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STABILITY ANALYSIS OF SECOND-ORDER FLUID FLOW MODELS IN A STATIONARY ERGODIC ENVIRONMENT

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In this paper, we study the stability of a fluid queue with an infinite-capacity buffer. The input and service rates are governed by a stochastic process, called the environment process, and are allowed to depend on the fluid level in the buffer. The variability of the traffic is modeled by a Brownian motion and a local variance function, which also depends on the fluid level in the buffer. The behavior of this second-order fluid flow model is described by a reflected stochastic differential equation, and, under stationarity and ergodicity assumptions on the environment process, we obtain stability conditions for this general fluid queue.

1. Introduction. Fluid flow models are widely used in the performance evaluation of high-speed communication networks. Typically, the fluid represents information stored in a buffer and waiting for transmission in a network. The arrival and service processes are modulated by a random external environment, and the quantity of interest is the behavior of the buffer level. A lot of papers, see, among others, Anick, Mitra and Sondhi (1982), Mitra (1988), Stern and Elwalid (1991), Kulkarni (1997), Prabhu (1997), Sericola and Tuffin (1999) and the references therein, considered the case where the random external environment is a continuous-time Markov chain (CTMC) with a finite or infinite state space. In these papers, the input and output rates of the buffer are both piecewise constant, depending on the state of the Markov process. Extensions of this model can be found in Lam and Lee (1997), where the authors deal with the case where the input and output rates of the buffer may also depend on the buffer level. In Asmussen (1995) and Karandikar and Kulkarni (1995), new models, called second-order models, were introduced by adding a "white noise" factor, which represents, in practice, the variability of the traffic during the transmission periods. The fluid level was thus described by a reflected Brownian motion modulated by a CTMC X. When the CTMC X is in state i, the fluid level is modeled by a reflected Brownian motion with drift b_i and variance parameter σ_i^2. In the state of the art [Kulkarni (1997)], the author mentioned as further work the importance of the study of such models with arrivals and services depending on the fluid level.

In this paper, we consider a fluid flow model driven by a stationary ergodic process (not necessarily a Markov process), where the arrival and service rates are

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allowed to depend on the fluid level in the buffer. The variability of the traffic is modeled by a Brownian motion and a local variance mapping, which also depends on the fluid level in the buffer. This generalization leads, in particular, to the use of Itô's stochastic integral, and the well-known differential equations used in Prabhu (1997) become stochastic differential equations with reflection at 0. The fluid level process with these level-dependent drift and variance coefficients is, essentially, a reflected Brownian motion that has been altered, at each instant, by changing its drift and its variance.

The main result of this paper is obtaining, under stationarity and ergodicity assumptions, stability conditions for such models, that is, conditions for the existence of the limiting behavior for the fluid level in the buffer. The stability of such queues was already studied, for example, in Sigman and Ryan (2000) and Atar, Budhiraja and Dupuis (2001) when the buffer level did not depend on an external environment. The idea for proving stability is, in this case, to exploit the Markovian nature of the buffer level at time t and to prove its positive recurrence under the assumptions, roughly speaking, that the drift is negative for high values of the buffer level and that the local variance is uniformly nondegenerated. Such criteria of positive recurrence cannot be used in the present paper, because our governing process is not necessarily a Markov process.

To the best of our knowledge, there are no stability results concerning fluid queues in a general, non-Markovian, random environment. According to Kulkarni [(1997), page 333] there is no general theory for such a case. This kind of general environment is very important for stability results because traffic arriving at a queue has generally already traversed parts of the network and has thus lost its Markov property. Moreover, our results allow us to study the stability of queues fed by general sources.

The remainder of the paper is organized as follows. The model and notation are introduced in the next section. In Section 3, we solve Lindley's equation corresponding to that model. The main results, concerning the stability of the fluid queue, are proved in Sections 4 and 6. Section 5 contains some properties of the solution to Lindley's equation used in Section 6.

2. Model and notation. Consider an infinite-capacity buffer where fluid enters and exits according to the behavior of a general stochastic process denoted by $X = \{X(t),\ t \geq 0\}$. Let $Q(t)$ denote the amount of fluid in the buffer at time t. The input and service rates in the buffer are denoted, respectively, by $\lambda(X(t), Q(t))$ and $\mu(X(t), Q(t))$. They are both nonnegative and depend on the environment and the fluid level at time t. Their difference is called the local drift and is denoted by

$$b\big(X(t), Q(t)\big) = \lambda\big(X(t), Q(t)\big) - \mu\big(X(t), Q(t)\big).$$

The variability of the traffic at time t is represented by a Brownian motion $\{B_t\}$ and a local variance mapping $\sigma(X(t), Q(t))$. The behavior of the process $\{Q(t)\}$ is

thus described by the following reflected stochastic differential equation (RSDE):

$$dQ(t) = b\big(X(t), Q(t)\big)\,dt + \sigma\big(X(t), Q(t)\big)\,dB_t + dL(t), \tag{1}$$

so-called because 0 is a reflecting barrier, where the stochastic process $L = \{L(t), t \geq 0\}$ is an increasing process introduced to prevent $\{Q(t)\}$ from being negative. Equation (1) is very compact and covers a broad range of situations.

Suppose, for instance, that we do not take into account the random part of (1) by taking $\sigma(X(t), Q(t)) = 0$. In that case, we obtain the following classical differential equation [see Prabhu (1997)]:

$$\frac{d}{dt}Q(t) = \begin{cases} b\big(X(t), Q(t)\big), & \text{if } Q(t) > 0, \\ b\big(X(t), Q(t)\big)^+, & \text{if } Q(t) = 0, \end{cases}$$

by setting $dL(t) = b(X(t), Q(t))^- \mathbb{1}_{\{Q(t)=0\}}\,dt$, where $x^+ = \max(x, 0)$ and $x^- = (-x)^+$.

Let us now establish more rigorously the setting in which we will work from now on.

2.1. *Mathematical framework.* Let $(\Omega, \mathcal{F}, P)$ be a probabilistic space. We denote by $\{X(t), t \in \mathbb{R}\}$ a stationary stochastic process with values in $\mathcal{X} \subset \mathbb{R}^d$, cadlag, and by $\{B_t, t \in \mathbb{R}\}$ a standard Brownian motion with real values. The processes $\{X(t)\}$ and $\{B_t\}$ are supposed to be independent.

Let $\{\theta_t : (\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, P), t \in \mathbb{R}\}$ be the shift operator operating on $\{X(t), t \in \mathbb{R}\}$ and on the increments of $\{B_t, t \in \mathbb{R}\}$, that is, such that, $\forall s, s', t \in \mathbb{R}$ with $s \geq s'$ and $\omega \in \Omega$, $X(s, \theta_t\omega) = X(s + t, \omega)$ and $B_s(\theta_t\omega) - B_{s'}(\theta_t\omega) = B_{s+t}(\omega) - B_{s'+t}(\omega)$. In practice, $\mathcal{F}$ is the σ-algebra generated by the random variables $X(t)$ and the increments $B_{t'} - B_t$ for $t' \geq t$ and t and t' in $\mathbb{R}$, so we may as well assume that the probability P is θ_t-invariant, that is, $P(\theta_t^{-1}A) = P(A)$ for all $t \in \mathbb{R}$ and $A \in \mathcal{F}$. We also suppose that $(\Omega, \mathcal{F}, P, \{\theta_t\})$ is ergodic.

For the sake of completeness, we recall how the probability space $(\Omega, \mathcal{F}, P, \{X(t)\}, \{B_t\}, \{\theta_t\})$ is constructed. Let $\{x(t), t \in \mathbb{R}\}$ be a stationary ergodic process defined on a probabilistic space $(\Omega^1, \mathcal{F}^1, \mu)$, where $\mathcal{F}^1 = \sigma(x(t), t \in \mathbb{R})$, and let $\{\beta_t, t \in \mathbb{R}\}$ be a standard Brownian motion (i.e., $\beta_0 = 0$) defined on $(\Omega^2, \mathcal{F}^2, \nu)$, where $\mathcal{F}^2 = \sigma(\beta_{s'} - \beta_s; s' \geq s, s', s \in \mathbb{R})$. We denote by θ_t^1 the shift operator defined on Ω^1 such that $x(s) \circ \theta_t^1(\omega^1) = x(s+t)(\omega^1)$ for all $(s, t, \omega^1) \in \mathbb{R} \times \mathbb{R} \times \Omega^1$, and by θ_t^2 the operator defined on Ω^2 such that $(\beta_{s'} - \beta_s) \circ \theta_t^2(\omega^2) = (\beta_{s'+t} - \beta_{s+t})(\omega^2)$ for all $(s', s, t, \omega^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega^2$ (i.e., θ_t^2 operates on the increments of $\{\beta_t\}$). Note that ν is θ_t^2-invariant for all t because $\{\beta_t\}$ has stationary increments and that $(\Omega^2, \mathcal{F}^2, \nu, \{\theta_t^2\})$ is ergodic because $\{\beta_t\}$ has independent increments. We already know that μ is θ_t^1-invariant and that $(\Omega^1, \mathcal{F}^1, \mu, \{\theta_t^1\})$ is ergodic.

We then set $\Omega = \Omega^1 \times \Omega^2$, $\mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2$, $P = \mu \otimes \nu$, $\theta_t = (\theta_t^1, \theta_t^2)$, $X(t,\omega) = x(t,\omega^1)$ and $B_t(\omega) = \beta_t(\omega^2)$ for all $t \in \mathbb{R}$ and $\omega = (\omega^1, \omega^2) \in \Omega$. The probability space $(\Omega, \mathcal{F}, P, \{\theta_t\})$ so defined is then ergodic and P is θ_t-invariant.

