W$, it follows that

$$\max_{r=1,3} \left|y_{t_o+t}(r) - \lambda_r \pi\left(m(r) + m(3) > 0\right)\right| \leq We^{-t},$$

and Gronwall's inequality yields that

$$\max_{r=1,3} \left|x_{t_o+t}(r) - y_{t_o+t}(r)\right| \leq \varepsilon t e^t$$

for all $t > 0$. Let $t(\varepsilon)$ solve $t(\varepsilon)e^{2t(\varepsilon)} = W/\varepsilon$ so that

$$\max_{r=1,3} \left|x_{t_o+t(\varepsilon)}(r) - \lambda_r \pi\left(m(r) + m(3) > 0\right)\right| \leq 2\varepsilon t(\varepsilon)e^{t(\varepsilon)}.$$

Since $\lim_{\varepsilon \searrow 0} \varepsilon t(\varepsilon)e^{t(\varepsilon)} = 0$, the claimed limit of $(x(1), x(3))$ follows due to the arbitrariness of t_o and ε. $\square$

3.3. *Totally overloaded network* ($\lambda_1 + \lambda_3 > W$, $\lambda_2 + \lambda_3 > W$).

PROOF OF THEOREM 1.1(c). We first consider the case $\lambda_2 < W$ and $\lambda_2 < \lambda_1$. The desired conclusion for this case is obtained by showing that $\liminf_{t\to\infty}(x_t(1) - x_t(2)) > 0$, as the proof of Theorem 1.1(b) then applies verbatim to conclude the proof here. Appeal to equality (3.2) to choose τ such that $x_t(2) < W$ for $t > \tau$, and let $t > \tau$ be such that $x_t(1) \leq \min\{\lambda_2, x_t(2)\}$. If $x_t(4) > 0$ or $x_t(1) + x_t(3) + x_t(4) < W$, then $\dot{x}_t(1) \geq \lambda_1 - \lambda_2 > 0$ via conditions (2.2) and (2.4). Otherwise, if $x_t(4) = 0$ and $x_t(1) + x_t(3) = W$, then it is necessary that $x_t(1) = x_t(2)$ and one can appeal to Lemma 3.4 to take $\dot{x}_t(4) = 0$ and $\dot{x}_t(1) + \dot{x}_t(3) = 0$. Equalities (3.1), (3.3) and (3.4) then imply that $\dot{x}_t(1) = \lambda_1 \pi_{x_t}(p_1) + \lambda_2 \pi_{x_t}(p_2)$, and in particular, that $\dot{x}_t(1) \geq 0$. If $\dot{x}_t(1) = 0$, then $\pi_{x_t}(m(1) + m(3) > 0) < 1$ via the hypothesis $x_t(1) \leq \lambda_2 < \lambda_1$, so the process m^{x_t} is ergodic on the set $\hat{E}_{\{1,3\}}$ and $\pi_{x_t}(\hat{E}_{\{1,3\}})$ is positive. This further implies that $\lambda_1 \pi_{x_t}(p_1) > 0$ and leads to a contradiction. Therefore $\dot{x}_t(1) > 0$ for almost all $t > \tau$ such that $x_t(1) \leq \min\{\lambda_2, x_t(2)\}$ and $\liminf_{t\to\infty}(x_t(1) - \min\{\lambda_2, x_t(2)\}) > 0$. Since $\limsup_{t\to\infty} x_t(2) \leq \lambda_2$, it follows that $\liminf_{t\to\infty}(x_t(1) - x_t(2)) > 0$.

The remaining case is proved via a stochastic ordering argument. For $t > 0$, $X_t(2)$ is stochastically nondecreasing in λ_2; therefore $x_*(2)$ is nondecreasing in λ_2. For $\lambda_1 < W$ the conclusion of the previous paragraph implies that $\lim_{\lambda_2 \nearrow \lambda_1} x_*(2) = \lambda_1$. Since $x_*(2) \leq \lambda_2$, it follows that if $\lambda_1 = \lambda_2 < W$, then $x_*(2) = \lambda_2$, and by symmetry $x_*(1) = \lambda_1$. $X_t(4)$ is stochastically nonincreasing in λ_2, so $x_*(4) = 0$ and in turn $x_*(3) = W - \lambda_1$ in the case $\lambda_1 = \lambda_2 < W$. A similar monotonicity argument establishes the theorem in the case $\min\{\lambda_1, \lambda_2\} \geq W$. $\square$