Let $(\mathcal{F}_t)_{t\in\mathbb{R}}$ be a right-continuous filtration such that $\{B_t\}$ and $\{X(t)\}$ are $(\mathcal{F}_t)$-adapted. For some subset A of $\mathbb{R}^k$, we denote by $\mathcal{B}(A)$ the σ-algebra generated by the open sets of A.

Let us suppose that:

- the mapping $b : (\mathcal{X} \times [0,+\infty), \mathcal{B}(\mathcal{X} \times [0,+\infty))) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable and verifies $\sup_{x\geq 0}[b(X(0),x)]^+ \in L^1(\Omega)$ and $[b(X(0),0)]^+ \in L^2(\Omega)$;
- the mapping $\sigma : (\mathcal{X} \times [0,+\infty), \mathcal{B}(\mathcal{X} \times [0,+\infty))) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ is measurable and verifies $\sup_{x\geq 0}[\sigma(X(0),x)]^2 \in L^1(\Omega)$;
- the mappings b and σ are both Lipschitz with respect to the second variable, that is, $\exists C > 0$, $\forall y \in \mathcal{X}$, $\forall x, x' \geq 0$, $|b(y,x) - b(y,x')| + |\sigma(y,x) - \sigma(y,x')| \leq C|x - x'|$.

Under the Lipschitz assumptions, for a fixed $u \in \mathbb{R}$ and for $y \geq 0$, the following RSDE

$$
\begin{aligned}
&dQ(t) = b\big(X(t), Q(t)\big)\,dt + \sigma\big(X(t), Q(t)\big)\,dB_t + dL(t) \qquad \text{for } t \geq u,\\
&Q(u) = y,\\
&Q(t) \geq 0 \qquad \text{for } t \geq u,\\
&L(t) = \int_u^t \mathbb{1}_{\{Q(s)=0\}}\,dL(s) \qquad \text{for } t \geq u
\end{aligned}
\tag{2}
$$

admits a unique solution couple $(Q(t), L(t))_{t\geq u}$, $(\mathcal{F}_t)$-adapted, where $\{Q(t)\}$ and $\{L(t)\}$ are both continuous and nonnegative and $\{L(t)\}$ is increasing. The proof of this result can be found in Skorokhod (1961) or El Karoui and Chaleyat-Maurel (1978). We see that the process $\{L(t)\}$ is linked to the process $\{Q(t)\}$, interfering only when $Q(t) = 0$ and "prodding it upward" whenever $Q(t)$ has a tendency to go downward while approaching 0, that is, whenever, intuitively, "$b(X(t),0)\,dt + \sigma(X(t),0)\,dB_t \leq 0$."

It is well known [see, e.g., El Karoui and Chaleyat-Maurel (1978), page 118] that, for a fixed $u \in \mathbb{R}$, $L(t)$ can be written as

$$
L(t) = \sup_{s\in[u,t]} \left(-y - \int_u^s \big[b\big(X(v), Q(v)\big)\,dv + \sigma\big(X(v), Q(v)\big)\,dB_v\big] \right)^+. \tag{3}
$$

Roughly speaking, since $L(t)$ is increasing only when $Q(t) = 0$, we formally have the (nonrigorous) expression "$dL(t) = [b(X(t), Q(t))\,dt + \sigma(X(t), Q(t))\,dB_t]^- \times \mathbb{1}_{\{Q(t)=0\}}$."

Moreover, in the case where $b(X(0),0) \in L^2(\Omega)$ and $\sigma(X(0),0) \in L^2(\Omega)$, it is standard that $Q(t)$ and $L(t)$ admit finite moments of order 2 for all $t \geq 0$. More

precisely, using Gronwall's inequality argument as well as the Burkholder–Davis–Gundy inequalities, we obtain that, for all $t \geq 0$,

$$E\left(\sup_{s\in[0,t]} |Q(s)|^2\right) < +\infty \quad \text{and} \quad E\big(L(t)^2\big) < \infty.$$

We recall the definition of the stability [see Loynes (1962)] of the fluid queue.

DEFINITION 2.1. The queue is said to be stable if there exists an almost surely finite random variable W such that $Q(t)$ converges in distribution to W when t tends to ∞.

3. Lindley's stationarity equation. As usual, a key to state a stability criterion is to solve an equation of the Lindley type [see Loynes (1962)]. The unique solution to the RSDE (2) is, in fact, a function of the real number u and the nonnegative real number y, the driving Brownian motion and the environment process being kept the same as u and y vary. To make visible the dependence on u and y, we denote by $(Q_u^y(t), L_u^y(t))_{t\geq u}$ the solution to the RSDE. However, when we consider the solution with $y = 0$, we denote it simply by $(Q_u(t), L_u(t))_{t\geq u}$. Thus, $Q_u(t)$ is the amount of fluid in the buffer at time t when the buffer is empty at time u. We set by convention $Q_u(t) = 0$ and $L_u(t) = 0$ for $t \leq u$. We then have the following results.

PROPOSITION 3.1. *Let $(Y_t^1, k_t^1)_{t\geq u}$ and $(Y_t^2, k_t^2)_{t\geq u}$ be two solutions to the following RSDEs*:

$$\begin{aligned} &dY_t = b^i\big(X(t), Y_t\big)\,dt + \sigma\big(X(t), Y_t\big)\,dB_t + dk_t \qquad \textit{for } t \geq u, \\ &Y_t \geq 0 \qquad \textit{for } t \geq u, \\ &k_t = \int_u^t \mathbb{1}_{\{Y_s=0\}}\,dk_s \qquad \textit{for } t \geq u, \end{aligned} \tag{4}$$

$i = 1, 2$, with $b^1 \geq b^2$ and b^i, $i = 1, 2$, Lipschitz with respect to the second variable. If $Y_u^1 \geq Y_u^2$, then we have, for all $t \geq u$:

1. $Y_t^1 \geq Y_t^2$;
2. $\forall h \geq 0, 0 \leq k_{t+h}^1 - k_t^1 \leq k_{t+h}^2 - k_t^2$.

PROOF. The first point is a comparison theorem for RSDEs and is quite standard [see, e.g., El Karoui and Chaleyat-Maurel (1978)].

However, although the second point is rather straightforward, it has, to our knowledge, never been mentioned. Roughly speaking, it states that, if Y^1 is above Y^2, then Y^1 hits 0 less often than Y^2.

More precisely, let $\tau = \inf\{v \geq u \mid Y_v^1 = 0\}$, with the convention $\inf \varnothing = +\infty$. Then, by strong uniqueness to the RSDE, we have $Y_t^1 = Y_t^2$ and $k_t^1 - k_\tau^1 = k_t^2 - k_\tau^2$

for all $t \geq \tau$. Besides, $k^1_{t'} - k^1_t = 0$ for $u \leq t \leq t' \leq \tau$, as $Y^1 > 0$ on $[u, \tau)$. It is then easy to check that, whether t lies in $[u, \tau)$ or in $[\tau, +\infty)$, we have $k^1_{(t+h)\wedge\tau} - k^1_t \leq k^2_{(t+h)\wedge\tau} - k^2_t$, with the notation $a \wedge b = \min(a, b)$. Hence, for $h \geq 0$,

$$\begin{aligned} 0 \leq k^1_{t+h} - k^1_t &= k^1_{t+h} - k^1_{(t+h)\wedge\tau} + k^1_{(t+h)\wedge\tau} - k^1_t \\ &\leq k^2_{t+h} - k^2_{(t+h)\wedge\tau} + k^2_{(t+h)\wedge\tau} - k^2_t \\ &= k^2_{t+h} - k^2_t. \end{aligned}$$

This completes the proof. □

This proposition is used to prove the following lemma.

LEMMA 3.2. *Let u, t and t' be arbitrary numbers with $u \leq t \leq t'$. For every $h > 0$, we have*:

1. $0 \leq Q_u(t) \leq Q_{u-h}(t)$;
2. $0 \leq L_{u-h}(t') - L_{u-h}(t) \leq L_u(t') - L_u(t)$.

PROOF. Consider the RSDE (4) with $Y_u = 0$ and $b^1 = b^2 = b$. From (2), the unique solution to this equation is given by $Y^2_t = Q_u(t)$ and $k^2_t = L_u(t)$.

Let $h \geq 0$ and consider again the RSDE (4) with $Y_u = Q_{u-h}(u)$. From (2), the unique solution to this equation is given by $Y^1_t = Q_{u-h}(t)$ and $k^1_t = L_{u-h}(t) - L_{u-h}(u)$.

We have $Y^2_u = 0 \leq Q_{u-h}(u) = Y^1_u$, so we can apply Proposition 3.1, which leads to

$$Q_{u-h}(t) = Y^1_t \geq Y^2_t = Q_u(t),$$

and, since $t' \geq t$, we get $L_{u-h}(t') - L_{u-h}(t) = k^1_{t'} - k^1_t \leq k^2_{t'} - k^2_t = L_u(t') - L_u(t)$. □

The stationary equation of the Lindley type is given in the following proposition.

PROPOSITION 3.3. *Suppose that $E(\sup_{x\geq 0} |b(X(0), x)|) < +\infty$. Then there exist a nonnegative random variable W a.s. finite or infinite and a process $\{L(t, v)\}_{t\geq v}$ such that, whenever W is finite and for $t \geq v$,*

$$\begin{aligned} W \circ \theta_t = W \circ \theta_v &+ \int_v^t b(X(s), W \circ \theta_s)\, ds \\ &+ \int_v^t \sigma(X(s), W \circ \theta_s)\, dB_s + L(t, v), \qquad P\text{-a.s.}, \end{aligned} \tag{5}$$

$$L(t, v) = \int_{s=v}^{s=t} \mathbb{1}_{\{W\circ\theta_s = 0\}}\, dL(s, v).$$

Besides, the process $\{L(t,v)\}_{t\geq v}$ *is increasing in* t, *decreasing in* v *and compatible with the flow, that is, for all* $t \geq v$ *and* $r \in \mathbb{R}$, $L(t,v)\circ\theta_r = L(t+r, v+r)$.

PROOF. Since $(Q_u(s), L_u(s))_{s\geq u}$ is the solution to (2) with $y=0$, then, for every $t\in\mathbb{R}$, $(Q_u(s)\circ\theta_t, L_u(s)\circ\theta_t)_{s\geq u}$ is the solution to

$$\begin{aligned}
&dQ(s) = b\big(X(s)\circ\theta_t, Q(s)\big)\,ds + \sigma\big(X(s)\circ\theta_t, Q(s)\big)\,d_s(B_s\circ\theta_t)\\
&\qquad\qquad + dL(s), \qquad s\geq u,\\
&Q(u)=0,\\
&Q(s)\geq 0, \qquad s\geq u,\\
&L(s) = \int_u^s \mathbb{1}_{\{Q(x)=0\}}\,dL(x), \qquad s\geq u.
\end{aligned} \tag{6}$$

The processes $\{X(t)\}$ and the increments of $\{B_t\}$ being compatible with the flow $\{\theta_t\}$, it follows that system (6) is the same as

$$\begin{aligned}
&dQ(s) = b\big(X(s+t), Q(s)\big)\,ds + \sigma\big(X(s+t), Q(s)\big)\,d_sB_{s+t}\\
&\qquad\qquad + dL(s), \qquad s\geq u,\\
&Q(u)=0,\\
&Q(s)\geq 0, \qquad s\geq u,\\
&L(s) = \int_u^s \mathbb{1}_{\{Q(x)=0\}}\,dL(x), \qquad s\geq u.
\end{aligned}$$

Thus, by strong uniqueness, we have

$$\begin{aligned}
\forall t,s\in\mathbb{R}, \qquad &\big(Q_u(s)\circ\theta_t, L_u(s)\circ\theta_t\big)\\
&= \big(Q_{u+t}(s+t), L_{u+t}(s+t)\big), \qquad P\text{-a.s.}
\end{aligned} \tag{7}$$

Let u, t and t' be real numbers such that $u\leq t\leq t'$. From Lemma 3.2, $Q_u(0)$ increases when u decreases and the difference $L_u(t')-L_u(t)$ decreases when u decreases. We thus define the following limits:

$$W = \lim_{u\searrow-\infty} Q_u(0) \quad\text{and}\quad L(t',t) = \lim_{u\searrow-\infty}\big(L_u(t')-L_u(t)\big).$$

Intuitively, W is the fluid level in the buffer at time 0 when the buffer is empty at time $-\infty$.

The rest of the proof is divided in two steps.

Step 1. We show that the event $\{W=+\infty\}$ is θ_t-invariant for every $t\in\mathbb{R}$. We detail the proof only for nonnegative values of t, the other case being similar.

Let u, v and t be real numbers such that $u+t\leq v\leq t$ with $u\leq -t$ and $t\geq 0$.

We have

$$
\begin{aligned}
Q_u(0)\circ\theta_t &= Q_{u+t}(t)\\
&= Q_{u+t}(v) + \int_v^t b\big(X(s), Q_{u+t}(s)\big)\,ds + \int_v^t \sigma\big(X(s), Q_{u+t}(s)\big)\,dB_s\\
&\quad + L_{u+t}(t) - L_{u+t}(v)\\
&= Q_{u+t-v}(0)\circ\theta_v + \int_v^t b\big(X(s), Q_{u+t-s}(0)\circ\theta_s\big)\,ds\\
&\quad + \int_v^t \sigma\big(X(s), Q_{u+t-s}(0)\circ\theta_s\big)\,dB_s\\
&\quad + L_{u+t}(t) - L_{u+t}(v),
\end{aligned}
\tag{8}
$$

where the first and third equalities are due to (7), and the second one is obtained by using (2). If $v=0$, we obtain $u+t\le 0\le t$ and

$$
\begin{aligned}
Q_u(0)\circ\theta_t &= Q_{u+t}(0) + \int_0^t b\big(X(s), Q_{u+t-s}(0)\circ\theta_s\big)\,ds\\
&\quad + \int_0^t \sigma\big(X(s), Q_{u+t-s}(0)\circ\theta_s\big)\,dB_s + L_{u+t}(t) - L_{u+t}(0).
\end{aligned}
\tag{9}
$$

Consider separately the three terms on the right-hand side of (9).

The term $\int_0^t b(X(s), Q_{u+t-s}(0)\circ\theta_s)\,ds$ is uniformly lower bounded with respect to u because $E(\sup_{x\ge 0}|b(X(0),x)|) < +\infty$, which implies that $\sup_{x\ge 0}|b(X(0),x)| < +\infty$ a.s.

From Lemma 3.2, we have, for $u+t<0$, $L_{u+t}(t)-L_{u+t}(0)\le L_0(t)-L_0(0) = L_0(t)$. Thus, $L_{u+t}(t)-L_{u+t}(0)$ is uniformly bounded with respect to u a.s. for $u+t<0$. Besides, for any $u\le 0$,

$$
\begin{aligned}
&E\left(\left(\int_0^t \sigma\big(X(s), Q_u(s)\big)\,dB_s\right)^2\right)\\
&\quad = E\left(\int_0^t \big[\sigma\big(X(s), Q_u(s)\big)\big]^2\,ds\right)\\
&\quad \le E\left(\int_0^t \sup_{x\ge 0}\big[\sigma\big(X(s), x\big)\big]^2\,ds\right)\\
&\quad = tE\left(\sup_{x\ge 0}\big[\sigma\big(X(0), x\big)\big]^2\right).
\end{aligned}
$$

Thus, $(\int_0^t \sigma(X(s), Q_u(s))\,dB_s)_{u\le 0}$ is bounded in $L^2(\Omega)$. So there exists a subsequence $(u_k)_{k\in\mathbb{N}}$, which depends on ω, tending to $-\infty$ as $k\to\infty$ such that $(\int_0^t \sigma(X(s), Q_{u_k}(s))\,dB_s)_{k\in\mathbb{N}}$ is bounded a.s.

Suppose that, for a fixed ω, we have $W(\omega)=+\infty$. We thus have, by the definition of W, $\lim_{k\to\infty} Q_{u_k+t}(0,\omega) = W(\omega) = +\infty$. By writing (9) with u_k

instead of u and by taking the limit when $k \to \infty$, we get $\lim_{k\to\infty} Q_{u_k}(0, \theta_t\omega) = +\infty$. By the definition of W, we obtain that $W(\theta_t\omega) = +\infty$. This means that the event $\{W = +\infty\}$ is θ_t-invariant and thus, by ergodicity, we have either $P(W = +\infty) = 0$ or $P(W = +\infty) = 1$.

Step 2. Suppose that W is finite a.s. and consider the relation (8). As the function b is continuous with respect to the second variable, and since $Q_{u+t-s}(0) \circ \theta_s$ converges increasingly to $W \circ \theta_s$ when u tends to $-\infty$, we get

$$\int_v^t b\big(X(s), Q_{u+t-s}(0) \circ \theta_s\big)\, ds \underset{u\searrow-\infty}{\longrightarrow} \int_v^t b\big(X(s), W \circ \theta_s\big)\, ds.$$

As σ is continuous with respect to the second variable and $\sup_{x\geq 0}[\sigma(X(0), x)]^2 \in L^1(\Omega)$,

$$\int_v^t \sigma\big(X(s), Q_{u+t-s}(0) \circ \theta_s\big)\, dB_s \underset{u\searrow-\infty}{\longrightarrow} \int_v^t \sigma\big(X(s), W \circ \theta_s\big)\, dB_s \qquad \text{in } L^2(\Omega).$$

We may consider, by taking a subsequence, that this convergence holds almost surely.

By definition, we have $L_{u+t}(t) - L_{u+t}(v) \to L(t, v)$ a.s. when $u \to -\infty$ and so, by taking the limit when $u \to -\infty$ in (8), we obtain the desired relation

$$W \circ \theta_t = W \circ \theta_v + \int_v^t b\big(X(s), W \circ \theta_s\big)\, ds + \int_v^t \sigma\big(X(s), W \circ \theta_s\big)\, dB_s + L(t, v).$$

Now, noticing that, for $u \leq w \leq v$, $\mathbb{1}_{\{Q_u(s)=0\}}\, d(L_u(s) - L_u(v)) = \mathbb{1}_{\{Q_u(s)=0\}} \times d(L_w(s) - L_w(v))$ [indeed, $L_w(s)$ increases when $Q_w(s)$ hits 0 and, a fortiori, when $Q_u(s)$ hits 0], we may write, for $u \leq w \leq v$,

$$\begin{aligned} L_u(t) - L_u(v) &= \int_{s=v}^{s=t} \mathbb{1}_{\{Q_u(s)=0\}}\, d\big(L_u(s) - L_u(v)\big) \\ &= \int_{s=v}^{s=t} \mathbb{1}_{\{Q_u(s)=0\}}\, d\big(L_w(s) - L_w(v)\big). \end{aligned}$$

Letting $u \to -\infty$, we get $L(t, v) = \int_{s=v}^{s=t} \mathbb{1}_{\{W\circ\theta_s=0\}}\, d(L_w(s) - L_w(v))$. Finally, letting $w \to -\infty$, we get

$$L(t, v) = \int_{s=v}^{s=t} \mathbb{1}_{\{W\circ\theta_s=0\}}\, dL(s, v).$$

Let us finish by showing that the process $\{L(t, v)\}_{t\geq v}$ is compatible with $\{\theta_t,\ t \in \mathbb{R}\}$. For $t \geq v$, $r \in \mathbb{R}$ and $u \leq v$, we have $L_u(t) \circ \theta_r - L_u(v) \circ \theta_r = L_{u+r}(t + r) - L_{u+r}(v + r)$, which, when $u \to -\infty$, yields $L(t, v) \circ \theta_r = L(t + r, v + r)$ as desired. $\square$

PROPOSITION 3.4. *The amount of fluid* $\{Q_0(t)\}_{t\geq 0}$ *converges in distribution to* W, *that is*,

$$Q_0(t) \underset{t\to+\infty}{\overset{\mathcal{D}}{\longrightarrow}} W.$$

PROOF. Since, for all $u \leq 0$, $Q_u(0) = Q_0(-u) \circ \theta_u$, we have that $Q_u(0) \overset{\mathcal{D}}{=} Q_0(-u)$. Then, by letting $u \to -\infty$, we get $Q_0(t) \overset{\mathcal{D}}{\to}_{t\to+\infty} W$. $\square$

Note that, in the case where $W = +\infty$ a.s., we have

$$\forall x \geq 0, \qquad P\big(Q_0(t) > x\big) = P\big(Q_{-t}(0) > x\big) \underset{t\to\infty}{\longrightarrow} 1.$$

4. Stability of the initially empty queue. To establish the results of this paper, we need the two following technical lemmas.