4. Matching assignment. Under Matching Assignment the limiting trajectory $(x_t : t \geq 0)$ complies with

$$\dot{x}_t(1) = \lambda_1 \pi_{x_t}\big(m(1) + m(3) > 0\big) - x_t(1), \tag{4.1}$$

$$\dot{x}_t(2) = \lambda_2 \pi_{x_t}\big(m(2) + m(3) > 0\big) - x_t(2), \tag{4.2}$$

$$\dot{x}_t(3) = \lambda_3 \pi_{x_t}\big(m(3) > 0\big) - x_t(3), \tag{4.3}$$

$$\begin{aligned} \dot{x}_t(4) &= 2W - x_t(1) - x_t(2) - x_t(3) - 2x_t(4) - \lambda_3 \pi_{x_t}\big(m(3) > 0\big) \\ &\quad - \lambda_1 \pi_{x_t}\big(m(1) = 0, m(3) > 0\big) - \lambda_2 \pi_{x_t}\big(m(2) = 0, m(3) > 0\big), \end{aligned} \tag{4.4}$$

where π_{x_t} is an equilibrium distribution for the Markov process m^{x_t} that takes values in E and has transitions as given in Table 4.

4.1. *Underloaded network* $(\lambda_1 + \lambda_3 < W,\ \lambda_2 + \lambda_3 < W)$.

PROOF OF THEOREM 1.2(a). Assume without loss of generality that $\lambda_1 \geq \lambda_2$. Fix positive $\varepsilon < W - \lambda_1 - \lambda_3$, and appeal to equalities (4.1)–(4.3) to choose $\tau > 0$ such that if $t > \tau$, then $x_t(r) < \lambda_r + \varepsilon/2$ for $r = 1, 2, 3$. If $t > \tau$ is such that $x_t(4) < W - \lambda_1 - \lambda_3 - \varepsilon$, then $x_t(1) + x_t(3) + x_t(4) < W$ and $x_t(2) + x_t(3) + x_t(4) < W$, so conditions (2.2) and (2.3) imply $\pi_{x_t}(m(1) = 0, m(3) > 0) = \pi_{x_t}(m(2) = 0, m(3) > 0) = 0$. In turn, $\dot{x}_t(4) \geq 2(W - \max\{x_t(1), x_t(2)\} - \max\{x_t(3), \lambda_3\} - x_t(4)) > 0$ via equality (4.4). Since ε can be chosen arbitrarily small, it follows that $\liminf_{t\to\infty} x_t(4) \geq W - \lambda_1 - \lambda_3 > 0$, and thus by (2.2)–(2.4), $\lim_{t\to\infty} x_t(r) = \lambda_r$ for $r = 1, 2, 3$. In turn, $\limsup_{t\to\infty} x_t(4) \leq W - \lambda_1 - \lambda_3$ via the state space constraint $x_t \in S$ for $t \geq 0$. This completes the proof. $\square$

4.2. *Partially overloaded network* $(\lambda_1 + \lambda_3 > W,\ \lambda_2 + \lambda_3 < W)$.

LEMMA 4.1. *Lemma* 3.3 *holds for Matching Assignment as well.*

PROOF. Let $S_o = \{x \in S : x(2) + x(3) < x(1) + x(3) = W,\ x(2) < \lambda_1\}$. For each $x \in S_o$, appeal to the proof of Theorem 2.4 of Zachary (1995) to identify a Lyapounov function $f(\cdot, x) : \hat{E}_{\{1,3\}} \mapsto \mathbb{R}_+$ such that $D_{x,x} f(m) < -\varepsilon'$ whenever $m \in E_{\{1,3\}}$ and $f(m, x) > A$, for some positive numbers ε', A. The form of this function as given in Zachary (1995) suffices to verify that $f(\cdot, x)$ satisfies conditions (i), (ii), (iv) of Proposition 2.2. Since $(\mu_j^x(m) : m \in E, j \in J)$ is uniformly continuous in x, there exist positive numbers δ, ε with $\varepsilon \leq \varepsilon'$ such that for all $y \in S$ with $|x - y| < \delta$, $D_{x,y} f(m) < -\varepsilon$ whenever $m \in E_{\{1,3\}}$ and $f(m, x) > A$. Proposition 2.2 now implies the desired conclusion. $\square$