LEMMA 4.1. *Let* V *be a nonnegative, a.s. finite random variable such that, for every* $t \in \mathbb{R}$, $V \circ \theta_t - V \in L^1(\Omega)$. *Then* $E(V \circ \theta_t - V) = 0$ *for every* $t \in \mathbb{R}$.

PROOF. See, for instance, Baccelli and Brémaud [(1994), page 77]. $\square$

We omit the proof of the following lemma, which is easy to check.

LEMMA 4.2. *If* b *is lower bounded, for all* $t, u \in \mathbb{R}$ *such that* $u \leq t$, *both* $\sup_{x\geq 0} |b(X(0), x)|$ *and* $\int_u^t \sup_{x\geq 0} |b(X(v), x)|\, dv$ *are integrable.*

The following theorem establishes that the buffer does not fill indefinitely as long as, on average, the drift is negative when the fluid level in the buffer is high. The second assertion means that, since the drift is, on average, positive, the fluid builds up in the buffer.

THEOREM 4.3. *Suppose that the queue is initially empty.*

1. *If* $E(\limsup_{x\to\infty} b(X(0), x)) < 0$, *then the queue is stable.*
2. *If* $E(\inf_{x\geq 0} b(X(0), x)) > 0$, *then*, $\forall x \geq 0$, $\lim_{t\to\infty} P(Q(t) > x) = 1$.

PROOF. Note that the assumption $\sup_{x\geq 0}[b(X(0), x)]^+ \in L^1(\Omega)$ implies that $\limsup_{x\to\infty} b(X(0), x)$ is semi-integrable, that is, $E(\limsup_{x\to\infty} b(X(0), x))$ exists, but could be equal to $-\infty$ (which could happen if the service rate is not integrable). We are going to prove Theorem 4.3 first when b is lower bounded, then in the general case.

Step 1. Suppose that b is lower bounded. For the first assertion, we prove that if $W = +\infty$ then $E(\limsup_{x\to\infty} b(X(0), x)) \geq 0$.

Note that we will often take expectations of $Q_u(t)$ and $L_u(t)$ in this part of the proof. This is entirely justified by the fact that b lower bounded, along with the assumption $[b(X(0), 0)]^+ \in L^2(\Omega)$, ensures that $b(X(0), 0)$ is in $L^2(\Omega)$, and so $Q_u(t)$ and $L_u(t)$ admit finite moments of order 2 as shown at the end of Section 2.

So, suppose that $W = +\infty$ a.s. and let $u < 0$ and $t \geq 0$. Since P is θ_t-invariant, we have $E(Q_u(0) \circ \theta_t) = E(Q_u(0))$. From Lemma 3.2, $Q_u(0)$ is decreasing in u, and thus we have $E(Q_u(0) - Q_{u+t}(0)) \geq 0$. Combining these two relations, we obtain

$$E\big(Q_u(0) \circ \theta_t - Q_{u+t}(0)\big) = E\big(Q_u(0) - Q_{u+t}(0)\big) \geq 0.$$

Replacing the value of $Q_u(0) \circ \theta_t - Q_{u+t}(0)$ by its expression in (9), we get, since the martingale part has a null expectation,

$$0 \leq E\left(\int_0^t b\big(X(s), Q_{u+t-s}(0) \circ \theta_s\big)\, ds\right) + E\big(L_{u+t}(t) - L_{u+t}(0)\big). \tag{10}$$

Observing that $Q_{v+t-s}(0) \circ \theta_s$ increases from $Q_{u+t-s}(0) \circ \theta_s$ to $+\infty$ when v decreases from u to $-\infty$, we obtain

$$\begin{aligned} b\big(X(s), Q_{u+t-s}(0) \circ \theta_s\big) &\leq \sup_{v \leq u} b\big(X(s), Q_{v+t-s}(0) \circ \theta_s\big) \\ &= \sup_{x \geq Q_{u+t-s}(0) \circ \theta_s} b\big(X(s), x\big) \\ &\leq \sup_{x \geq 0} b\big(X(s), x\big), \end{aligned}$$

and $\int_0^t \sup_{x\geq 0} b(X(s), x)\, ds \in L^1(\Omega)$ from Lemma 4.2. By the definition of W, $\lim_{u\to-\infty} Q_u(0) = W = +\infty$. Thus, using the dominated convergence theorem, we get

$$E\left(\int_0^t \sup_{x \geq Q_{u+t-s}(0) \circ \theta_s} b\big(X(s), x\big)\, ds\right) \underset{u \searrow -\infty}{\longrightarrow} E\left(\int_0^t \limsup_{x\to+\infty} b\big(X(s), x\big)\, ds\right).$$

We have seen in the previous section that the difference $L_{u+t}(t) - L_{u+t}(0)$ converges decreasingly to a limit $L(t, 0)$ when $u \searrow -\infty$. Since $W = +\infty$, we obtain

$$L(t, 0) = \int_{s=0}^{s=t} \mathbb{1}_{\{W \circ \theta_s = 0\}}\, dL(s, 0) = 0.$$

Note that $L_{u+t}(t) - L_{u+t}(0)$ is integrable, and thus $E(L_{u+t}(t) - L_{u+t}(0)) \to_{u\to-\infty} E(L(t, 0)) = 0$. Using these results, (10), at $t = 1$, leads to

$$0 \leq E\left(\int_0^1 \limsup_{x\to+\infty} b\big(X(s), x\big)\, ds\right) = E\left(\limsup_{x\to+\infty} b\big(X(0), x\big)\right)$$

by Fubini's theorem and the stationarity of the process $\{X(t)\}$. This completes the proof of the first assertion.

To prove the second assertion, let us suppose that $W < +\infty$ a.s. and let $t \geq 0$.

In that case, since $\sup_{x\geq 0}[\sigma(X(0),x)]^2 \in L^1(\Omega)$, $\int_0^t \sigma(X(s), W\circ\theta_s)\,dB_s$ is integrable. Concerning b, we have, from Lemma 4.2,

$$E\left(\left|\int_0^t b\big(X(s), W\circ\theta_s\big)\,ds\right|\right) \leq E\left(\int_0^t \big|b\big(X(s), W\circ\theta_s\big)\big|\,ds\right)$$
$$\leq E\left(\int_0^t \sup_{x\geq 0}\big|b\big(X(s),x\big)\big|\,ds\right) < +\infty.$$

Moreover, $L(t,0)$ is also integrable since

$$0 \leq L(t,0) = \lim_{u\searrow -\infty, u\leq 0}[L_u(t) - L_u(0)] \leq L_0(t) - L_0(0) = L_0(t),$$

with $L_0(t) \in L^1(\Omega)$. It thus follows from (5), for $v = 0$, that $W\circ\theta_t - W$ is integrable and, from Lemma 4.1, we obtain $E(W\circ\theta_t - W) = 0$. This leads, again using Fubini's theorem and the stationarity of the process $\{X(t)\}$, to

$$\begin{aligned} 0 = E(W\circ\theta_t - W) &= E\left(\int_0^t b\big(X(s), W\circ\theta_s\big)\,ds\right) + E(L(t,0)) \\ &= \int_0^t E\big(b(X(s), W\circ\theta_s)\big)\,ds + E(L(t,0)) \qquad (11)\\ &= tE\big(b(X(0), W)\big) + E(L(t,0)). \end{aligned}$$

Now, since $L(t,0) \geq 0$, we obtain at $t = 1$ the contradiction

$$0 \geq E\big(b(X(0), W)\big) \geq E\left(\inf_{x\geq 0} b\big(X(0), x\big)\right),$$

which completes the proof, when b is lower bounded.

Step 2. This is the general case, where b is not necessarily lower bounded. Let us show the first point. Suppose that $E(\limsup_{x\to\infty} b(X(0),x)) < 0$. For all $N \in \mathbb{N}$, we denote by $(Q_u^N(t), L_u^N(t))_{t\geq u}$ the solution to the following RSDE:

$$\begin{aligned} &dQ(t) = \big[b\big(X(t), Q(t)\big) \vee (-N)\big]\,dt + \sigma\big(X(t), Q(t)\big)\,dB_t + dL(t), \qquad t \geq u,\\ &Q(u) = 0,\\ &Q(t) \geq 0, \qquad t \geq u,\\ &L(t) = \int_u^t \mathbb{1}_{\{Q(s)=0\}}\,dL(s), \qquad t \geq u. \end{aligned}$$

As $b(\cdot,\cdot) \vee (-N)$ is greater than or equal to b, Proposition 3.1 implies that $Q_u(t) \leq Q_u^N(t)$, $t \geq u$. Besides, by hypothesis,

$$\left|\limsup_{x\to+\infty}\big[b\big(X(0),x\big)\big]^+\right| \leq \sup_{x\geq 0}\big[b\big(X(0),x\big)\big]^+ \in L^1(\Omega)$$

and

$$\limsup_{x\to+\infty}\big[b\big(X(0),x\big)\vee(-N)\big]\underset{N\to+\infty}{\longrightarrow}\limsup_{x\to+\infty}b\big(X(0),x\big).$$

Thus, by the monotone convergence theorem,

$$E\bigg(\limsup_{x\to+\infty}\big[b\big(X(0),x\big)\vee(-N)\big]\bigg)\underset{N\to+\infty}{\longrightarrow}E\bigg(\limsup_{x\to+\infty}b\big(X(0),x\big)\bigg)<0.$$

This means, in particular, that, for N sufficiently large,

$$E\bigg(\limsup_{x\to+\infty}\big[b\big(X(0),x\big)\vee(-N)\big]\bigg)<0.$$

For such an N, the mapping $b(\cdot,\cdot)\vee(-N)$ is lower bounded and since $\sup_{x\geq 0}[b(X(0),x)\vee(-N)]\in L^1(\Omega)$, we may apply the results of Step 1. Thus, $Q_u^N(0)$ converges to some almost surely finite random variable, say W^N, when $u\to-\infty$. So

$$W=\lim_{u\to-\infty}Q_u(0)\leq\lim_{u\to-\infty}Q_u^N(0)=W^N<+\infty\qquad\text{a.s.}$$

and the queue is stable.

Consider now the second point. By hypothesis, $E(\inf_{x\geq 0}b(X(0),x))$ is positive and finite, so we have $\inf_{x\geq 0}b(X(0),x)\in L^1(\Omega)$ and thus

$$\begin{aligned}\sup_{x\geq 0}\big[b\big(X(0),x\big)\big]^-&=-\inf_{x\geq 0}\big[b\big(X(0),x\big)\wedge 0\big]\\&=-\bigg(\inf_{x\geq 0}b\big(X(0),x\big)\wedge 0\bigg)=\bigg(\inf_{x\geq 0}b\big(X(0),x\big)\bigg)^-,\end{aligned}$$

which implies that $\sup_{x\geq 0}[b(X(0),x)]^-\in L^1(\Omega)$. This, along with the hypothesis that $\sup_{x\geq 0}[b(X(0),x)]^+$ is in $L^1(\Omega)$, yields $\sup_{x\geq 0}|b(X(0),x)|\in L^1(\Omega)$.

We then use Proposition 3.3, which states that the buffer content converges to either a finite or an infinite r.v. W. We conclude in the same way as we did in Step 1. Let us suppose that $W<+\infty$ a.s. Then, since $W\circ\theta_t-W$ is integrable, we likewise find the contradiction

$$0\geq E\big(b\big(X(0),W\big)\big)\geq E\bigg(\inf_{x\geq 0}b\big(X(0),x\big)\bigg).\qquad\square$$

5. Properties of the Lindley equation. In this section, we give a few properties of the stationary process $\{W\circ\theta_t,\ t\in\mathbb{R}\}$ and its associate process $\{L(t,v),t\geq v\}$, which will be used in the next section.

Let us first define the following limits, which belong to $[0,+\infty]$:

$$L_W=\lim_{t\to-\infty}\uparrow L(0,t)\quad\text{and}\quad K_W=\lim_{t\to\infty}\uparrow L(t,0).$$

LEMMA 5.1. *$L_W = 0$ a.s. or $L_W = +\infty$ a.s.*

PROOF. We are first going to show that the event $\{L_W = +\infty\}$ is invariant under the flow. Suppose that, for some $\omega \in \Omega$, $L_W(\omega) = \lim_{t\to-\infty} L(0,t)(\omega) = +\infty$ and let $s \in \mathbb{R}$. Then, for all $t \le 0$, $L(0,t)(\theta_s\omega) = L(s, t+s)(\omega) = L(0, t+s)(\omega) - L(0,s)(\omega)$. By letting $t \to -\infty$, $L(0, t+s)(\omega)$ tends to $L_W(\omega) = +\infty$. Hence,

$$\lim_{t\to-\infty} L(0,t)(\theta_s\omega) = L_W(\theta_s\omega) = +\infty.$$

The event $\{L_W = +\infty\}$ is then invariant, so it is of probability 0 or 1.

Let us suppose now that $L_W < +\infty$ a.s. Then, as, for all $s \le t$,

$$L(0,t) = L(0,s) - L(t,s) = L(0,s) - L(0, s-t) \circ \theta_t,$$

we get, by letting $s \to -\infty$, $L(0,t) = L_W - L_W \circ \theta_t$. We saw that $L(0,t)$ is integrable [indeed, $0 \le L(0,t) \le L_t(0)$ with $L_t(0) \in L^1(\Omega)$]. Applying Lemma 4.1, we get that $E(L(0,t)) = 0$. Since $L(0,t)$ is nonnegative, this yields that $L(0,t) = 0$ a.s. By letting $t \to -\infty$, we thus get $L_W = 0$ a.s. $\square$

One gets similarly the following lemma.

LEMMA 5.2. *$K_W = 0$ a.s. or $K_W = +\infty$ a.s.*

Moreover, one can easily check that $P(K_W = 0) = P(L_W = 0)$.

The next corollaries point out the connection between L_W and K_W and the hitting of the level zero for the stationary process $\{W \circ \theta_t,\ t \in \mathbb{R}\}$.

COROLLARY 5.3. *Suppose that $L_W = +\infty$ on a set F^- of probability* 1. *Then, for all $\omega \in F^-$, there exists an $s^- = s^-(\omega)$ such that $W \circ \theta_{s^-}(\omega) = 0$.*

Besides this, $s^-(\omega)$ can be chosen less than or equal to S for any $S \in \mathbb{R}$.

PROOF. Let $\omega \in F^-$. As $L_W(\omega) = \lim_{t\to-\infty} L(0,t)(\omega) = +\infty$, then $L(0,t)(\omega) > 0$ for $t \le T$ for some $T = T(\omega)$. Hence, there exists some $s^- = s^-(\omega) \in [T, 0]$ such that $W \circ \theta_{s^-} = 0$. One can point out that, since $L_W(\theta_S\omega) = \lim_{t\to-\infty} L(0,t)(\theta_S\omega) = \lim_{t\to-\infty} L(t+S, S)(\omega) = +\infty$, one gets by the same method that there is an $s^- = s^-(\omega) \le S$ such that $W \circ \theta_{s^-} = 0$. $\square$

COROLLARY 5.4. *Suppose that $L_W = +\infty$ on F^- of probability* 1. *For each ω in F^-, there exists a sequence $(s_n^-)_{n\in\mathbb{N}} = (s_n^-(\omega))_{n\in\mathbb{N}}$ of decreasing real numbers tending to $-\infty$ as $n \to +\infty$ such that $W \circ \theta_{s_n^-} = 0$ for all n.*

PROOF. Let $\omega \in F^-$ and let $s_0^-(\omega) = s^-(\omega)$ as in Corollary 5.3. We proceed by induction on n. If $s_0^-, \dots, s_n^-$ exist, then, using Corollary 5.3 with $S = s_n^-(\omega) - 1$, we get an $s_{n+1}^-(\omega)$ that satisfies $W \circ \theta_{s_{n+1}^-} = 0$. One can show by induction that $s_n^- \le -n$, which, in particular, implies that $s_n^- \to -\infty$ when $n \to \infty$. $\square$

Note that this corollary means intuitively that, under the condition $L_W = +\infty$, the buffer empties infinitely often between time $t = -\infty$ and time $t = 0$.

The following result is proved similarly using K_W.

COROLLARY 5.5. *Suppose that $K_W = +\infty$ on a set F^+ of probability* 1. *For each ω in F^+, there exists a sequence $(s_n^+)_{n\in\mathbb{N}} = (s_n^+(\omega))_{n\in\mathbb{N}}$ of increasing real numbers tending to $+\infty$ as $n \to +\infty$ such that $W \circ \theta_{s_n^+} = 0$ for all n.*

Finally, the combination of Corollary 5.4 and Corollary 5.5 yields the following theorem.

THEOREM 5.6. *Suppose that $L_W = +\infty$ a.s.* (*or, equivalently, that $K_W = +\infty$ a.s.*). *Then there exists a set F of probability* 1 *such that, for each $\omega \in F$, there exists a sequence $(s_n)_{n\in\mathbb{Z}} = (s_n(\omega))_{n\in\mathbb{Z}}$ of increasing numbers tending to $+\infty$ as $n \to +\infty$ and tending to $-\infty$ as $n \to -\infty$ such that $W \circ \theta_{s_n} = 0$ for all $n \in \mathbb{Z}$.*

The proof of this result is straightforward by setting F as the intersection of F^+ and F^- of Corollaries 5.4 and 5.5, and the sequence $(s_n)_{n\in\mathbb{Z}}$ is obtained by merging $(s_n^+)_{n\in\mathbb{N}}$ and $(s_n^-)_{n\in\mathbb{N}}$. Theorem 5.6 says that the condition $L_W = +\infty$ (or $K_W = +\infty$) implies that $W \circ \theta_t$ is 0 infinitely often between $t = -\infty$ and $t = +\infty$.

The next corollary is a key result for the rest of the paper.