PROOF OF THEOREM 1.2(b). Lemma 3.2 and its proof are valid for Matching Assignment as well, so the proof of Theorem 1.1(b) applies here verbatim with the proper interpretation of π in the context of Matching Assignment. $\square$

4.3. *Totally overloaded network* ($\lambda_1 + \lambda_3 \geq W,\ \lambda_2 + \lambda_3 \geq W$).

PROOF OF THEOREM 1.2(c). The proof of Theorem 1.1(c) applies verbatim with the proper interpretation of p_1 and p_2 in the context of Matching Assignment. □

Acknowledgments. This work was done when the author was with Bilkent University, Ankara, Turkey. The author is grateful to a referee for carefully reading an earlier draft of the paper.

REFERENCES

ALANYALI, M. (1999). Three resource management policies for a resource with partition constraints. *Adv. in Appl. Probab.* **31** 795–818.

ALANYALI, M. and HAJEK, B. (1997). Analysis of simple algorithms for dynamic load balancing. *Math. Oper. Res.* **22** 840–871.

BEAN, N. G., GIBBENS, R. J. and ZACHARY, S. (1995). Asymptotic analysis of single resource loss systems in heavy traffic, with applications to integrated networks. *Adv. in Appl. Probab.* **27** 273–292.

BEAN, N. G., GIBBENS, R. J. and ZACHARY, S. (1997). Dynamic and equilibrium behavior of controlled loss networks. *Ann. Appl. Probab.* **7** 873–885.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York.

FAYOLLE, G., MALYSHEV, V. A. and MENSHIKOV, M. V. (1995). *Topics in the Constructive Theory of Countable Markov Chains*. Cambridge Univ. Press.

HAJEK, B. (1982). Hitting time and occupation time bounds implied by drift analysis with applications. *Adv. in Appl. Probab.* **14** 502–525.

HUNT, P. J. (1995). Pathological behavior in loss networks. *J. Appl. Probab.* **32** 519–533.

HUNT, P. J. and KURTZ, T. G. (1994). Large loss networks. *Stochastic Process. Appl.* **53** 363–378.

HUNT, P. J. and LAWS, C. N. (1997). Optimization via trunk reservation in single resource loss systems under heavy traffic. *Ann. Appl. Probab.* **7** 1058–1079.

KATZELA, I. and NAGHSHINEH, M. (1996). Channel assignment schemes for cellular mobile communication systems: A comprehensive survey. *IEEE Personal Communications Magazine* **3**, June 10–31.

KELLY, F. P. (1985). Stochastic models of computer communication systems. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **47** 379–395.

KELLY, F. P. (1991). Loss networks. *Ann. Appl. Probab.* **1** 319–378.

PALLANT, D. L. and TAYLOR, P. G. (1995). Modeling handovers in cellular mobile networks with dynamic channel allocation. *Oper. Res.* **43** 33–42.

ROYDEN, H. L. (1988). *Real Analysis*. Macmillan, New York.

ZACHARY, S. (1995). On two-dimensional Markov chains in the positive quadrant with partial spatial homogeneity. *Markov Process. Related Fields* **1** 267–280.

ZACHARY, S. and ZIEDINS, I. (2002). A refinement of the Hunt–Kurtz theory of large loss networks, with an application to virtual partitioning. *Ann. Appl. Probab.* **12** 1–22.

ZANG, H., JUE, J. P. and MUKHERJEE, B. (2000). A review of routing and wavelength assignment approaches for wavelength-routed optical WDM networks. *Optical Networks Magazine*. **1** 47–60.

DEPARTMENT OF ELECTRICAL
AND COMPUTER ENGINEERING
AND
THE CENTER FOR INFORMATION
AND SYSTEMS ENGINEERING
BOSTON UNIVERSITY
8 SAINT MARY'S STREET
BOSTON, MASSACHUSETTS 02215
E-MAIL: alanyali@bu.edu