COROLLARY 5.7. *Suppose that $L_W = +\infty$ a.s. Then, for all u, $Q_u(t) = 0$ infinitely often. Besides, coupling between $(Q_u(t))_{t\ge u}$ and $\{W \circ \theta_t, t \in \mathbb{R}\}$ occurs; that is, for $t \ge T = T(\omega)$ great enough, we have $Q_u(t) = W \circ \theta_t$.*

PROOF. Let us recall that, for all u and $t \ge u$, $Q_u(t) \le W \circ \theta_t$. By Theorem 5.6, there exists an r.v. $T = s_n \ge u$ such that $W \circ \theta_T = 0$ [where $(s_n)_{n\in\mathbb{Z}}$ is the sequence introduced in Theorem 5.6]. Thus, for all $v \le u$, we have that $0 \le Q_u(T) \le Q_v(T) \le W \circ \theta_T = 0$, and then, $\forall\, v \le u$, $Q_v(T) = Q_u(T) = 0$. By the strong uniqueness of system (2), we then have that, for all $t \ge T$, $Q_v(t) = Q_u(t)$. By letting $v \to -\infty$, we get that, $\forall\, t \ge T$, $Q_u(t) = W \circ \theta_t$. Thus, coupling occurs at time T. Since $\{W \circ \theta_t,\ t \ge T\}$ hits 0 infinitely often, so does $(Q_u(t))_{t\ge u}$. $\square$

Theorem 5.6 and Corollary 5.7 give us information as to whether the buffer empties infinitely often in the stationary regime owing to L_W. We provide here

two criteria ensuring that $L_W = +\infty$. We then know that, if one of these criteria is met, then $\{W \circ \theta_t, t \in \mathbb{R}\}$ hits 0 infinitely often.

Let us consider the following conditions:

(C1) $E(\sup_{x\geq 0} b(X(0), x)) < 0$;
(C2) b is upper bounded, $E(\limsup_{x\to\infty} b(X(0), x)) < 0$ and $c = \inf_{z\in\mathcal{X}, x\geq 0} \sigma(z, x)^2 > 0$.

PROPOSITION 5.8. *If either* (C1) *or* (C2) *is fulfilled, then* $L_W = +\infty$ *a.s.* (*or, equivalently,* $K_W = +\infty$ *a.s.*).

PROOF. As in Theorem 4.3, the proof is divided into two steps. In Step 1, we prove the result when b is lower bounded, and the general case is studied in Step 2.

Step 1. Suppose that b is lower bounded. Suppose that (C1) is satisfied. Note in passing that $E(\limsup_{x\to\infty} b(X(0), x))$ is less than or equal to $E(\sup_{x\geq 0} b(X(0), x))$ so (C1) implies that the initially empty queue is stable (i.e., $W < +\infty$). The function b being lower bounded, (11) is applicable and yields at $t = 1$ the equality $E(L(1, 0)) = -E(b(X(0), W))$. Now, since $-E(b(X(0), W)) \geq -E(\sup_{x\geq 0} b(X(0), x)) > 0$, then necessarily $L(1, 0) > 0$ on some nonnegligible set. This implies that necessarily $L_W = +\infty$ a.s.

Suppose that (C2) is satisfied. Then W is a.s. finite because

$$E\left(\limsup_{x\to\infty} b\big(X(0), x\big)\right) < 0.$$

Intuitively, the condition $E(\limsup_{x\to\infty} b(X(0), x)) < 0$ means that $W \circ \theta_t$ is "forced downward" when reaching high values, while $\sup_{z\in\mathcal{X}, x\geq 0} b(z, x) < +\infty$ and the nondegeneracy condition ($c > 0$) ensure that $\{W \circ \theta_t, t \in \mathbb{R}\}$ behaves like a Brownian motion and (at least when not far from 0) eventually hits 0.

Let us take an $A \geq 0$ great enough such that $E(\sup_{x\geq A} b(X(0), x)) < 0$ [indeed, since we have $E(\limsup_{x\to\infty} b(X(0), x)) < 0$ and b is upper bounded, we are allowed to do so]. Let us define the concave mapping $h : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h : x \mapsto \begin{cases} 1 - \exp(-\alpha x), & \text{if } x \in [0, A], \\ \alpha \exp(-\alpha A)(x - A) + 1 - \exp(-\alpha A), & \text{if } x > A, \end{cases} \tag{12}$$

where $\alpha > 0$ will be chosen later on. Let us suppose that $L_W = 0$. This implies that $L(s, 0) = 0$ for all $s \geq 0$.

As h is only $\mathcal{C}^1$, Itô's formula [see, e.g., Karatzas and Shreve (1997)] cannot be used straightforwardly. However, a standard approximation technique of approaching h by a sequence of $\mathcal{C}^2$ mappings can be used to get the following

equality:

$$
\begin{aligned}
h(W\circ\theta_1)-h(W)=\int_0^1 &\big[h'(W\circ\theta_s)b\big(X(s),W\circ\theta_s\big)\\
&\qquad+\tfrac{1}{2}h''(W\circ\theta_s)\sigma\big(X(s),W\circ\theta_s\big)^2\big]\,ds\\
&+\int_0^1 h'(W\circ\theta_s)\sigma\big(X(s),W\circ\theta_s\big)\,dB_s,
\end{aligned}
\tag{13}
$$

where h'' is defined by $h''(x)=-\alpha^2\exp(-\alpha x)\mathbb{1}_{\{x\le A\}}$ (note that h'' is equal to the left derivative of h'). The term involving $L(s,0)$ is equal to 0 by hypothesis. Here, h' and h'' are bounded, so $h(W\circ\theta_1)-h(W)$ is integrable; since it is also finite, Lemma 4.1 again states that $E(h(W\circ\theta_1)-h(W))=0$. Let us now consider the right-hand side of (13). We have

$$
\begin{aligned}
&\int_0^1\big[h'(W\circ\theta_s)b\big(X(s),W\circ\theta_s\big)+\tfrac{1}{2}h''(W\circ\theta_s)\sigma\big(X(s),W\circ\theta_s\big)^2\big]\,ds\\
&\quad=\int_0^1\big[h'(W\circ\theta_s)b\big(X(s),W\circ\theta_s\big)\\
&\qquad\qquad+\tfrac{1}{2}h''(W\circ\theta_s)\sigma\big(X(s),W\circ\theta_s\big)^2\big]\mathbb{1}_{\{W\circ\theta_s\le A\}}\,ds\\
&\qquad+\int_0^1\big[h'(W\circ\theta_s)b\big(X(s),W\circ\theta_s\big)\\
&\qquad\qquad+\tfrac{1}{2}h''(W\circ\theta_s)\sigma\big(X(s),W\circ\theta_s\big)^2\big]\mathbb{1}_{\{W\circ\theta_s>A\}}\,ds\\
&\quad=\int_0^1\exp\big(-\alpha(W\circ\theta_s)\big)\big[\alpha b\big(X(s),W\circ\theta_s\big)\\
&\qquad\qquad-\tfrac{1}{2}\alpha^2\sigma\big(X(s),W\circ\theta_s\big)^2\big]\mathbb{1}_{\{W\circ\theta_s\le A\}}\,ds\\
&\qquad+\int_0^1\alpha\exp(-\alpha A)\mathbb{1}_{\{W\circ\theta_s>A\}}b\big(X(s),W\circ\theta_s\big)\,ds.
\end{aligned}
\tag{14}
$$

Let us set

$$
\eta(\alpha)=\alpha\sup_{z\in\mathcal{X},x\ge0}b(z,x)-\tfrac{1}{2}\alpha^2\inf_{z\in\mathcal{X},x\ge0}\sigma(z,x)^2.
$$

For the moment, let us take α large enough such that $\eta(\alpha)<0$. Then

$$
\begin{aligned}
&\int_0^1\exp\big(-\alpha(W\circ\theta_s)\big)\big[\alpha b\big(X(s),W\circ\theta_s\big)-\tfrac{1}{2}\alpha^2\sigma\big(X(s),W\circ\theta_s\big)^2\big]\mathbb{1}_{\{W\circ\theta_s\le A\}}\,ds\\
&\quad\le\int_0^1\exp(-\alpha A)\eta(\alpha)\mathbb{1}_{\{W\circ\theta_s\le A\}}\,ds
\end{aligned}
$$

and

$$\int_0^1 \alpha \exp(-\alpha A)\mathbb{1}_{\{W\circ\theta_s>A\}} b\big(X(s), W\circ\theta_s\big)\,ds$$
$$\le \int_0^1 \alpha \exp(-\alpha A)\mathbb{1}_{\{W\circ\theta_s>A\}} \sup_{x\ge A} b\big(X(s), x\big)\,ds.$$

Hence,

$$\begin{aligned}
&\int_0^1 \exp\big(-\alpha(W\circ\theta_s)\big)\Big[\alpha b\big(X(s), W\circ\theta_s\big) - \frac{1}{2}\alpha^2\sigma\big(X(s), W\circ\theta_s\big)^2\Big]\mathbb{1}_{\{W\circ\theta_s\le A\}}\,ds \\
&\quad + \int_0^1 \alpha \exp(-\alpha A)\mathbb{1}_{\{W\circ\theta_s>A\}} b\big(X(s), W\circ\theta_s\big)\,ds \\
&\quad \le \int_0^1 \Big[\exp(-\alpha A)\eta(\alpha)\mathbb{1}_{\{W\circ\theta_s\le A\}} \\
&\qquad\qquad + \alpha \exp(-\alpha A)\mathbb{1}_{\{W\circ\theta_s>A\}} \sup_{x\ge A} b\big(X(s), x\big)\Big]\,ds \\
&\quad \le \int_0^1 \alpha \exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha} \vee \sup_{x\ge A} b\big(X(s), x\big)\Big]\,ds,
\end{aligned}$$

where we have used the notation $a \vee b = \max(a, b)$. Looking back at (14), we then have

$$\begin{aligned}
&\int_0^1 \Big[h'(W\circ\theta_s) b\big(X(s), W\circ\theta_s\big) + \frac{1}{2}h''(W\circ\theta_s)\sigma\big(X(s), W\circ\theta_s\big)^2\Big]\,ds \\
&\qquad \le \int_0^1 \alpha \exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha} \vee \sup_{x\ge A} b\big(X(s), x\big)\Big]\,ds.
\end{aligned}$$

Now, taking the expectation on both sides of (13), we get

$$\begin{aligned}
0 &\le E\Big(\int_0^1 \alpha \exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha} \vee \sup_{x\ge A} b\big(X(s), x\big)\Big]\,ds\Big) \\
&= \alpha \exp(-\alpha A) E\Big(\frac{\eta(\alpha)}{\alpha} \vee \sup_{x\ge A} b\big(X(0), x\big)\Big).
\end{aligned} \tag{15}$$

Note that $E(\sup_{x\ge A} b(X(0), x)) < 0$ and $\eta(\alpha)/\alpha \to -\infty$ as $\alpha \to \infty$, so for α great enough we have

$$E\Big(\frac{\eta(\alpha)}{\alpha} \vee \sup_{x\ge A} b\big(X(0), x\big)\Big) < 0.$$

For such an α, (15) becomes a contradiction. Thus, $L_W = +\infty$.

Step 2. This is the general case, where b is not necessarily lower bounded. As in the proof of Theorem 4.3, we use the same notation W^N, corresponding to the Lindley random variable for the drift $b(\cdot,\cdot) \vee (-N)$, $N \in \mathbb{N}$. We also denote by $\{L^N(t,v)\}_{t\geq v}$ the corresponding associated process.

Suppose that (C1) is fulfilled. Let N be great enough such that

$$E\left(\sup_{x\geq 0}\left[b\big(X(0),x\big) \vee (-N)\right]\right) < 0.$$

Then, applying Step 1, we have $E(L^N(1,0)) > 0$. Since $W \circ \theta_t \leq W^N \circ \theta_t$ for all t, we then have that $L^N(1,0) \leq L(1,0)$ and so $L(1,0) > 0$ on some nonnegligible set, which leads to $L_W = +\infty$.

Suppose that (C2) is fulfilled. Let us pick N such that

$$E\left(\limsup_{x\to+\infty}\left[b\big(X(0),x\big) \vee (-N)\right]\right) < 0.$$

Using Step 1, we again have $L^N(1,0) > 0$. We conclude in the same way. $\square$

6. The queue is not initially empty. We saw, for monotonicity reasons, how important it is that the queue is initially empty in the previous sections. One may wonder if Theorem 4.3 is still valid when the queue level is initially any nonnegative real number y. The answer is unfortunately negative. Consider the simple case where the buffer level $(Q(t))_{t\geq 0}$ verifies the following simple model without any external environment:

$$\begin{aligned} dQ(t) &= Q(t)\big(1-Q(t)\big)\,dB_t, \\ Q(0) &= 1/2. \end{aligned} \tag{16}$$

In that case, the comparison theorem for SDEs [see, e.g., Karatzas and Shreve (1997)] implies that $Q(t) \in (0,1)$ for all $t \geq 0$. Note that the increasing process $(L(t))_{t\geq 0}$ that forces $Q(t)$ to be nonnegative is not necessary here [in other words, $L(t) = 0$ for all t]. We see that the drift b is equal to 0, but this may be made consistent with Theorem 4.3 by defining b as being equal to 0 on $[0,1]$ and being equal to say -1 on $(2,+\infty)$, so that the condition $\limsup_{x\to+\infty} b(x) < 0$ may be fulfilled. Again, the fact that b is not equal to 0 on $[1,+\infty)$ does not matter, as $Q(t)$ will take its values in $(0,1)$.

It is quite standard that $Q(t)$ does not converge in distribution as t tends to ∞. This is due to the fact that $Q(t)$ is Markovian (the dependence on a stationary environment having disappeared) and that the diffusion coefficient $\sigma(x) = x(1-x)$ is equal to 0 at points 0 and 1, making $Q(t)$ not positive recurrent. [For a detailed study of the positive recurrence of Markovian diffusion processes in dimension 1, see, e.g., Chapter 5, Section 7 of Rogers and Williams (1987).]

The limiting distribution of the buffer content may not, in general, exist, however, and without any further assumptions, all we have is a result of tightness.

From here, as the initial conditions play a part, we use the notation $Q_u^y(t)$ to denote the buffer level at time t when it is equal to y at time u, and $L_u^y(t)$ denotes its associate process.

Consider the solution $(Q_u^{y,y}(t), L_u^{y,y}(t))_{t\geq u}$ to the following SDE reflected at y:

$$\begin{aligned}
&dQ_u^{y,y}(t) = b\big(X(t), Q_u^{y,y}(t)\big)\,dt + \sigma\big(X(t), Q_u^{y,y}(t)\big)\,dB_t + dL_u^{y,y}(t), \qquad t \geq u,\\
&Q_u^{y,y}(u) = y,\\
&Q_u^{y,y}(t) \geq y, \qquad t \geq u,\\
&L_u^{y,y}(t) = \int_u^t \mathbb{1}_{\{Q_u^{y,y}(s)=y\}}\,dL_u^{y,y}(s), \qquad t \geq u.
\end{aligned}$$

THEOREM 6.1. *If* $E(\limsup_{x\to\infty} b(X(0), x)) < 0$, *then the buffer level starting at* y *at time* 0 $(Q_0^y(t))_{t\geq 0}$ *is tight, and there exists an a.s. finite random variable* W^y *such that, for all* $t \geq 0$,

$$0 \leq Q_0^y(t) \leq W^y \circ \theta_t.$$

The notion of tightness for $(Q_0^y(t))_{t\geq 0}$ can be compared here to the notion of substability for classical queues, introduced in Loynes (1962).

PROOF OF THEOREM 6.1. The proofs of Proposition 3.3 and Theorem 4.3 were made when reflection for $Q_u(t)$ occurred at 0 and $Q_u(u) = 0$. When reflection for $Q_u^{y,y}(t)$ occurs at y, and since $Q_u^{y,y}(u) = y$, we similarly get that $Q_u^{y,y}(0)$ [resp. $Q_u^{y,y}(t)$] converges nondecreasingly to some (not necessarily finite) r.v. W^y (resp. $W^y \circ \theta_t$) as u tends to $-\infty$, and that $E(\limsup_{x\to\infty} b(X(0), x)) < 0$ implies that $W^y < +\infty$ a.s.

Besides, $Q_0^{y,y}(t)$ is above $Q_0^y(t)$ (being reflected at $y \geq 0$), and we then easily get that

$$0 \leq Q_0^y(t) \leq Q_0^{y,y}(t) = Q_{-t}^{y,y}(0) \circ \theta_t \leq W^y \circ \theta_t.$$

Tightness then follows from the fact that $P(Q_0^y(t) > x) \leq P(W^y \circ \theta_t > x) = P(W^y > x)$ for all $x \geq 0$. $\square$

We now give sufficient conditions for the existence of the limiting distribution independently of the initial condition. These conditions will, of course, rule out situations as in SDE (16), where bounds cannot be crossed by trajectories. In particular, unless the drift b is not (on average) negative, then σ will not be allowed to be 0. Under these conditions, the limiting distribution is that of W (i.e., the one corresponding to the initially empty queue).

THEOREM 6.2. *Suppose that either* (C1) *or* (C2) *is fulfilled. Then*:

1. $(Q_0^y(t))_{t\geq 0}$ *reaches* 0 *a.s*;

2. $Q_0^y(t) = W \circ \theta_t$ *for* $t \geq T_y = T_y(\omega)$ *great enough* (*coupling effect*);
3. $Q_0^y(t) \overset{\mathcal{D}}{\to} W$ *when* $t \to \infty$ (*i.e., the queue is stable*).

Note that if $\{X(t), t \in \mathbb{R}\}$ is Markovian, then proving that $(Q_0^y(t))_{t\geq 0}$ reaches 0 leads easily, by using the Markov property and the result of Section 4, to the convergence in distribution of $Q_0^y(t)$ to W. However, since the environment is not Markovian, things are in our case more complicated, and the results of Section 5 will be used.

Note that it is easy to show that the condition $E(\inf_{x\geq 0} b(X(0), x)) > 0$ still implies that, $\forall x \geq 0$, $\lim_{t\to\infty} P(Q_0^y(t) > x) = 1$. Indeed, since $Q_0^y(0) = y \geq 0 = Q_0(0)$, we have that $Q_0^y(t) \geq Q_0(t)$ and, $\forall x \geq 0$, $\lim_{t\to\infty} P(Q_0(t) > x) = 1$ from Theorem 4.3.

PROOF OF THEOREM 6.2. Let us prove the first point under condition (C1). Since $\sup_{x\geq 0} b(X(t), x) \geq b(X(t), \cdot)$, by Proposition 3.1, $Q_0^y(t)$ is less than or equal to $\overline{Q}(t)$, where $(\overline{Q}(t), \overline{L}(t))_{t\geq 0}$ is the solution to the following RSDE:

$$\begin{aligned}
&d\overline{Q}(t) = \sup_{x\geq 0} b\big(X(t), x\big)\, dt + \sigma\big(X(t), \overline{Q}(t)\big)\, dB_t + d\overline{L}(t),\\
&\overline{Q}(0) = y,\\
&\overline{Q}(t) \geq 0,\\
&\overline{L}(t) = \int_0^t \mathbb{1}_{\{\overline{Q}(s)=0\}}\, d\overline{L}(s).
\end{aligned}$$

Hence, it suffices to show that $\overline{Q}(t)$ reaches 0 a.s. So let us suppose that $\overline{Q}(t) > 0$ for all $t \geq 0$ on some nonnegligible set F. Let us first note that $((1/t)\int_0^t \sigma(X(s), \overline{Q}(s))\, dB_s)_{t>0}$ converges in $L^2(\Omega)$ toward 0 as $t \to +\infty$, which implies that there exists some sequence $(t_k)_{k\in\mathbb{N}}$ tending to $+\infty$ as $k \to +\infty$ such that $(1/t_k)\int_0^{t_k} \sigma(X(s), \overline{Q}(s))\, dB_s$ converges to 0 a.s. Then, on F and for all k, we have

$$\frac{\overline{Q}(t_k)}{t_k} = \frac{y}{t_k} + \frac{1}{t_k}\int_0^{t_k} \sup_{x\geq 0} b\big(X(s), x\big)\, ds + \frac{1}{t_k}\int_0^{t_k} \sigma\big(X(s), \overline{Q}(s)\big)\, dB_s,$$

since $\overline{L}(t) = 0$ for all t on F. By the ergodic theorem, we have the almost sure convergence

$$\frac{1}{t_k}\int_0^{t_k} \sup_{x\geq 0} b\big(X(s), x\big)\, ds \underset{k\to\infty}{\longrightarrow} E\Big(\sup_{x\geq 0} b\big(X(0), x\big)\Big).$$

Hence, on F,

$$\frac{\overline{Q}(t_k)}{t_k} \underset{k\to\infty}{\longrightarrow} E\Big(\sup_{x\geq 0} b\big(X(0), x\big)\Big).$$

Now, since $E(\sup_{x\geq 0} b(X(0),x)) < 0$, $\overline{Q}(t_k) < 0$ for k great enough on F, which is a contradiction.

Let us now prove that (C2) implies that $Q_0^y(t) = 0$ for some t. Let us denote by τ the first hitting time of $(Q_0^y(t))_{t\geq 0}$ of 0, that is, $\tau = \inf\{t > 0 \mid Q_0^y(t) = 0\}$, with the usual convention $\inf \varnothing = +\infty$, and let us set likewise, for all $n \in \mathbb{N}$, $\tau_n = \tau \wedge n$. Let us define the mapping h as (12), and

$$\eta(\alpha) = \alpha \sup_{z\in\mathcal{X}, x\geq 0} b(z,x) - \tfrac{1}{2}\alpha^2 \inf_{z\in\mathcal{X}, x\geq 0} \sigma(z,x)^2.$$

As in the proof of Proposition 5.8, the choices of A and α are such that $\eta(\alpha) < 0$ and $E((\eta(\alpha)/\alpha) \vee \sup_{x\geq A} b(X(0),x)) < 0$. Then, similarly to (13) and with the same definition for h'', we have the Itô-like formula

$$\begin{aligned} &h(Q_0^y(\tau_n)) - h(y) \\ &\quad = \int_0^{\tau_n} \big[h'(Q_0^y(s))b\big(X(s), Q_0^y(s)\big) + \tfrac{1}{2}h''(Q_0^y(s))\sigma\big(X(s), Q_0^y(s)\big)^2\big]\,ds \\ &\qquad + \int_0^{\tau_n} h'(Q_0^y(s))\sigma\big(X(s), Q_0^y(s)\big)\,dB_s, \end{aligned}$$

which is, for our purpose, more conveniently rewritten as

$$\begin{aligned} &h(Q_0^y(\tau_n)) - h(y) \\ &\quad = \int_0^n \mathbb{1}_{\{s\leq\tau_n\}}\big[h'(Q_0^y(s))b\big(X(s), Q_0^y(s)\big) \\ &\qquad\qquad + \tfrac{1}{2}h''(Q_0^y(s))\sigma\big(X(s), Q_0^y(s)\big)^2\big]\,ds \\ &\qquad + \int_0^n \mathbb{1}_{\{s\leq\tau_n\}} h'(Q_0^y(s))\sigma\big(X(s), Q_0^y(s)\big)\,dB_s. \end{aligned} \tag{17}$$

Following the same pattern as in the proof of Proposition 5.8, we get the following equalities and inequalities:

$$\begin{aligned} &\int_0^n \mathbb{1}_{\{s\leq\tau_n\}}\Big[h'(Q_0^y(s))b\big(X(s), Q_0^y(s)\big) + \frac{1}{2}h''(Q_0^y(s))\sigma\big(X(s), Q_0^y(s)\big)^2\Big]\,ds \\ &\quad = \int_0^n \mathbb{1}_{\{s\leq\tau_n\}} \exp\big(-\alpha Q_0^y(s)\big)\Big[\alpha b\big(X(s), Q_0^y(s)\big) \\ &\qquad\qquad - \frac{1}{2}\alpha^2\sigma\big(X(s), Q_0^y(s)\big)^2\Big]\mathbb{1}_{\{Q_0^y(s)\leq A\}}\,ds \\ &\qquad + \int_0^n \mathbb{1}_{\{s\leq\tau_n\}}\alpha\exp(-\alpha A)\mathbb{1}_{\{Q_0^y(s)>A\}} b\big(X(s), Q_0^y(s)\big)\,ds \\ &\quad \leq \int_0^n \mathbb{1}_{\{s\leq\tau_n\}} \exp(-\alpha A)\eta(\alpha)\mathbb{1}_{\{Q_0^y(s)\leq A\}} \end{aligned}$$

$$
\begin{aligned}
&\quad + \mathbb{1}_{\{s\le\tau_n\}}\alpha\exp(-\alpha A)\mathbb{1}_{\{Q_0^y(s)>A\}}\sup_{x\ge A} b\big(X(s),x\big)\,ds\\
&\le \int_0^n \mathbb{1}_{\{s\le\tau_n\}}\alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\\
&= \int_0^n \alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\,\mathbb{1}_{\{\tau=+\infty\}}\\
&\quad + \int_0^n \mathbb{1}_{\{s\le\tau_n\}}\alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\,\mathbb{1}_{\{\tau<+\infty\}}.
\end{aligned}
$$

Thus, (17) yields the inequality

$$
\begin{aligned}
&h(Q_0^y(\tau_n)) - h(y)\\
&\quad\le \int_0^n \alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\,\mathbb{1}_{\{\tau=+\infty\}}\\
&\quad + \int_0^n \mathbb{1}_{\{s\le\tau_n\}}\alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\,\mathbb{1}_{\{\tau<+\infty\}}\\
&\quad + \int_0^n \mathbb{1}_{\{s\le\tau_n\}}h'(Q_0^y(s))\sigma\big(X(s),\,Q_0^y(s)\big)\,dB_s.
\end{aligned}
\tag{18}
$$

Now we may suppose (up to a subsequence) that $(1/n)\int_0^n \mathbb{1}_{\{s\le\tau_n\}}h'(Q_0^y(s))\times\sigma(X(s),\,Q_0^y(s))\,dB_s$ converges a.s. to 0 as $n\to+\infty$. Besides, $(1/n)\int_0^n \mathbb{1}_{\{s\le\tau_n\}}\times[\eta(\alpha)/\alpha\vee\sup_{x\ge A} b(X(s),x)]\,ds\,\mathbb{1}_{\{\tau<+\infty\}}$ is equivalent to $(1/n)\int_0^\tau[\eta(\alpha)/\alpha\vee\sup_{x\ge A} b(X(s),x)]\,ds\,\mathbb{1}_{\{\tau<+\infty\}}$ when $n\to+\infty$, and so tends to 0. Dividing by n in (18), and since $h(Q_0^y(\tau_n))$ is nonnegative, we get

$$
\begin{aligned}
\frac{-h(y)}{n} &\le \frac{1}{n}\int_0^n \alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\,\mathbb{1}_{\{\tau=+\infty\}}\\
&\quad + \frac{1}{n}\int_0^n \mathbb{1}_{\{s\le\tau_n\}}\alpha\exp(-\alpha A)\Big[\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(s),x\big)\Big]ds\,\mathbb{1}_{\{\tau<+\infty\}}\\
&\quad + \frac{1}{n}\int_0^n \mathbb{1}_{\{s\le\tau_n\}}h'(Q_0^y(s))\sigma\big(X(s),\,Q_0^y(s)\big)\,dB_s.
\end{aligned}
$$

Letting $n\to\infty$ and using the ergodic theorem, we get

$$
0\le\alpha\exp(-\alpha A)E\Big(\frac{\eta(\alpha)}{\alpha}\vee\sup_{x\ge A} b\big(X(0),x\big)\Big)\mathbb{1}_{\{\tau=+\infty\}}.
$$

Since

$$
E\Big(\Big(\frac{\eta(\alpha)}{\alpha}\Big)\vee\sup_{x\ge A} b\big(X(0),x\big)\Big)<0,
$$

this necessarily implies $\mathbb{1}_{\{\tau=+\infty\}} = 0$; that is, τ is finite, which means that $Q_0^y(t)$ eventually reaches 0. This completes the proof of the first point.

Let us now prove the second point. If $Q_0^y(t)$ reaches 0 at time $\tau = \tau(\omega)$, then, as $\forall t \geq 0$, $Q_0(t) \leq Q_0^y(t)$ [the initial condition for $Q_0^y(t)$ being y above 0], we have $Q_0(\tau) = Q_0^y(\tau) = 0$. By strong uniqueness, we thus have $Q_0(t) = Q_0^y(t)$ for $t \geq \tau$. Under (C1) or (C2), Proposition 5.8 states that L_W (and K_W) are infinite, so Corollary 5.7 is applicable; that is, for some T, we have $Q_0(t) = W \circ \theta_t$, $t \geq T$. By setting $T_y = T \vee \tau$, we get, for all $t \geq T_y$, $Q_0^y(t) = Q_0(t) = W \circ \theta_t$. Once the second point is established, the third point is rather straightforward. All we have to prove is that, for all bounded continuous mappings f from $\mathbb{R}_+$ to $\mathbb{R}$, $E(f(Q_0^y(t)))$ converges to $E(f(W))$. Indeed, for all $t \geq 0$ and by invariance,

$$E\big(f(Q_0^y(t))\big) = E\big(f(Q_0^y(t)) - f(W \circ \theta_t)\big) + E(f(W)).$$

But since, from second point, $f(Q_0^y(t)) - f(W \circ \theta_t)$ tends to 0 as $t \to \infty$, by the dominated convergence theorem, we get that $E(f(Q_0^y(t)) - f(W \circ \theta_t))$ tends to 0 as t tends to ∞. $\square$

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